

MANIFOLDS

1. TOPOLOGICAL MANIFOLDS

We use the countability and separation vocabulary of the notes on the Basic Concepts of Topology.

Definition 1.1. Let X be a topological space. A *chart* on X is a pair (U, φ) in which $U \subseteq X$ is open and

$$\varphi: U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$, for some $n \geq 0$. For U non-empty, the *dimension* of the chart is the dimension of the Euclidean space containing $\varphi(U)$ as an open subset. We call U the *domain* of the chart and φ its *coordinate map*, and we say the chart is *at* p if $p \in U$.

X is *locally Euclidean* if every point of X lies in the domain of a chart, of whatever dimension, and *locally Euclidean of dimension n* if every point of X lies in the domain of a chart of dimension n .

Note that $\varphi(U)$ is required to be open, so that Definition 2.1 below applies to functions defined on it, and that for any two charts (U, φ) , (V, ψ) the sets $\varphi(U \cap V)$ and $\psi(U \cap V)$ are again open, $U \cap V$ being open in both U and V .

A chart of dimension n , $\varphi: U \rightarrow \varphi(U)$, gives a system of local coordinates $(x^1, \dots, x^n)$ on the open set $U \subseteq X$. Namely x^i is the i^{th} coordinate function of $\varphi: U \rightarrow \varphi(U) \subset \mathbb{R}^n$.

Definition 1.2. A *topological manifold* is a second countable Hausdorff space X that is locally Euclidean. It is a *topological n -manifold*, or equivalently is a *manifold of dimension n* , if it is locally Euclidean of dimension n .

Remark 1.3. Each connected component of a manifold has a well-defined dimension, invariant under homeomorphism. That the dimension is determined at all is not a formal consequence of the definition: it is the theorem of *invariance of domain*, which says that an injective continuous map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ is an open map. It follows that a non-empty open subset of $\mathbb{R}^n$ can be homeomorphic to an open subset of $\mathbb{R}^m$ only if $m = n$.

Hence if p lies in the domains of charts of dimensions n and m , say (U, φ) and (V, ψ) , then $\varphi(U \cap V)$ and $\psi(U \cap V)$ are homeomorphic open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, so $m = n$: every point p of a locally Euclidean space has a well-defined $\dim_p X$, the dimension of any chart at p . The function $p \mapsto \dim_p X$ is locally constant, being constant on the domain of any chart, and therefore constant on each connected component of X . A topological manifold is a topological n -manifold exactly when $\dim_p X = n$ for every p . A disjoint union of manifolds of different dimensions, such as $\mathbb{R} \sqcup \mathbb{R}^2$, is a topological manifold that does not have a dimension.

Remark 1.4. Neither point-set hypothesis follows from being locally Euclidean, and each is there for a reason. The line with two origins — two copies of $\mathbb{R}$ glued along $\mathbb{R} \setminus \{0\}$ —

is locally Euclidean and second countable, but it is not Hausdorff, and in it a sequence can converge to two different points. The long line is locally Euclidean and Hausdorff but not second countable — indeed not paracompact — and it carries no Riemannian metric.

We will be especially interested in smooth manifolds, as defined in the next three sections. They are manifolds which carry extra structure, an atlas of smooth charts, that allows one to do calculus on them.

2. SMOOTH FUNCTIONS ON OPEN SUBSETS OF $\mathbb{R}^n$

Definition 2.1. Let $U \subseteq \mathbb{R}^n$ be open. A function $f: U \rightarrow \mathbb{R}$ is *smooth*, or of class C^∞ , if all its iterated partial derivatives

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}, \quad k \geq 0, \quad 1 \leq i_1, \dots, i_k \leq n,$$

exist and are continuous at every point of U . A map $F: U \rightarrow \mathbb{R}^m$ is *smooth* if each of its component functions $F_1, \dots, F_m$ is smooth.

Smoothness is a local condition: F is smooth on U if and only if every point of U has a neighborhood on which F is smooth. Compositions of smooth maps are smooth, by the chain rule, and this is the only fact about C^∞ that the definitions of the next two sections actually use.

Definition 2.2. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. A *diffeomorphism* $F: U \rightarrow V$ is a bijection such that both F and F^{-1} are smooth.

3. SMOOTH COMPATIBILITY AND ATLASES

Definition 3.1. Two charts (U, φ) and (V, ψ) on X are *smoothly compatible* if the *transition map*

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a diffeomorphism between open subsets of Euclidean space. If $U \cap V = \emptyset$ the condition is vacuous, and the charts are compatible.

