

THE HUREWICZ THEOREMS

These notes continue *Pseudomanifolds*, whose oriented pseudomanifolds with boundary are the geometric representatives of homology classes used throughout, and they rest as well on *Homology, Singular Homology, Further Properties of Singular Homology*, shortened to *Further Properties*, and *The homotopy category, and the homotopy groups*. References to the companions are given as *Theorem such-and-such of Pseudomanifolds*, and similarly for the others; all other references are internal.

We keep the notation of *Pseudomanifolds*: $|K|$ is the geometric realization of a finite simplicial complex K ; for points $p_0, \dots, p_k$ of a convex set C , $\langle p_0, \dots, p_k \rangle: \Delta^k \rightarrow C$ is the affine singular simplex with $e_j \mapsto p_j$, and $\iota_n = \langle e_0, \dots, e_n \rangle$ is the identity of Δ^n read as a singular simplex; $C_*^{\text{or}}(K)$ is the oriented simplicial chain complex, $H_*^{\text{or}}(K)$ its homology, and $\Theta_\lambda: C_*^{\text{or}}(K) \rightarrow C_*(|K|)$ the comparison chain map attached to a total order λ of the vertices, recalled in §3 of those notes. The homotopy extension property, Lemma 6.4 of *The homotopy category, and the homotopy groups*, is used constantly below and is cited without further attribution.

§1 proves the Hurewicz theorem identifying $\pi_n(X, x_0)$ with $H_n(X)$ in the first degree in which either can be non-zero, §2 proves the relative theorem for a pair, and §3 draws from it the criterion that a map of simply connected spaces inducing isomorphisms on homology in a range does the same on homotopy groups.

1. THE HUREWICZ THEOREM

This is the first place where the pseudomanifolds of the companion pay: the theorem identifies homology, in the first degree in which it can be non-zero, with a homotopy group.

By Definition 4.1 of *The homotopy category, and the homotopy groups*, $\pi_n(X, x_0)$ is the set of homotopy classes rel ∂I^n of maps of pairs $\varphi: (I^n, \partial I^n) \rightarrow (X, x_0)$, a group for $n \geq 1$ and abelian for $n \geq 2$. For $n \geq 1$ let $\mu_n \in H_n(I^n, \partial I^n)$ be the fundamental class of the cube, Definition 3.4 of *Further Properties*, determined by the standard orientation of $\mathbb{R}^n$. For $n = 0$ this reads: I^0 is a point and $\partial I^0 = \emptyset$, so such a φ is a point of X and $\pi_0(X, x_0)$ is the set of path components, pointed by that of x_0 . We put $\mu_0 \in H_0(I^0, \partial I^0) = H_0(\text{pt})$ equal to the class of the point; this is a convention, the local classes requiring $n \geq 1$.

Let j denote the inclusion of pairs $(X, \emptyset) \rightarrow (X, \{x_0\})$. In the reduced long exact sequence of the pair, Theorem 6.4 of *Singular Homology*, every group $\tilde{H}_i(\{x_0\})$ vanishes, so

$$j_*: \tilde{H}_n(X) \longrightarrow H_n(X, \{x_0\})$$

is an isomorphism for every $n \geq 0$; for $n \geq 1$ the source is $H_n(X)$.

Definition 1.1. The *Hurewicz map* is

$$h: \pi_n(X, x_0) \longrightarrow \tilde{H}_n(X), \quad h[\varphi] = j_*^{-1}(\varphi_* \mu_n),$$

where $\varphi_*: H_n(I^n, \partial I^n) \rightarrow H_n(X, \{x_0\})$. It is a homomorphism for $n \geq 1$.

It is well defined: homotopic maps of pairs induce the same homomorphism on relative homology, by Theorem 6.6 of *Singular Homology*. It is natural in (X, x_0) , both classes being carried along by g_* for a pointed map g . In degree 0 it unwinds to $h[y] = [y] - [x_0] \in \tilde{H}_0(X)$ for a point y of X , the classical description. From now on we restrict to $n \geq 1$.

That h is a homomorphism is seen as follows. Write $I_-^n = \{t_1 \leq \frac{1}{2}\}$ and $I_+^n = \{t_1 \geq \frac{1}{2}\}$, so that the product on π_n is $[\varphi][\psi] = [\varphi \cdot \psi]$, where $\varphi \cdot \psi$ is $\varphi \circ e_-$ on I_-^n and $\psi \circ e_+$ on I_+^n , with $e_\pm: I_\pm^n \rightarrow I^n$ the affine homeomorphisms doubling the first coordinate; both have positive determinant. The map $\varphi \cdot \psi$ carries $I^n \setminus (\text{int } I_-^n \cup \text{int } I_+^n)$ to x_0 , so $(\varphi \cdot \psi)_* \mu_n$ may be computed in

$$H_n(I^n, I^n \setminus (\text{int } I_-^n \cup \text{int } I_+^n)) \cong H_n(I_-^n, \partial I_-^n) \oplus H_n(I_+^n, \partial I_+^n),$$

the isomorphism coming from the relative Mayer–Vietoris sequence, Theorem 7.19 of *Singular Homology*, applied to $X = I^n$ with $Y = I^n \setminus (\text{int } I_-^n \cup \text{int } I_+^n) = \partial I^n \cup M$, where $M = \{t_1 = \frac{1}{2}\}$, and to the enlarged halves $\{t_1 \leq \frac{1}{2} + \varepsilon\}$, $\{t_1 \geq \frac{1}{2} - \varepsilon\}$, whose interiors cover: their intersection is a slab $M \times [-\varepsilon, \varepsilon]$ and it meets Y in $M \cup (\partial M \times [-\varepsilon, \varepsilon])$, a deformation retract of the slab, so the corresponding relative groups vanish and the remaining two terms give the displayed sum; each enlarged half deformation retracts to the corresponding half as a pair. There μ_n has components the fundamental classes of the two halves: each is detected by a local class at an interior point, where μ_n restricts to o_x by Definition 3.2 of *Further Properties*. Since $e_\pm$ carry those fundamental classes to μ_n , by Lemma 3.3 of those notes applied to $e_\pm^{-1}$, the two components contribute $\varphi_* \mu_n$ and $\psi_* \mu_n$.

