

LIMITS AND COLIMITS

A limit is the universal object mapping *into* a diagram, a colimit the universal object mapped *out of* one. Products, intersections, kernels, inverse limits, pullbacks are all limits; disjoint unions, quotients, cokernels, direct limits, pushouts are all colimits. The point of the definitions is that a single universal property proves at one stroke the things one would otherwise prove separately in each case — above all uniqueness, which is Theorem 4.3 below. We use the vocabulary of the companion notes on categories and functors.

1. LIMITS OVER THE NATURAL NUMBERS

Before the general definitions, here are the two cases that motivate them, in which the shape is the ordered set of natural numbers, read in one direction or the other.

Inverse limits.

Definition 1.1. An *inverse system* of sets is a sequence of sets and maps

$$\cdots \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0.$$

Its *inverse limit* is

$$\varprojlim_n X_n = \left\{ (x_n)_{n \geq 0} \in \prod_{n \geq 0} X_n : f_n(x_n) = x_{n-1} \text{ for all } n \geq 1 \right\},$$

with $\pi_m: \varprojlim_n X_n \rightarrow X_m$ the m -th coordinate.

The maps π_m satisfy $f_m \circ \pi_m = \pi_{m-1}$ for $m \geq 1$, and this construction is universal among such families:

Proposition 1.2. Let T be a set and let $\tau_m: T \rightarrow X_m$ be maps with $f_m \circ \tau_m = \tau_{m-1}$ for all $m \geq 1$. Then there is a unique map $t: T \rightarrow \varprojlim_n X_n$ with $\pi_m \circ t = \tau_m$ for every m .

Proof. If such a t exists then the m -th coordinate of $t(y)$ is $\pi_m(t(y)) = \tau_m(y)$, so t is forced: $t(y) = (\tau_m(y))_{m \geq 0}$. That formula does define a map into the inverse limit, since $f_m(\tau_m(y)) = \tau_{m-1}(y)$ says exactly that the family $(\tau_m(y))_m$ satisfies the compatibility condition; and it visibly satisfies $\pi_m \circ t = \tau_m$. $\square$

Groups, rings and spaces. Suppose now that the X_n are groups and the f_n homomorphisms. Then $\prod_n X_n$ is a group under coordinate-wise multiplication, and $\varprojlim_n X_n$ is a subgroup of it: if (x_n) and (y_n) are compatible then so is $(x_n y_n^{-1})$, because f_n is a homomorphism. If the X_n are rings and the f_n ring homomorphisms, the same computation makes $\varprojlim_n X_n$ a subring of $\prod_n X_n$, with unit the constant sequence $(1, 1, \dots)$. If the X_n are topological spaces and the f_n continuous, give $\prod_n X_n$ the product topology and $\varprojlim_n X_n$ the subspace topology.

In each case the projections π_m are morphisms of the same kind, and Proposition 1.2 holds verbatim: the map $t(y) = (\tau_m(y))_m$ is the only candidate, and it is a homomorphism

because each coordinate is one, and continuous because a map into a product is continuous exactly when all its coordinates are — and it lands in the subspace $\varprojlim_n X_n$. So the universal property characterizes the inverse limit in **Grp**, in **Ring** and in **Top**, not merely in **Set**.

Remark 1.3 (the topology). The topology on $\varprojlim_n X_n$ deserves a word, because two descriptions of it coincide, and in the discrete case there is a convenient third.

- (i) It is the subspace topology from the product topology on the product.
- (ii) It is the *initial* topology for the projections: the coarsest topology making every π_m continuous. The sub-basis of the topology is the collection of sets of the form $\pi_m^{-1}(U)$ with $U \subseteq X_m$ open. When, as here, the index set is a chain, these sub-basic sets are already closed under finite intersection, and hence form a basis: if $m \leq m'$ and $g = f_{m+1} \circ \cdots \circ f_{m'}: X_{m'} \rightarrow X_m$, then $\pi_m = g \circ \pi_{m'}$, so

$$\pi_m^{-1}(U) \cap \pi_{m'}^{-1}(U') = \pi_{m'}^{-1}(g^{-1}(U) \cap U').$$

- (iii) When each X_n is discrete — the case of interest for towers of finite groups or finite rings — a basis is given by the sets $\pi_n^{-1}(x)$, $x \in X_n$. Two points of the limit are close when they agree at a late stage.