Remark 3.2. The two charts are not required in advance to have the same dimension, but compatibility forces it when they overlap. This follows from invariance of domain, as in the case of topological manifolds. But in the smooth case one does not need to rely on that non-obvious theorem, for it is obvious that a diffeomorphism between non-empty open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ forces $n = m$: simply observe that the differential of such a map at any point is a linear isomorphism, its inverse being the differential of the inverse map. Since $\psi \circ \varphi^{-1}$ is a homeomorphism to begin with, both charts being homeomorphisms onto open sets, the content of the definition is that it and its inverse $\varphi \circ \psi^{-1}$ are smooth where they are defined.

Definition 3.3. A *smooth atlas* on X is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of pairwise smoothly compatible charts whose domains cover X .

The relation “smoothly compatible” is reflexive and symmetric but *not* transitive on charts, which is why the definition asks for the members of an atlas to be pairwise compatible. On atlases, however, the following works.

Definition 3.4. Two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ on X are *equivalent*, written $\mathcal{A} \sim \mathcal{B}$, if $\mathcal{A} \cup \mathcal{B}$ is again a smooth atlas; equivalently, if every chart of $\mathcal{A}$ is smoothly compatible with every chart of $\mathcal{B}$.

Lemma 3.5. $\sim$ is an equivalence relation on the set of smooth atlases on X .

Proof. Reflexivity and symmetry are immediate. For transitivity, suppose $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$, and let $(U, \varphi) \in \mathcal{A}$ and $(W, \chi) \in \mathcal{C}$. We must show $\chi \circ \varphi^{-1}$ is smooth on $\varphi(U \cap W)$; the same argument applied with the roles reversed handles $\varphi \circ \chi^{-1}$.

Smoothness is local, so it is enough to prove it near an arbitrary point $\varphi(p)$ with $p \in U \cap W$. Since the domains of $\mathcal{B}$ cover X there is $(V, \psi) \in \mathcal{B}$ with $p \in V$. On the open set $\varphi(U \cap V \cap W)$, which contains $\varphi(p)$,

$$\chi \circ \varphi^{-1} = (\chi \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1}),$$

a composition of smooth maps: the right factor is smooth because $\mathcal{A} \sim \mathcal{B}$, the left because $\mathcal{B} \sim \mathcal{C}$. $\square$

The same computation identifies a canonical representative in each class.

Proposition 3.6. Every smooth atlas $\mathcal{A}$ on X is contained in a unique maximal smooth atlas, namely

$$\mathcal{A}^* = \{\text{charts on } X \text{ smoothly compatible with every chart of } \mathcal{A}\}.$$

Moreover $\mathcal{A} \sim \mathcal{B}$ if and only if $\mathcal{A}^* = \mathcal{B}^*$, so the equivalence classes of atlases correspond bijectively to the maximal atlases.

Proof. Certainly $\mathcal{A} \subseteq \mathcal{A}^*$ and the domains of $\mathcal{A}^*$ cover X . To see that $\mathcal{A}^*$ is an atlas we must check that any two of its charts $(W, \chi), (W', \chi')$ are compatible, and this is the argument of Lemma 3.5: near a point $\chi(p)$ with $p \in W \cap W'$ choose $(U, \varphi) \in \mathcal{A}$ with $p \in U$ and factor $\chi' \circ \chi^{-1} = (\chi' \circ \varphi^{-1}) \circ (\varphi \circ \chi^{-1})$. Any atlas containing $\mathcal{A}$ consists of charts compatible with those of $\mathcal{A}$, hence is contained in $\mathcal{A}^*$; so $\mathcal{A}^*$ is maximal and is the unique maximal atlas containing $\mathcal{A}$. The last statement follows, since $\mathcal{A} \sim \mathcal{B}$ says exactly that $\mathcal{A} \cup \mathcal{B}$ is an atlas, i.e. that $\mathcal{A} \subseteq \mathcal{B}^*$ and $\mathcal{B} \subseteq \mathcal{A}^*$; by uniqueness both $\mathcal{A}^*$ and $\mathcal{B}^*$ are then the maximal atlas containing $\mathcal{A} \cup \mathcal{B}$. $\square$

Definition 3.7. A *smooth structure* on a space X is an equivalence class of smooth atlases on X , or equivalently, by Proposition 3.6, a maximal smooth atlas. A chart belonging to the maximal atlas is called a *smooth chart*, and its coordinate map a system of *local coordinates*.