Theorem 1.2 (Hurewicz). *Let $n \geq 2$ and let X be a path-connected space with $\pi_i(X, x_0) = 0$ for $1 \leq i \leq n-1$. Then*

- (i) $\tilde{H}_i(X) = 0$ for $i < n$; and
- (ii) $h: \pi_n(X, x_0) \rightarrow H_n(X)$ is an isomorphism.

Remark 1.3. The case $n = 1$ is different and is due to Poincaré: for any path-connected X the map $h: \pi_1(X, x_0) \rightarrow H_1(X)$ is surjective with kernel the commutator subgroup, so that $H_1(X)$ is the abelianization of $\pi_1(X, x_0)$. The hypothesis $n \geq 2$ in Theorem 1.2 is what makes π_n abelian, and the hypothesis that X be $(n-1)$ -connected is essential: for $X = S^1 \vee S^2$ the group π_2 is infinitely generated while $H_2(X) \cong \mathbb{Z}$.

Lemma 1.4. *Let $n \geq 1$, let X be path connected with $\pi_j(X, x_0) = 0$ for $1 \leq j \leq n-1$, let K be a finite simplicial complex and let $f: |K| \rightarrow X$ be a map. Write K^j for the j -skeleton of K , the subcomplex of simplices of dimension at most j . Then f is homotopic to a map constant with value x_0 on $|K^{n-1}|$, by a homotopy that may be taken stationary on every simplex of K on which f is already constant with value x_0 , and hence on every subcomplex of such simplices.*

Proof. We deform f over the skeleta. Since X is path connected, the images of the finitely many vertices of K can be joined to x_0 by paths, and by Lemma 6.4, applied to K and its subcomplex K^0 , the resulting homotopy of $f|_{K^0}$ extends to a homotopy of f ; so we may assume that f is constant with value x_0 on K^0 .

Suppose $1 \leq j \leq n-1$ and that f is constant with value x_0 on K^{j-1} . Let s be a j -simplex of K . Its characteristic map identifies $f|_s$ with a map of pairs $(\Delta^j, \partial \Delta^j) \rightarrow (X, x_0)$, and

composing with the radial homeomorphism of pairs $(I^j, \partial I^j) \rightarrow (\Delta^j, \partial \Delta^j)$ turns that into a representative of an element of $\pi_j(X, x_0)$, in the sense of Definition 4.1 of *The homotopy category, and the homotopy groups*. That group vanishes; transporting the null-homotopy back, $f|_s$ is homotopic rel ∂s to the constant map at x_0 . These homotopies are stationary on K^{j-1} , so together they form a homotopy of $f|_{K^j}$, which extends to a homotopy of f by Lemma 6.4; we may therefore assume that f is constant with value x_0 on K^j . After the step $j = n - 1$ the map is constant with value x_0 on $|K^{n-1}|$.

The homotopies just chosen are arbitrary, and at a vertex already carried to x_0 one may take the constant path, while on a simplex on which f is already constant with value x_0 one may take the stationary homotopy. Let S be the set of simplices of K on which f is constant with value x_0 ; it is a subcomplex, a face of such a simplex being again one. With those choices the homotopy prescribed at each stage is stationary on $|S|$, so it may be prescribed on $|K^j \cup S|$ instead, the two prescriptions agreeing where they meet, and extended over $|K|$ by Lemma 6.4 applied to the pair $(K, K^j \cup S)$. Such an extension restricts to the prescription on $|S|$, hence is stationary there; and the map at the end of the stage is again constant with value x_0 on $|S|$, so the choice may be repeated. The deformation is therefore stationary on $|S|$, and hence on every subcomplex of simplices on which f is constant with value x_0 . $\square$

Proof of Theorem 1.2(i). For $i = 0$ the assertion is the path-connectedness of X : it gives $H_0(X) \cong \mathbb{Z}$, hence $\tilde{H}_0(X) = 0$.

Let $0 < i < n$ and let $\alpha \in H_i(X)$. By Corollary 5.2 of *Pseudomanifolds* there are an oriented closed i -dimensional pseudomanifold M and a map $f: |M| \rightarrow X$ with $f_*[M] = \alpha$. Since $\dim M = i \leq n - 1$ we have $|M^{n-1}| = |M|$, so Lemma 1.4 makes f homotopic to a constant map. A constant map induces 0 on reduced homology, so $\alpha = f_*[M] = 0$. Hence $\tilde{H}_i(X) = 0$ for $i < n$. $\square$

It remains to prove that h is an isomorphism. Two preliminaries: a chain-level description of h , and the additivity of the classes of the faces of a simplex.

First some notation. A map of pairs $\tau: (\Delta^n, \partial \Delta^n) \rightarrow (X, \{x_0\})$ is assigned a class in $\pi_n(X, x_0)$ by reading it off using the convex body Δ^n , as in Definition 9.8 of *The homotopy category, and the homotopy groups*: it is the class of $\alpha(\tau, T)$ for an affine linear map T of positive determinant carrying Δ^n into I^n , and by Lemma 9.7 it is independent of which such T is taken. We write it $\kappa(\tau)$, the element of the homotopy group that τ represents, as opposed to $[\tau]$, which is the element of relative homology that τ represents. More generally, $\kappa(g)$ is the class read off from a map of pairs $g: (U, \partial U) \rightarrow (X, x_0)$, where $U \subseteq \mathbb{R}^n$ is compact and convex with non-empty interior. For a map on the cube itself κ returns the ordinary homotopy class, Remark 9.9. All the classes attached to maps on Δ^n below are understood in this sense, and by Corollary 9.12 of those notes every element of $\pi_n(X, x_0)$ is of the form $\kappa(\tau)$.