If every X_n is Hausdorff then $\varprojlim_n X_n$ is *closed* in $\prod_n X_n$: the condition $f_n \circ \pi_n = \pi_{n-1}$ is the equalizer of two continuous maps into a Hausdorff space, hence closed, and the limit is the intersection of these closed sets. Combining this with Tychonoff's theorem: an inverse limit of compact Hausdorff spaces is compact Hausdorff, and it is non-empty as soon as every X_n is. Compactness is needed for that last point: an inverse limit of non-empty infinite discrete spaces can be empty.

Example 1.4. $\varprojlim_n \mathbb{Z}/p^n$, for the reduction maps, is the ring $\mathbb{Z}_p$ of p -adic integers; $\varprojlim_n k[x]/(x^n)$ is the ring $k[[x]]$ of formal power series. In both, an element is by definition a compatible sequence of truncations, which is Definition 1.1 verbatim.

Give each $\mathbb{Z}/p^n$ the discrete topology. By Remark 1.3 the sets $\pi_n^{-1}(x)$ form a basis for $\mathbb{Z}_p$, and these are the cosets of $p^n \mathbb{Z}_p$: the topology is the p -adic one, in which $p^n \rightarrow 0$. Each $\mathbb{Z}/p^n$ is finite, hence compact Hausdorff, so $\mathbb{Z}_p$ is a compact Hausdorff — indeed totally disconnected — topological ring. The same argument gives every inverse limit of finite groups — a *profinite* group — its usual compact totally disconnected topology. Galois groups of infinite extensions are the standard examples.

Direct limits.

Definition 1.5. A *direct system* of sets is a sequence

$$X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} X_2 \longrightarrow \cdots,$$

and its *direct limit* is the quotient

$$\varinjlim_n X_n = \left(\coprod_{n \geq 0} X_n \right) / \sim,$$

where $x \in X_m$ is identified with $g_{n-1} \cdots g_m(x) \in X_n$ for every $n \geq m$, with $\iota_m: X_m \rightarrow \varinjlim_n X_n$ the composite of the inclusion and the quotient map.

Concretely, two elements become equal in the direct limit if and only if they become equal at some finite stage. The maps ι_m satisfy $\iota_{m+1} \circ g_m = \iota_m$, and again the construction is universal:

Proposition 1.6. *Let T be a set and let $\tau_m: X_m \rightarrow T$ be maps with $\tau_{m+1} \circ g_m = \tau_m$ for all m . Then there is a unique map $t: \varinjlim_n X_n \rightarrow T$ with $t \circ \iota_m = \tau_m$ for every m .*

Proof. Every element of the direct limit is $\iota_m(x)$ for some m and some $x \in X_m$, so if t exists it must satisfy $t(\iota_m(x)) = \tau_m(x)$, which determines it. For existence, the recipe $\iota_m(x) \mapsto \tau_m(x)$ is well defined: if $\iota_m(x) = \iota_{m'}(x')$ then x and x' have a common image at some stage $n \geq m, m'$, and the compatibility $\tau_n \circ g_{n-1} \cdots g_m = \tau_m$ gives $\tau_m(x) = \tau_n(\text{common image}) = \tau_{m'}(x')$. $\square$

Example 1.7. $\mathbb{R}^\infty = \varinjlim_n \mathbb{R}^n$ for the inclusions $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$, and $S^\infty = \varinjlim_n S^n$. For a CW complex X with skeleta $X^{(n)}$ one has $X = \varinjlim_n X^{(n)}$ as a space, and $H_*(X) = \varinjlim_n H_*(X^{(n)})$. For the maps $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \cdots$ the direct limit is $\mathbb{Z}[1/p]$.

Remark 1.8 (the topology, again). Direct limits can also be formed in **Grp**, **Ab**, **Ring** and **Top** by giving the set above the evident structure: one operates on representatives at a common stage, which is possible because the system is a chain, and the universal property is proved by the argument just given. The direct limit $\varinjlim_n X_n$ carries the quotient topology from the co-product, the *final*, or weak, topology, in which a set is open exactly when its preimage in every X_n is open. Thus, a map out of $\varinjlim_n X_n$ is continuous precisely when its restriction to each X_n is.