In practice one specifies a smooth structure by writing down any convenient atlas, usually a small one, and invoking Proposition 3.6.

For this it is convenient to recognize the following.

Lemma 3.8. Let $\varphi: U \rightarrow \varphi(U)$ be a chart on X and let $U' \subseteq U$ be open. Then $\varphi|_{U'}: U' \rightarrow \varphi(U')$ is again a chart; and if φ is in a maximal smooth atlas, then $\varphi|_{U'}$ is also in that maximal atlas.

Proof. Since φ is a homeomorphism onto an open set, $\varphi(U')$ is open and $\varphi|_{U'}$ is a homeomorphism onto it, so $\varphi|_{U'}$ is a chart. For any chart (W, χ) of the maximal atlas the transition map $\chi \circ (\varphi|_{U'})^{-1}$ is the restriction of $\chi \circ \varphi^{-1}$ to the open set $\varphi(U' \cap W)$, hence a diffeomorphism, so $\varphi|_{U'}$ is compatible with every chart of the atlas. $\square$

In particular every point of a smooth manifold has arbitrarily small smooth chart domains.

4. SMOOTH MANIFOLDS

Definition 4.1. A *smooth manifold* is a second countable Hausdorff space M together with a smooth structure on it; that is, together with an equivalence class of smooth atlases.

Remark 4.2. By Remark 3.2, or by Remark 1.3 applied to the underlying topological space, any two smooth charts at a point of a smooth manifold M have the same dimension. So there is a function on M assigning to each point the dimension of any smooth chart at it, and this function is locally constant, being constant on the domain of any chart; it is therefore constant on each connected component of M . We call M a *smooth n -manifold*, or a *smooth manifold of dimension n* , and write $\dim M = n$, when that function is constantly n ; a non-empty connected smooth manifold is a smooth n -manifold for exactly one n .

Example 4.3 (Euclidean space and open subsets). $\mathbb{R}^n$ with the single chart $(\mathbb{R}^n, \text{id})$ is a smooth manifold of dimension n . If M is a smooth manifold and $U \subseteq M$ is open, the charts of M with domain contained in U form an atlas on U , making U a smooth manifold, having at each of its points the same dimension it has in M ; it is second countable and Hausdorff as a subspace.

Example 4.4 (the sphere). Let $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$, a second countable Hausdorff space as a subspace of $\mathbb{R}^{n+1}$. Let $N = (0, \dots, 0, 1)$ and $S = -N$, and let

$$\begin{aligned}\sigma_N: S^n \setminus \{N\} &\rightarrow \mathbb{R}^n, & \sigma_N(x) &= \frac{(x_1, \dots, x_n)}{1 - x_{n+1}}, \\ \sigma_S: S^n \setminus \{S\} &\rightarrow \mathbb{R}^n, & \sigma_S(x) &= \frac{(x_1, \dots, x_n)}{1 + x_{n+1}}\end{aligned}$$

be stereographic projection from the two poles. Each is a homeomorphism onto $\mathbb{R}^n$, and the two domains cover S^n . The transition map is

$$\sigma_S \circ \sigma_N^{-1}(u) = \frac{u}{|u|^2}, \quad u \in \mathbb{R}^n \setminus \{0\},$$

which is smooth and is its own inverse. So the two charts are smoothly compatible and form an atlas: S^n is a smooth manifold of dimension n .

Example 4.5 (products). If M and N are smooth manifolds of dimensions m and n , then $M \times N$ is second countable and Hausdorff, and the products $(U \times V, \varphi \times \psi)$ of charts form an atlas, since $(\varphi' \times \psi') \circ (\varphi \times \psi)^{-1} = (\varphi' \circ \varphi^{-1}) \times (\psi' \circ \psi^{-1})$ is smooth. So $M \times N$ is a smooth manifold of dimension $m + n$; the torus $S^1 \times S^1$ is an example.

5. SMOOTH MAPS

Definition 5.1. Let M and N be smooth manifolds. A map $f: M \rightarrow N$ is *smooth* if for every $p \in M$ there are smooth charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subseteq V$ such that

$$\psi \circ f \circ \varphi^{-1}: \varphi(U) \longrightarrow \psi(V)$$

is smooth. A *smooth function* on M is a smooth map $M \rightarrow \mathbb{R}$; these form an $\mathbb{R}$ -algebra $C^\infty(M)$ under pointwise operations.