Lemma 1.5. *Let $n \geq 1$ and let $\tau: (\Delta^n, \partial \Delta^n) \rightarrow (X, \{x_0\})$ be a map of pairs. Then*

$$j_* h(\kappa(\tau)) = [\tau],$$

where $[\tau]$ is the class in $H_n(X, \{x_0\})$ of the relative cycle determined by τ .

Proof. Let T be an affine linear map of positive determinant carrying Δ^n into I^n , put $D = T(\Delta^n)$ and write $\alpha = \alpha(\tau, T)$, so that by definition $\kappa(\tau) = \kappa(\alpha)$ and α is $\tau \circ T^{-1}$ on D

and x_0 on $I^n \setminus D$. It carries $I^n \setminus \text{int } D$ to x_0 , so it is also a map of pairs $(I^n, I^n \setminus \text{int } D) \rightarrow (X, \{x_0\})$ and $\alpha_* \mu_n$ may be computed after passing to $H_n(I^n, I^n \setminus \text{int } D)$. There, by Lemma 3.3 of *Further Properties*, the image of μ_n is the image of $T_*[l_n]$ under the map of pairs $(D, \partial D) \rightarrow (I^n, I^n \setminus \text{int } D)$. Hence

$$j_* h(\kappa(\tau)) = \alpha_* \mu_n = (\alpha|_D)_* T_*[l_n] = (\tau \circ T^{-1})_* T_*[l_n] = \tau_*[l_n] = [\tau].$$

□

Lemma 1.6. *Let $n \geq 2$, let M be an oriented closed n -dimensional pseudomanifold, let λ be a total order of its vertices, and let $g: |M| \rightarrow X$ be constant with value x_0 on M^{n-1} . For an n -simplex s of M let σ_s and ε_s be as in Theorem 3.2 of Pseudomanifolds. Then*

$$g_*[M] = h\left(\sum_s \varepsilon_s \kappa(g \circ \sigma_s)\right),$$

the sum being formed in $\pi_n(X, x_0)$.

Proof. Each σ_s carries $\partial \Delta^n$ into M^{n-1} , so $g \circ \sigma_s$ is a map of pairs $(\Delta^n, \partial \Delta^n) \rightarrow (X, \{x_0\})$ and $\kappa(g \circ \sigma_s)$ is defined. By Theorem 3.2 of those notes the class $[M]$ is represented by the cycle $c_\lambda = \sum_s \varepsilon_s \sigma_s$, so $g_*[M]$ is represented by $\sum_s \varepsilon_s (g \circ \sigma_s)$. Applying j_* and Lemma 1.5,

$$j_* g_*[M] = \sum_s \varepsilon_s [g \circ \sigma_s] = \sum_s \varepsilon_s j_* h(\kappa(g \circ \sigma_s)) = j_* h\left(\sum_s \varepsilon_s \kappa(g \circ \sigma_s)\right),$$

and j_* is injective. □

Proof of surjectivity in Theorem 1.2(ii). Let $\alpha \in H_n(X)$. By Corollary 5.2 of *Pseudomanifolds* there are an oriented closed n -dimensional pseudomanifold M and a map $f: |M| \rightarrow X$ with $f_*[M] = \alpha$. By Lemma 1.4, applied to M , we may deform f so that it is constant with value x_0 on M^{n-1} ; this does not change $f_*[M]$. Lemma 1.6 now gives $\alpha = h(\sum_s \varepsilon_s \kappa(f \circ \sigma_s))$, so α is in the image of h . □

Lemma 1.7. *Let $n \geq 2$ and let $G: \partial \Delta^{n+1} \rightarrow X$ be constant with value x_0 on the $(n-1)$ -skeleton of Δ^{n+1} . If G extends to a map $\Delta^{n+1} \rightarrow X$, then*

$$\sum_{i=0}^{n+1} (-1)^i \kappa(G \circ \delta^i) = 0 \quad \text{in } \pi_n(X, x_0).$$

Proof. Write $\gamma_i = \kappa(G \circ \delta^i)$ for $0 \leq i \leq n+1$, and write t_i for the i -th face of Δ^{n+1} , so that δ^i carries Δ^n affinely onto t_i ; the result to be proved is then $\sum_{i=0}^{n+1} (-1)^i \gamma_i = 0$. Let $\widehat{G}: \Delta^{n+1} \rightarrow X$ be an extension of G .

Two hemispheres. Put $E = t_0 \cup \dots \cup t_n$, so that $\partial \Delta^{n+1} = E \cup t_{n+1}$ and $E \cap t_{n+1} = \partial t_{n+1}$. Let b be the barycenter of t_{n+1} and let

$$p: \Delta^{n+1} \longrightarrow t_{n+1}$$

be the affine linear map that sends e_i to e_i for every $i < n+1$ and sends e_{n+1} to b ; it does land in t_{n+1} , an affine linear map carrying Δ^{n+1} into the convex hull of the images of its vertices. This is the Schlegel diagram of Δ^{n+1} based at t_{n+1} , in its parallel form.