Remark 1.9. Both constructions are instances of one pattern: a diagram indexed by the ordered set $\mathbb{N}$, and an object receiving compatible maps to it (or emitting compatible maps from it) universally. We view an ordered set as a special case of a category: its objects are the natural numbers, and between m and n there is exactly one arrow when $m \leq n$ and none otherwise.

The remaining sections generalize what we have just done, making it a definition for an arbitrary indexing category. The inverse and direct limits above become the cases $\mathcal{J} = (\mathbb{N}, \geq)$ and $\mathcal{J} = (\mathbb{N}, \leq)$ of Definitions 4.1 and 5.1.

2. THE CASE OF DIRECTED SETS

Definition 2.1. A *partially ordered set* is a set with a binary relation $x \leq y$ satisfying (i) $x \leq x$ for all x ; (ii) if $x \leq y$ and $y \leq x$, then $x = y$, and (iii) if $x \leq y$ and $y \leq z$ then $x \leq z$. A partially ordered, directed set is a non-empty partially ordered set such that for any $x_1, \dots, x_k$ there is y with $x_i \leq y$ for all i .

Definition 2.2. Let I be a partially ordered, directed set. A *projective I system* in a category $\mathcal{C}$ is the following data: For each $i \in I$ we have an object A_i of $\mathcal{C}$ and for $i \leq j$ we have a morphism $f_{j,i}: A_j \rightarrow A_i$ with the provisos that $f_{i,i} = \text{Id}_{A_i}$ and if $i \leq j \leq k$ then $f_{j,i} \circ f_{k,j} = f_{k,i}$. [In other words, make I into a category with exactly one morphism from i to j when $i \leq j$ and none otherwise; then a projective I -system in $\mathcal{C}$ is a functor $I^{\text{op}} \rightarrow \mathcal{C}$.]

Definition 2.3. Given a partially ordered, directed set I and a projective I -system $\{A_i, f_{j,i}\}$ in $\mathcal{C}$, a *projective limit* is an object L of $\mathcal{C}$ together with morphisms $\rho_i: L \rightarrow A_i$ such that $\rho_i = f_{j,i} \circ \rho_j$ whenever $i \leq j$, universal with respect to this property. That is, given any object T of $\mathcal{C}$ and morphisms $\tau_i: T \rightarrow A_i$ with $\tau_i = f_{j,i} \circ \tau_j$ whenever $i \leq j$, there is a unique morphism $t: T \rightarrow L$ with $\rho_i \circ t = \tau_i$ for every i .

As usual, the question is one of existence, since uniqueness up to canonical isomorphism follows from the universal property (the argument is that of Theorem 4.3).

There are analogous definitions for inductive systems.

Definition 2.4. Let I be a partially ordered directed set and $\mathcal{C}$ a category. An *inductive I -system* in $\mathcal{C}$ is an object, A_i , of $\mathcal{C}$ for each $i \in I$ and a morphism $f_{i,j}: A_i \rightarrow A_j$ for each $i \leq j$ subject to $f_{i,i} = \text{Id}_{A_i}$ and $f_{j,k} \circ f_{i,j} = f_{i,k}$ whenever $i \leq j \leq k$. An *inductive limit*, or *colimit*, of an inductive I -system is an object C of $\mathcal{C}$ and morphisms $\rho_i: A_i \rightarrow C$ for every i such that $\rho_j \circ f_{i,j} = \rho_i$ whenever $i \leq j$, universal for this property: given any object T and morphisms $\tau_i: A_i \rightarrow T$ with $\tau_j \circ f_{i,j} = \tau_i$ whenever $i \leq j$, there is a unique $t: C \rightarrow T$ with $t \circ \rho_i = \tau_i$ for every i .

We write $\varprojlim_{i \in I} A_i$ and $\varinjlim_{i \in I} A_i$ for these. For $I = \mathbb{N}$ they are the constructions of §1, and in **Set**, **Grp**, **Ab**, **Ring** and **Top** they exist over any directed I , by the same recipes: the projective limit is the set of families $(x_i) \in \prod_i A_i$ with $f_{j,i}(x_j) = x_i$ for $i \leq j$, and the inductive limit is $(\coprod_i A_i)/\sim$, where $x \in A_i$ and $y \in A_j$ are identified when $f_{i,k}(x) = f_{j,k}(y)$ for some $k \geq i, j$. Directedness of I is what makes the latter an equivalence relation, and what allows one to operate on representatives brought to a common stage.