A smooth map is continuous: near p we have $f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$ on U , a composition of continuous maps. The definition does not depend on the charts chosen, which is exactly what compatibility was arranged for.

Lemma 5.2. *If $f: M \rightarrow N$ is smooth then for every pair of smooth charts (U', φ') of M and (V', ψ') of N the map $\psi' \circ f \circ \varphi'^{-1}$ is smooth on the open set $\varphi'(U' \cap f^{-1}(V'))$.*

Proof. The set in question is open because f is continuous. Fix a point $\varphi'(p)$ in it and choose charts (U, φ) , (V, ψ) as in Definition 5.1 for this p . On a small enough neighborhood of $\varphi'(p)$,

$$\psi' \circ f \circ \varphi'^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \varphi'^{-1}),$$

a composition of three smooth maps: the outer two are transition maps of the smooth structures of N and M , and the middle one is smooth by hypothesis. Smoothness being local, the conclusion follows. $\square$

The following is an easy exercise.

Proposition 5.3. *Identity maps are smooth, and a composition of smooth maps is smooth. Hence smooth manifolds and smooth maps form a category $\mathbf{Man}^\infty$, and there is a forgetful functor $\mathbf{Man}^\infty \rightarrow \mathbf{Top}$.*

Definition 5.4. A *diffeomorphism* is an isomorphism in $\mathbf{Man}^\infty$: a smooth bijection $f: M \rightarrow N$ whose inverse is smooth. Two smooth structures on the same space X are *the same* if the identity map of X is a diffeomorphism from one to the other, that is, if the atlases are equivalent.

A *local diffeomorphism* is a smooth map $f: M \rightarrow N$ with the property that for each point $x \in M$ there are open neighborhoods U of x in M and V of $f(x)$ in N such that the restriction of f to U induces a diffeomorphism from U to V .

Example 5.5. A map $f: M \rightarrow \mathbb{R}$ is smooth if and only if $f \circ \varphi^{-1}$ is smooth for every chart of some atlas, since $\mathbb{R}$ has the single chart $(\mathbb{R}, \text{id})$. For $M = S^n$ and the atlas of Example 4.4, this is a condition on two functions on $\mathbb{R}^n$. The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth: in the chart σ_N it is $u \mapsto (2u, |u|^2 - 1)/(|u|^2 + 1)$.

Example 5.6 (distinct atlases, same manifold). On $X = \mathbb{R}$ the atlas $\{(\mathbb{R}, \text{id})\}$ and the atlas $\{(\mathbb{R}, t \mapsto t^3)\}$ are *not* equivalent: the transition map $t \mapsto t^{1/3}$ is not differentiable at 0. So they define different smooth structures on the same space. The resulting manifolds are nonetheless diffeomorphic, by $t \mapsto t^{1/3}$; “different structure” and “different manifold” are not the same thing.

Example 5.7 (free actions of finite groups). Let M be a smooth manifold, let G be a finite group, and let $G \times M \rightarrow M$ be a free smooth action, so that each $g \in G$ acts as a diffeomorphism of M and $gx = x$ for some x forces $g = 1$. Then the quotient space M/G inherits a unique structure of a smooth manifold in such a way that the quotient map $M \rightarrow M/G$ is a local diffeomorphism.

Indeed, the quotient map $q: M \rightarrow M/G$ is open, since $q^{-1}(q(U)) = \bigcup_{g \in G} gU$ is open whenever U is. Hence M/G is second countable: if $\{B_n\}$ is a countable base of M , then $\{q(B_n)\}$ is a countable family of open sets, and it is a base, since a point $q(x)$ of an open set $W \subseteq M/G$ has $x \in B_n \subseteq q^{-1}(W)$ for some n , whence $q(x) \in q(B_n) \subseteq W$. And M/G

is Hausdorff: if $q(x) \neq q(y)$ then the orbits Gx and Gy are disjoint finite sets, so, M being Hausdorff, they have disjoint open neighborhoods U and V ; replacing U by $\bigcap_{g \in G} gU$ and V by $\bigcap_{g \in G} gV$, which are open, still contain Gx and Gy , and are still disjoint, we may take U and V to be G -invariant. Then $q(U)$ and $q(V)$ are open neighborhoods of $q(x)$ and $q(y)$, and they are disjoint: a point of both would give $u \in U$ and $g \in G$ with $gu \in V$, and gu lies in U as well, U being invariant.