Since p fixes every vertex of t_{n+1} it is the identity on t_{n+1} , and in particular on ∂t_{n+1} . It carries t_i , for $i \leq n$, affinely onto the simplex

$$u_i = \langle e_0, \dots, \widehat{e_i}, \dots, e_n, b \rangle,$$

the cone from b over the i -th codimension-one face of t_{n+1} , and these are the simplices of the subdivision of t_{n+1} obtained by coning its boundary to b : in barycentric coordinates $u_i = \{x \in t_{n+1} : x_i = \min_j x_j\}$, so the u_i cover t_{n+1} , are convex with non-empty interiors, and have pairwise disjoint interiors. Thus p induces a piecewise linear isomorphism between E and that subdivision; in particular $p|_E : E \rightarrow t_{n+1}$ is a homeomorphism, affine on each t_i and the identity on ∂t_{n+1} . Write $K = t_{n+1}$, let $\psi_+ : K \rightarrow t_{n+1}$ be the identity, and let $\psi_- : K \rightarrow E$ be the inverse of $p|_E$; these agree on ∂K , both being the identity there. Since G is constant with value x_0 on the $(n-1)$ -skeleton, and in particular on ∂t_{n+1} , the composites $G \circ \psi_{\pm}$ are maps of pairs $(K, \partial K) \rightarrow (X, x_0)$, and so carry classes $\kappa(G \circ \psi_{\pm})$ in $\pi_n(X, x_0)$, read off using the convex body K as in Definition 9.8 of *The homotopy category, and the homotopy groups*.

The two hemispheres carry the same class. For $x \in K$ the segment from $\psi_-(x)$ to $\psi_+(x)$ lies in Δ^{n+1} , that set being convex, so

$$H(x, u) = \widehat{G}((1-u)\psi_-(x) + u\psi_+(x))$$

is defined and continuous. It runs from $G \circ \psi_-$ to $G \circ \psi_+$, and for $x \in \partial K$ it is stationary at x_0 , since there $\psi_-(x) = \psi_+(x)$ and $G(x) = x_0$. So the two are homotopic as maps of pairs and

$$\kappa(G \circ \psi_-) = \kappa(G \circ \psi_+).$$

This is the only use made of the extension $\widehat{G}$.

Computing the two sides. The map $G \circ \psi_+$ is $G|_{t_{n+1}}$, whose class read through the affine identification δ^{n+1} is γ_{n+1} ; so $\kappa(G \circ \psi_+) = \sigma_{n+1}\gamma_{n+1}$, where $\sigma_{n+1} = \pm 1$ is the sign of the determinant of that identification, by Lemma 9.10 and part (iii) of Lemma 9.7 of *The homotopy category, and the homotopy groups*. The map $G \circ \psi_-$ is constant with value x_0 on the $(n-1)$ -skeleton of the subdivision $K = u_0 \cup \dots \cup u_n$, the map p carrying the $(n-1)$ -skeleton of $\partial\Delta^{n+1}$ to it; so Theorem 9.14 of *The homotopy category, and the homotopy groups* gives

$$\kappa(G \circ \psi_-) = \sum_{i=0}^n \sigma_i \gamma_i,$$

the i -th term being read through the affine identification $p \circ \delta^i$ of Δ^n with u_i , and $\sigma_i = \pm 1$ the sign of its determinant, by the same two results.

The signs. All the identifications in sight are affine, so the signs can be computed. Read classes on K through the orientation of the hyperplane of t_{n+1} for which δ^{n+1} is positive; thus $\sigma_{n+1} = 1$. For $i \leq n$ the identification $p \circ \delta^i$ carries the ordered vertices $e_0, \dots, e_n$ of Δ^n to

$$e_0, \dots, \widehat{e_i}, \dots, e_n, b,$$

since δ^i omits e_i and p fixes $e_0, \dots, e_n$ and carries e_{n+1} to b . Its sign is that of the determinant of the matrix whose rows are the barycentric coordinates of those points with respect to $e_0, \dots, e_n$. All rows but the last are standard basis vectors, and in the expansion along the last row $b = (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ only the i -th column contributes, every other minor having the zero row left by the deleted basis vector; the minor there is the identity, so the determinant is $(-1)^{n+i}/(n+1)$. Hence

$$\sigma_i = (-1)^{n+i} \quad (i \leq n), \quad \sigma_{n+1} = 1,$$

and the equality of the two classes reads

$$\sum_{i=0}^n (-1)^{n+i} \gamma_i = \gamma_{n+1}.$$

Multiplying by $(-1)^n$ gives $\sum_{i=0}^n (-1)^i \gamma_i = (-1)^n \gamma_{n+1}$, that is, $\sum_{i=0}^{n+1} (-1)^i \gamma_i = 0$. $\square$

Proof of injectivity in Theorem 1.2(ii). Suppose that ξ is an element of $\pi_n(X, x_0)$ with $h(\xi) = 0$. As explained before Lemma 1.5, represent ξ by a map of pairs

$$\tau: (\Delta^n, \partial\Delta^n) \longrightarrow (X, \{x_0\}).$$

If τ is the constant simplex there is nothing to prove, so assume it is not. By Lemma 1.5 the relative class $[\tau]$ vanishes, so $\tau = \partial d + e$ with d in $C_{n+1}(X)$ and e in $C_n(\{x_0\})$; the latter group is infinite cyclic on the constant simplex c , so

$$\partial d = \tau - mc \quad \text{for some } m \in \mathbb{Z}.$$

Write $d = \sum_{j=1}^N \rho_j \sigma_j$ with $\rho_j = \pm 1$, call the pairs (j, i) with $0 \leq i \leq n+1$ the facets, and attach to the facet (j, i) the singular simplex $\sigma_j \circ \delta^i$ and the sign $\eta_{j,i} = \rho_j (-1)^i$, as in Step 2 of the proof of Theorem 5.1 of *Pseudomanifolds*. Grouping the facets by the singular simplices they carry, the signed count is 1 at τ , it is $-m$ at c , and it is 0 at every other singular n -simplex. Hence the facets may be paired off, each pair carrying the same singular simplex with opposite signs, as far as the signs allow, leaving unpaired $|m|$ facets carrying c , all of one sign, and a single facet (j_0, i_0) carrying τ , of sign $+1$. There may be many more facets carrying τ or c than these; but this is the count of the unpaired facets.