Remark 2.5. The rest of this material is more advanced and we will not make use of it. It is included for those who are interested in more general constructions of limits.

3. DIAGRAMS AND CONES

The examples above are special cases of a much more general construction.

Definition 3.1. Let $\mathcal{J}$ be a small category, the *shape*. A *diagram of shape $\mathcal{J}$* in a category $\mathcal{C}$ is a functor $F: \mathcal{J} \rightarrow \mathcal{C}$. We write $F_j = F(j)$ for an object j of $\mathcal{J}$ and $F_\alpha: F_j \rightarrow F_k$ for a morphism $\alpha: j \rightarrow k$.

Definition 3.2. A *cone* on F with vertex T is a family of morphisms $(\tau_j: T \rightarrow F_j)_{j \in \text{Ob } \mathcal{J}}$ such that for every $\alpha: j \rightarrow k$ in $\mathcal{J}$ the triangle with vertices T, F_j, F_k commutes:

$$F_\alpha \circ \tau_j = \tau_k.$$

Dually, a *cocone* on F with vertex T is a family $(\tau_j: F_j \rightarrow T)_j$ with $\tau_k \circ F_\alpha = \tau_j$ for every $\alpha: j \rightarrow k$. We write $\text{Cone}(T, F)$ and $\text{Cocone}(F, T)$ for the sets of these.

A cone with vertex T is the same thing as a natural transformation $\Delta_T \Rightarrow F$, where $\Delta_T: \mathcal{J} \rightarrow \mathcal{C}$ is the constant functor at T ; the naturality squares are exactly the commuting triangles. A morphism $t: T' \rightarrow T$ turns a cone (τ_j) on T into the cone $(\tau_j \circ t)$ on T' , so $T \mapsto \text{Cone}(T, F)$ is a contravariant functor $\mathcal{C} \rightarrow \mathbf{Set}$, and $T \mapsto \text{Cocone}(F, T)$ a covariant one.

4. LIMITS

Definition 4.1. A *limit* of $F: \mathcal{J} \rightarrow \mathcal{C}$ is a cone $(\pi_j: L \rightarrow F_j)$ with the universal property: for every cone $(\tau_j: T \rightarrow F_j)$ on F there is a *unique* morphism $t: T \rightarrow L$ with

$$\pi_j \circ t = \tau_j \quad \text{for every object } j \text{ of } \mathcal{J}.$$

We write $L = \varprojlim F$, and call the π_j the *projections*. $\mathcal{C}$ is *complete* if every diagram whose shape is a small category has a limit, and *finitely complete* if every diagram of finite shape does.

The universal property says exactly that the universal cone represents the functor of cones:

Proposition 4.2. A cone (π_j) with vertex L is a limit of F if and only if for every object T the map

$$\mathrm{Hom}_{\mathcal{C}}(T, L) \longrightarrow \mathrm{Cone}(T, F), \quad t \longmapsto (\pi_j \circ t)_j,$$

is a bijection. It is then natural in T .

Proof. The map is well defined: if (π_j) is a cone then so is $(\pi_j \circ t)$, since $F_\alpha \circ \pi_j \circ t = \pi_k \circ t$. Surjectivity of the map is the existence clause of Definition 4.1 and injectivity is the uniqueness clause, so the two statements agree verbatim. Naturality in T is the identity $(\pi_j \circ t) \circ u = \pi_j \circ (t \circ u)$ for $u: T' \rightarrow T$. $\square$

Here is the basic property, the one that makes the definition usable: a limit is unique in the strongest sense available in a category — not merely up to isomorphism, but up to a *unique* isomorphism compatible with the projections.

Theorem 4.3 (uniqueness of limits). Let $(\pi_j: L \rightarrow F_j)$ and $(\pi'_j: L' \rightarrow F_j)$ both be limits of F . Then there is a unique isomorphism $u: L' \rightarrow L$ with $\pi_j \circ u = \pi'_j$ for all j .