For the charts, G being finite and M Hausdorff, every $x \in M$ has an open neighborhood U with $gU \cap U = \emptyset$ for all $g \neq 1$; the quotient map restricts to a homeomorphism of U onto an open subset of M/G , and transporting the charts of M through these homeomorphisms gives an atlas whose transition maps are transition maps of M composed with the diffeomorphisms $x \mapsto gx$. For instance, the real projective n -space $\mathbb{R}P^n$, identified with $S^n/\{\pm 1\}$, naturally has the structure of a smooth manifold.

6. EXHAUSTIONS AND PARTITIONS OF UNITY

Second countability enters Definition 1.2 for the sake of this section. It yields an exhaustion of a manifold by compact sets, and with it partitions of unity, the device that assembles local constructions into global ones. Everything below is elementary and self-contained; in particular the metrization and paracompactness theorems quoted in the companion notes on separation, countability and paracompactness are not needed.

Lemma 6.1. *A topological manifold M has a countable base $D_1, D_2, \dots$ of open sets, each with compact closure contained in the domain of a chart.*

Proof. Let $C_1, C_2, \dots$ be a countable base of M and let S be the set of indices j for which $\overline{C_j}$ is compact and contained in the domain of a chart. The C_j with $j \in S$ are again a base. Indeed, let $x \in U$ with U open. By Lemma 3.8, restricting a chart at x to its intersection with U gives a chart (V, φ) at x with $V \subseteq U$; choose a Euclidean ball B about $\varphi(x)$ with $\overline{B}$ compact and contained in $\varphi(V)$, and then j with $x \in C_j \subseteq \varphi^{-1}(B)$. The set $\varphi^{-1}(\overline{B})$ is compact, hence closed, M being Hausdorff; so $\overline{C_j} \subseteq \varphi^{-1}(\overline{B}) \subseteq V$, and $\overline{C_j}$ is compact, being closed in a compact set. Thus $j \in S$ and $C_j \subseteq U$. $\square$

Proposition 6.2 (exhaustion by compact sets). *A topological manifold M has compact subsets $K_1, K_2, \dots$ with*

$$K_m \subseteq \text{int } K_{m+1} \text{ for every } m, \quad \bigcup_m K_m = M.$$

Proof. Let $D_1, D_2, \dots$ be as in Lemma 6.1 and put $K_1 = \overline{D_1}$. Given K_m , compact, finitely many D_j cover it; choose r_m greater than m and than every index so used, and put $K_{m+1} = \overline{D_1} \cup \dots \cup \overline{D_{r_m}}$, a finite union of compact sets. Then $K_m \subseteq D_1 \cup \dots \cup D_{r_m} \subseteq \text{int } K_{m+1}$, and $D_m \subseteq K_{m+1}$ for every m , so the K_m cover M . $\square$

Lemma 6.3 (bump functions). *Let (V, φ) be a smooth chart of a smooth manifold M , let $p \in V$, and let $r > 0$ be such that the closed Euclidean ball of radius $2r$ about $\varphi(p)$ lies in $\varphi(V)$. Put $B' = \varphi^{-1}(B(\varphi(p), r))$ and $B = \varphi^{-1}(B(\varphi(p), 2r))$. Then there is a smooth $\psi: M \rightarrow [0, 1]$ with $\psi \equiv 1$ on B' and $\text{supp } \psi \subseteq \overline{B}$, a compact subset of V .*

Proof. Let $h(t) = e^{-1/t}$ for $t > 0$ and $h(t) = 0$ for $t \leq 0$, a smooth function, and put

$$g(t) = \frac{h(2-t)}{h(2-t) + h(t-1)},$$

whose denominator is positive for every t ; then g is smooth, $g \equiv 1$ for $t \leq 1$ and $g \equiv 0$ for $t \geq 2$. Define $\psi(x) = g(|\varphi(x) - \varphi(p)|/r)$ for $x \in V$ and $\psi(x) = 0$ for $x \notin \overline{B}$. On V this is smooth: the Euclidean norm is smooth away from the origin, and near $\varphi(p)$ the function g is constantly 1. The two open sets V and $M \setminus \overline{B}$ cover M , since $\overline{B} \subseteq V$, and the two prescriptions agree on their intersection, where both vanish; so ψ is smooth on M . It is 1 on B' and vanishes off $\overline{B}$. $\square$

Theorem 6.4 (smooth partitions of unity). *Let M be a smooth manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M . Then there are countably many smooth functions $\varphi_i: M \rightarrow [0, 1]$ such that*