By Lemma 4.6 of those notes, gluing N copies $P_1, \dots, P_N$ of $\text{sd}^2 \Delta^{n+1}$ along the paired facets produces a finite simplicial complex W . The second barycentric subdivision is there only so that the result of the gluing is again a simplicial complex; it plays no further part in the argument, and the structure that matters is the coarser one. Let

$$q: \coprod_{j=1}^N \Delta_j^{n+1} \longrightarrow |W|$$

be the quotient map of that lemma and let q_j be its restriction to the j -th copy. The maps σ_j agree on identified points, so there is a unique map $F: |W| \rightarrow X$ with $F \circ q_j = \sigma_j$ for every j .

By Proposition 4.8 of those notes, W is in fact a normal $(n+1)$ -dimensional pseudomanifold with boundary, its free n -simplices being those lying on the unpaired facets — one carrying τ and $|m|$ carrying c . Nothing below uses this: the argument runs by studying the $q_j: \Delta_j^{n+1} \rightarrow |W|$ one at a time. But it is the same bookkeeping seen from the other side, and it is what makes the construction of §2 produce a pseudomanifold with boundary mapping its boundary into the subspace.

The plan is to apply Lemma 1.7 to each Δ_j^{n+1} . That lemma asks for a map constant with value x_0 on the $(n-1)$ -skeleton of Δ^{n+1} , the union of the codimension-two faces of the simplex, and σ_j has no reason to be one: the faces $\sigma_j \circ \delta^i$ need not carry $\partial\Delta^n$ to x_0 , so the classes we want are not yet defined. Deform F first. By Lemma 1.4, applied to W , there is a homotopy of F to a map $F_1: |W| \rightarrow X$ constant with value x_0 on $|W^{n-1}|$, stationary on every simplex on which F is already constant with value x_0 ; we fix such a homotopy. The image under q of an $(n-1)$ -face of a copy is sd^2 of an $(n-1)$ -simplex,

by Corollary 4.2 there, so all of its simplices have dimension at most $n - 1$ and it lies in $|W^{n-1}|$. Hence F_1 is constant with value x_0 on the image of every $(n - 1)$ -face of every copy Δ_j^{n+1} .

Fix j and put $G_j = F_1 \circ q_j: \Delta^{n+1} \rightarrow X$; the map q_j need not be injective, and nothing below uses that it is. By the last paragraph G_j is constant with value x_0 on the $(n - 1)$ -skeleton of Δ^{n+1} , so $\beta_{j,i} = \kappa(G_j \circ \delta^i) \in \pi_n(X, x_0)$ is defined for $0 \leq i \leq n + 1$; and G_j is an extension over Δ^{n+1} of its own restriction to $\partial\Delta^{n+1}$. So Lemma 1.7, applied to that restriction, gives $\sum_i (-1)^i \beta_{j,i} = 0$. Multiplying by ρ_j and summing over j ,

$$(1.1) \quad \sum_{j,i} \eta_{j,i} \beta_{j,i} = 0.$$

If the facets (j, i) and (j', i') are paired then $q_j \circ \delta^i = q_{j'} \circ \delta^{i'}$, the gluing being by $\delta^{i'} \circ (\delta^i)^{-1}$; so $G_j \circ \delta^i$ and $G_{j'} \circ \delta^{i'}$ are the same map, $\beta_{j,i} = \beta_{j',i'}$, while $\eta_{j,i} = -\eta_{j',i'}$, and those terms cancel in pairs. There remain the unpaired facets. If (j, i) is one of the $|m|$ unpaired facets carrying c , then F is constant with value x_0 on the image of that facet, so the deformation is stationary there, $G_j \circ \delta^i = c$, and $\beta_{j,i} = 0$.

This accounts for all but one of the facets, the remaining one being (j_0, i_0) : the terms indexed by the paired facets cancel against one another and those indexed by the unpaired facets carrying c vanish, so the sum over all facets other than (j_0, i_0) is zero, and it follows from (1.1) that $\eta_{j_0, i_0} \beta_{j_0, i_0} = 0$. As $\eta_{j_0, i_0} = +1$, this says $\beta_{j_0, i_0} = 0$.

On the other hand, F is constant with value x_0 on the image of the boundary of that facet, that boundary being carried to x_0 by τ ; the deformation is stationary there, so $G_{j_0} \circ \delta^{i_0}$ differs from τ only by a homotopy rel $\partial\Delta^n$, and $\beta_{j_0, i_0} = \kappa(\tau) = \xi$. Hence $\xi = 0$. $\square$

Corollary 1.8. *Let $m \geq 2$ and let Y be a path-connected, simply connected space with $\tilde{H}_i(Y) = 0$ for $i < m$. Then $\pi_i(Y, y_0) = 0$ for $i < m$.*

Proof. Induction on i . For $i = 1$ this is the hypothesis. Let $2 \leq i \leq m - 1$ and suppose $\pi_r(Y, y_0) = 0$ for $1 \leq r \leq i - 1$. Theorem 1.2, applied with $n = i$, makes $h: \pi_i(Y, y_0) \rightarrow H_i(Y)$ an isomorphism, and $H_i(Y) = \tilde{H}_i(Y) = 0$. $\square$

2. THE RELATIVE HUREWICZ THEOREM

Pseudomanifolds with boundary were introduced in *Pseudomanifolds* for this section. Throughout it (X, A) is a pair of spaces with a base point $x_0 \in A$, and $\pi_n(X, A, x_0)$ is the relative homotopy group of Definition 4.9 of *The homotopy category, and the homotopy groups*: the set of homotopy classes of maps of triples $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$, a group for $n \geq 2$ by Theorem 4.12 there and abelian for $n \geq 3$.