Proof. Since (π'_j) is a cone and (L, π_j) is a limit, there is a unique $u: L' \rightarrow L$ with $\pi_j \circ u = \pi'_j$. Symmetrically there is a unique $u': L \rightarrow L'$ with $\pi'_j \circ u' = \pi_j$. Then

$$\pi_j \circ (u \circ u') = \pi'_j \circ u' = \pi_j \quad \text{for every } j,$$

so $u \circ u': L \rightarrow L$ is a morphism factoring the cone (π_j) through (L, π_j) . The identity id_L does the same, and by the uniqueness clause of Definition 4.1 applied to the cone (π_j) itself there is only one such morphism; hence $u \circ u' = \mathrm{id}_L$. Symmetrically $u' \circ u = \mathrm{id}_{L'}$. So u is an isomorphism, and it was unique with the property $\pi_j \circ u = \pi'_j$ by construction. $\square$

Remark 4.4. The proof uses the universal property twice and nothing else. This is the pattern of every uniqueness statement in the subject: an object defined by a universal property is unique up to unique isomorphism, so one may speak of *the* limit, and any construction of it computes the same object.

Example 4.5 (what the shapes give). (i) $\mathcal{J}$ *discrete* (only identity morphisms): a cone is a family $\tau_j: T \rightarrow F_j$ with no conditions, and the limit is the *product* $\prod_j F_j$. The empty $\mathcal{J}$ gives a *terminal object*.

(ii) $\mathcal{J} = \{\bullet \rightrightarrows \bullet\}$: a diagram is a pair $f, g: A \rightarrow B$ and the limit is the *equalizer*, a map $e: E \rightarrow A$ with $fe = ge$ through which every such map factors uniquely. In **Set**, $E = \{a \in A : f(a) = g(a)\}$.

- (iii) $\mathcal{J} = \{\bullet \rightarrow \bullet \leftarrow \bullet\}$: the limit of $A \xrightarrow{u} C \xleftarrow{v} B$ is the *pullback* $A \times_C B$; in **Set** it is $\{(a, b) : u(a) = v(b)\}$.
- (iv) $\mathcal{J} = (\mathbb{N}, \geq)$: the limit of $\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$ is the *inverse limit* $\varprojlim X_n$.

5. COLIMITS

Everything dualizes: reverse all arrows.

Definition 5.1. A *colimit* of $F: \mathcal{J} \rightarrow \mathcal{C}$ is a cocone $(\iota_j: F_j \rightarrow L)$ such that for every cocone $(\tau_j: F_j \rightarrow T)$ there is a unique $t: L \rightarrow T$ with $t \circ \iota_j = \tau_j$ for all j . We write $L = \varinjlim F$ and call the ι_j the *structure maps*. $\mathcal{C}$ is *cocomplete* if every small diagram has a colimit.

Proposition 5.2 (duality). *A colimit of $F: \mathcal{J} \rightarrow \mathcal{C}$ is the same thing as a limit of the opposite diagram $F^{\text{op}}: \mathcal{J}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$. Consequently every statement about limits has a dual statement about colimits, proved by applying it in $\mathcal{C}^{\text{op}}$.*

Proof. A cocone on F with vertex T is literally a cone on F^{op} with vertex T regarded in $\mathcal{C}^{\text{op}}$, the direction of every morphism having been reversed, and the universal property of Definition 5.1 becomes that of Definition 4.1. $\square$

Corollary 5.3 (uniqueness of colimits). *Any two colimits of F are related by a unique isomorphism commuting with the structure maps.*

Proof. Apply Theorem 4.3 in $\mathcal{C}^{\text{op}}$ and use Proposition 5.2. $\square$

Example 5.4. Dually to Example 4.5: a discrete shape gives the *coproduct* $\coprod_j F_j$ — disjoint union in **Set**, free product in **Grp**, direct sum in **Ab** — and the empty shape gives an *initial object*; two parallel arrows give the *coequalizer*, which in **Set** is the quotient of B by the equivalence relation generated by $f(a) \sim g(a)$; the shape $\bullet \leftarrow \bullet \rightarrow \bullet$ gives the *pushout*; and $(\mathbb{N}, \leq)$ gives the *direct limit* $\varinjlim X_n$.

6. EXERCISES

Exercise 6.1. Show that a terminal object is the limit of the empty diagram, and deduce from Theorem 4.3 that any two terminal objects are isomorphic by a unique isomorphism.

Exercise 6.2. Prove that in **Set** the limit of $F: \mathcal{J} \rightarrow \mathbf{Set}$ is the set of families $(x_j) \in \prod_j F_j$ with $F_\alpha(x_j) = x_k$ for every $\alpha: j \rightarrow k$, and identify the projections.