- (i) *supp φ_i is compact and contained in some U_α , for every i ;*
- (ii) *the family $\{\text{supp } \varphi_i\}$ is locally finite; and*
- (iii) *$\sum_i \varphi_i = 1$, the sum being finite near each point by (ii).*

Proof. Let $K_1, K_2, \dots$ be an exhaustion as in Proposition 6.2 and set $K_0 = K_{-1} = \emptyset$. For $m \geq 1$ put

$$A_m = K_m \setminus \text{int } K_{m-1}, \quad W_m = \text{int } K_{m+1} \setminus K_{m-2}.$$

Then A_m is compact, being closed in K_m , and W_m is open; moreover $A_m \subseteq W_m$, since $A_m \subseteq K_m \subseteq \text{int } K_{m+1}$ while A_m misses $\text{int } K_{m-1} \supseteq K_{m-2}$. The A_m cover M : if m is least with $x \in K_m$ then $x \in A_m$.

Fix m and $x \in A_m$. Choose α with $x \in U_\alpha$, restrict a smooth chart at x , by Lemma 3.8, to obtain one, (V, φ) , with $V \subseteq U_\alpha \cap W_m$, and choose $r > 0$ with the closed ball of radius $2r$ about $\varphi(x)$ inside $\varphi(V)$. This gives sets $B'_x \subseteq B_x$ as in Lemma 6.3, with $\overline{B}_x$ compact and contained in V . Finitely many of the B'_x cover the compact set A_m ; collecting these over all m gives countably many pairs $B'_i \subseteq B_i$ such that the B'_i cover M , each $\overline{B}_i$ is compact and contained in some U_α , and $B_i \subseteq W_{m(i)}$, where $m(i)$ is the stage at which B_i was chosen.

The family $\{\overline{B}_i\}$ is locally finite. A point of M lies in $\text{int } K_m$ for some m , and $W_j \cap \text{int } K_m = \emptyset$ as soon as $j \geq m + 2$, since then $K_{j-2} \supseteq K_m$. So the neighbourhood $\text{int } K_m$ meets only those $\overline{B}_i$ with $m(i) \leq m + 1$, and there are finitely many of those.

Let ψ_i be the bump function of Lemma 6.3 for the pair $B'_i \subseteq B_i$. By local finiteness the sum $\Psi = \sum_i \psi_i$ is a finite sum near each point, hence smooth, and $\Psi \geq 1$ everywhere, every point lying in some B'_i , where $\psi_i = 1$. Put $\varphi_i = \psi_i/\Psi$. These are smooth, take values in $[0, 1]$, satisfy $\text{supp } \varphi_i = \text{supp } \psi_i \subseteq \overline{B}_i$, and sum to 1. $\square$

Corollary 6.5. *Every smooth manifold is paracompact, and so is every topological manifold: the sets B_i constructed in the proof form a locally finite open refinement of the given cover.*

Proof. Only Proposition 6.2, the existence of charts, and the first assertion of Lemma 3.8 were used to produce the B_i , and all of these are available for a topological manifold. $\square$

Remark 6.6. This is the only case of paracompactness used in these notes, and the route just taken is elementary: second countability enters through Proposition 6.2 and nowhere

else. The general theorems — that a regular second countable space is metrizable, and that a metrizable space is paracompact — are quoted in the companion notes, but they are not needed here; and they would yield only continuous partitions of unity, whereas Theorem 6.4 yields smooth ones, which is what a Riemannian metric or any other patching of local smooth data requires.

7. EXERCISES

Exercise 7.1. Show that smooth compatibility of charts is not a transitive relation, by exhibiting three charts on $\mathbb{R}$ of which the first is compatible with the second and the second with the third, but the first is not compatible with the third.

Exercise 7.2. Verify in detail that σ_N of Example 4.4 is a homeomorphism onto $\mathbb{R}^n$ with inverse $u \mapsto (2u, |u|^2 - 1)/(|u|^2 + 1)$, and that $\sigma_S \circ \sigma_N^{-1}(u) = u/|u|^2$.

Exercise 7.3. Show that $C^\infty(M)$ is an $\mathbb{R}$ -algebra and that a smooth map $f: M \rightarrow N$ induces an $\mathbb{R}$ -algebra homomorphism $f^*: C^\infty(N) \rightarrow C^\infty(M)$, $g \mapsto g \circ f$. Is $M \mapsto C^\infty(M)$ a functor? Of what variance?