Definition 2.1. Let $n \geq 1$. A map of triples φ is in particular a map of pairs $(I^n, \partial I^n) \rightarrow (X, A)$, so it carries the fundamental class μ_n of the cube into $H_n(X, A)$. The *relative Hurewicz map* is

$$h: \pi_n(X, A, x_0) \longrightarrow H_n(X, A), \quad h[\varphi] = \varphi_* \mu_n.$$

It is well defined, homotopic maps of triples being in particular homotopic maps of pairs, which induce the same map on relative homology by Theorem 6.6 of *Singular Homology*;

and it is natural for maps of triples $(X, A, x_0) \rightarrow (Y, B, y_0)$. For $A = \{x_0\}$ it is the composite j_*h of Definition 1.1.

For $n \geq 2$ it is a homomorphism, by the argument given after Definition 1.1 with (X, A) in place of $(X, \{x_0\})$. The one point to check is that $\varphi \cdot \psi$ still carries $I^n \setminus (\text{int } I_-^n \cup \text{int } I_+^n) = \partial I^n \cup M$ into A , where $M = \{t_1 = \frac{1}{2}\}$: the faces of I^n go into A because φ and ψ are maps of pairs, and M goes to x_0 because on it the left half is given by φ on the face $t_1 = 1$ and the right half by ψ on the face $t_1 = 0$, both of which lie in J^{n-1} when $n \geq 2$. The rest of the argument is unchanged, and the two components of μ_n contribute $\varphi_*\mu_n$ and $\psi_*\mu_n$ in $H_n(X, A)$.

Theorem 2.2 (relative Hurewicz). *Let $n \geq 2$, let X be path connected, and let $A \subseteq X$ be non-empty, path connected and simply connected, with $x_0 \in A$. Suppose $\pi_j(X, A, x_0) = 0$ for $1 \leq j \leq n-1$. Then*

- (i) $H_i(X, A) = 0$ for $i < n$; and
- (ii) $h: \pi_n(X, A, x_0) \rightarrow H_n(X, A)$ is an isomorphism.

Under these hypotheses the group $\pi_n(X, A, x_0)$ is abelian, so that the sums formed in it below are meaningful. For $n \geq 3$ this is Theorem 4.12 of *The homotopy category, and the homotopy groups*. For $n = 2$ it is not automatic, and here is where simple connectivity of A first enters: $\pi_1(A, x_0) = 1$, so by the exactness of

$$\pi_2(X, x_0) \longrightarrow \pi_2(X, A, x_0) \xrightarrow{\partial} \pi_1(A, x_0)$$

in Definition 4.14 of those notes the group $\pi_2(X, A, x_0)$ is a quotient of the abelian group $\pi_2(X, x_0)$. Beyond that, part (i) needs no more than path connectedness of A , while the rest of the proof uses simple connectivity through the read-off κ_A . See Remark 2.16.

Lemma 2.3 (compression). *Let $m \geq 1$, let X be path connected, let A be path connected, and suppose $\pi_j(X, A, x_0) = 0$ for $1 \leq j \leq m$. Let K be a finite simplicial complex, let L be a subcomplex, and let $f: (|K|, |L|) \rightarrow (X, A)$ be a map of pairs such that every simplex of K not carried by f into A has dimension at most m . Then f is homotopic, through maps of pairs $(|K|, |L|) \rightarrow (X, A)$, to a map with image in A . The homotopy may be taken stationary on every simplex of K on which f is already constant with value x_0 , and it carries into A , at every time, every simplex that f carries into A .*

Proof. Let K_A be the subcomplex of simplices s with $f(|s|) \subseteq A$; it contains L , and a face of such a simplex is again one.

First move every vertex to x_0 . For a vertex v with $f(v) \in A$ choose a path from $f(v)$ to x_0 inside A , which is path connected, and take it constant if $f(v) = x_0$; for a vertex with $f(v) \notin A$ choose a path in X . Extend these paths over $|K_A|$ inside A by Lemma 6.4, and then over $|K|$ inside X by Lemma 6.4 again. The result is a homotopy of f through maps of pairs, stationary on every simplex on which f was already constant with value x_0 and keeping $|K_A|$ inside A ; after it, $f(v) = x_0$ for every vertex v of K .

Now deform over the skeleta. Suppose $1 \leq j \leq m$ and that f carries $|K^{j-1}|$ into A ; the case $j = 1$ holds because the vertices are at x_0 . Let s be a j -simplex of K not in K_A . Then $f(|\partial s|) \subseteq A$, and the characteristic map of s identifies $f|_s$ with a map of pairs $(\Delta^j, \partial\Delta^j) \rightarrow (X, A)$. Since A is path connected, Proposition 7.3 of *The homotopy category, and the homotopy groups* makes the forgetful map $\pi_j(X, A, x_0) \rightarrow [(I^j, \partial I^j), (X, A)]$ onto, and

the source vanishes; so every map of pairs $(I^j, \partial I^j) \rightarrow (X, A)$ — and hence, the two pairs being homeomorphic, every map of pairs $(\Delta^j, \partial \Delta^j) \rightarrow (X, A)$ — is homotopic through maps of pairs to the constant map at x_0 . By Lemma 7.4 of those notes, $f|_s$ is therefore homotopic rel $|\partial s|$ to a map with image in A . These homotopies are rel the boundaries, so those belonging to different j -simplices agree on shared faces, and taken together with the stationary homotopy on the j -simplices of K_A they give a homotopy of f on $|K^j \cup K_A|$, stationary on $|K_A|$. Extending it over $|K|$ by Lemma 6.4 leaves $|K_A|$, and in particular $|L|$, where it was, so the homotopy is one of maps of pairs; and it is stationary on $|K^{j-1}|$. Now f carries $|K^j|$ into A .

After the step $j = m$ every simplex of dimension at most m is carried into A , and every simplex of larger dimension lies in K_A and has been carried into A throughout. So $f(|K|) \subseteq A$. $\square$

Proof of Theorem 2.2(i). Let $i < n$ and let $\alpha \in H_i(X, A)$. For $i = 0$ the group $H_0(X, A)$ vanishes because X is path connected and A is non-empty. So let $1 \leq i < n$. By Theorem 5.1 of *Pseudomanifolds* there are an oriented i -dimensional pseudomanifold with boundary K and a map of pairs $f: (|K|, |\partial K|) \rightarrow (X, A)$ with $f_*[K, \partial K] = \alpha$. Since $\dim K = i \leq n - 1$, Lemma 2.3 applied with $m = n - 1$ and $L = \partial K$ deforms f , through maps of pairs, to a map with image in A ; this does not change $f_*[K, \partial K]$. But then f factors as a map of pairs $(|K|, |\partial K|) \rightarrow (A, A) \rightarrow (X, A)$, and $H_i(A, A) = 0$, so $\alpha = 0$. $\square$

It remains to prove that h is an isomorphism. As in §1 we need a way of attaching an element of the homotopy group to a map on a simplex, and a chain-level description of h .

First the notation. A map of triples on the cube must send the whole of J^{n-1} , a set of dimension $n-1$, to x_0 , whereas a map on a simplex will send its boundary only into A . The base point is in fact immaterial here, by Proposition 7.3 of *The homotopy category, and the homotopy groups*: for A path connected and simply connected, forgetting it identifies $\pi_n(X, A, x_0)$ with the set $[(I^n, \partial I^n), (X, A)]$ of homotopy classes of maps of pairs.

The read-off is defined so as to run parallel to the absolute one of §1: carry Δ^n into the cube by an affine linear map of positive determinant, and fill in the rest of the cube. There the filling was by the base point, which a map of pairs into (X, A) does not supply; here it is done radially.

Let b be the barycenter of Δ^n and let

$$R: \mathbb{R}^n \longrightarrow \Delta^n$$

be the radial retraction from b : it is the identity on Δ^n , and it carries a point y outside Δ^n to the point where the segment from b to y meets $\partial \Delta^n$. Writing ν for the Minkowski gauge of Δ^n about b , so that $\Delta^n = \{\nu \leq 1\}$ and ν is continuous, one has $R(y) = b + (y - b) / \max(1, \nu(y))$; so R is continuous, and it carries $\mathbb{R}^n \setminus \text{int } \Delta^n$ onto $\partial \Delta^n$.

Call an affine linear map T of $\mathbb{R}^n$ *admissible* if $\det T > 0$ and $T(\Delta^n) \subseteq I^n$, as in Definition 9.4 of *The homotopy category, and the homotopy groups*; such maps exist. For a map of pairs $u: (\Delta^n, \partial \Delta^n) \rightarrow (X, A)$ and an admissible T set

$$\alpha_A(u, T) = u \circ R \circ T^{-1}: I^n \longrightarrow X.$$

On $T(\Delta^n)$ this is $u \circ T^{-1}$, and at every other point of I^n it is u of a point of $\partial \Delta^n$, hence a point of A . The set $T(\text{int } \Delta^n)$ is open and contained in I^n , hence contained in $\text{int } I^n$, so no point of ∂I^n lies in it; therefore $\alpha_A(u, T)$ carries ∂I^n into A and is a map of pairs

$(I^n, \partial I^n) \rightarrow (X, A)$. When $u(\partial \Delta^n) = \{x_0\}$ it is exactly the map $\alpha(u, T)$ of Definition 9.5 of those notes, namely $u \circ T^{-1}$ on $T(\Delta^n)$ and x_0 off it.

Write $[\iota_n]$ for the class of the identity singular simplex in $H_n(\Delta^n, \partial \Delta^n)$; by Example 2.8 of *Pseudomanifolds* it is the fundamental class $[\Delta^n, \partial \Delta^n]$ of the simplex with the orientation $[0, 1, \dots, n]$.

Definition 2.4. Let $n \geq 1$, let A be path connected and simply connected, and let $u: (\Delta^n, \partial \Delta^n) \rightarrow (X, A)$ be a map of pairs. Its *read-off class* $\kappa_A(u) \in \pi_n(X, A, x_0)$ is the class corresponding to that of $\alpha_A(u, T)$ in $[(I^n, \partial I^n), (X, A)]$ under the bijection of Proposition 7.3, for an admissible T ; by Lemma 2.5 below it does not depend on which one. No condition is imposed at any vertex.

Lemma 2.5. *The class $\kappa_A(u)$ is independent of the admissible map T and depends only on the homotopy class of u through maps of pairs; it vanishes if $u(\Delta^n) \subseteq A$; and every element of $\pi_n(X, A, x_0)$ is $\kappa_A(u)$ for some u .*

Proof. A homotopy u_s through maps of pairs gives the homotopy $\alpha_A(u_s, T) = u_s \circ R \circ T^{-1}$, through maps of pairs by the paragraph above.

Let T_0 and T_1 be admissible. For $n = 1$ the admissible maps form a convex set and $T_t = (1-t)T_0 + tT_1$ is a path of them. For $n \geq 2$ apply Proposition 9.2 of *The homotopy category, and the homotopy groups* to the single body $T_0(\Delta^n)$ and to the map $T_1 T_0^{-1}$, whose determinant is positive: it gives a path Θ_t of affine linear maps of positive determinant, from the identity to $T_1 T_0^{-1}$, with $\Theta_t(T_0(\Delta^n)) \subseteq I^n$ for every t , and then $T_t = \Theta_t \circ T_0$ is a path of admissible maps. In either case $(x, t) \mapsto u(R(T_t^{-1}x))$ is continuous, inversion being continuous on invertible affine linear maps, and at each time it is $\alpha_A(u, T_t)$, a map of pairs. So the class of $\alpha_A(u, T)$ does not depend on T .

If $u(\Delta^n) \subseteq A$ then $\alpha_A(u, T)$ has image in A as well. Being defined on the contractible space I^n , it is homotopic to a constant map through maps $I^n \rightarrow A$, and these are maps of pairs $(I^n, \partial I^n) \rightarrow (X, A)$; following that by a path in A from the constant value to x_0 , again through maps of pairs, gives a homotopy of maps of pairs from $\alpha_A(u, T)$ to the constant map at x_0 . So $\kappa_A(u) = 0$.

For the last statement fix an admissible T , put $D = T(\Delta^n)$ and $c = T(b)$, and let $f: (I^n, \partial I^n) \rightarrow (X, A)$ be a map of pairs whose class corresponds to the given element of $\pi_n(X, A, x_0)$ under Proposition 7.3. Each ray from c meets ∂I^n in one point and ∂D in one point; let $\theta: I^n \rightarrow D$ be the homeomorphism matching those two and scaling linearly along the ray, and put $u = f \circ \theta^{-1} \circ T$, a map of pairs $(\Delta^n, \partial \Delta^n) \rightarrow (X, A)$. Then $\alpha_A(u, T) = f \circ \theta^{-1} \circ \rho$, where $\rho = T \circ R \circ T^{-1}$ is the radial retraction of I^n onto D from c . On ∂I^n the maps ρ and θ agree, each carrying a point to where its ray from c meets ∂D , so $\theta^{-1} \circ \rho$ is the identity there; and I^n is convex, so the straight-line homotopy joins $\theta^{-1} \circ \rho$ to the identity of I^n without moving ∂I^n . Composing with f gives a homotopy of maps of pairs from $\alpha_A(u, T)$ to f , so $\kappa_A(u)$ is the given element. $\square$

Remark 2.6. The two read-offs are one construction: carry the body into the cube by an affine linear map of positive determinant, and fill in the rest of the cube radially. The absolute one asks that the whole of ∂U go to x_0 , the relative one only that $\partial \Delta^n$ go into A ; and if $\partial \Delta^n$ does go to the base point then the radial filling is constant, so that the construction is the α of §1. That is why Lemma 2.13 below is an identity of maps rather than a comparison.

Lemma 2.7. *Let $u: (\Delta^n, \partial\Delta^n) \rightarrow (X, A)$ be a map of pairs. Then*

$$h(\kappa_A(u)) = [u] \in H_n(X, A),$$

where $[u] = u_*[\iota_n]$ is the class of the relative cycle u .

Proof. Fix an admissible T , put $D = T(\Delta^n)$ and $v = \alpha_A(u, T)$. A map of triples φ representing $\kappa_A(u)$ is homotopic to v through maps of pairs, so the two induce the same map on H_n of the pairs and $h(\kappa_A(u)) = \varphi_*\mu_n = v_*\mu_n$. Now v carries $I^n \setminus \text{int } D$ into A , so it is also a map of pairs $(I^n, I^n \setminus \text{int } D) \rightarrow (X, A)$ and $v_*\mu_n$ may be computed after passing to $H_n(I^n, I^n \setminus \text{int } D)$. There, by Lemma 3.3 of *Further Properties*, the image of μ_n is the image of $T_*[\iota_n]$ under the map of pairs $(D, \partial D) \rightarrow (I^n, I^n \setminus \text{int } D)$. Hence

$$h(\kappa_A(u)) = v_*\mu_n = (v|_D)_*T_*[\iota_n] = (u \circ T^{-1})_*T_*[\iota_n] = u_*[\iota_n] = [u]. \quad \square$$

Note the contrast with Lemma 1.5: there the target of h was $\tilde{H}_n(X)$ and the comparison j_* had to be inverted, whereas here h already lands in $H_n(X, A)$ and nothing has to be inverted. This is what makes the sum formula below cost nothing.

Lemma 2.8 (relative sum formula). *Let $n \geq 2$, let K be an oriented n -dimensional pseudomanifold with boundary, let λ be a total order of its vertices, and let $f: (|K|, |\partial K|) \rightarrow (X, A)$ be a map of pairs carrying $|K^{n-1}|$ into A . For an n -simplex s of K let σ_s and ε_s be as in Theorem 3.2 of Pseudomanifolds. Then each $f \circ \sigma_s$ is a map of pairs $(\Delta^n, \partial\Delta^n) \rightarrow (X, A)$ and*

$$f_*[K, \partial K] = h\left(\sum_s \varepsilon_s \kappa_A(f \circ \sigma_s)\right).$$

Proof. Each σ_s carries $\partial\Delta^n$ into $|K^{n-1}|$, so $f \circ \sigma_s$ is a map of pairs. By Theorem 3.2 of those notes the class $[K, \partial K]$ is represented by the relative cycle $c_\lambda = \sum_s \varepsilon_s \sigma_s$, so $f_*[K, \partial K]$ is represented by $\sum_s \varepsilon_s (f \circ \sigma_s)$. Each summand is itself a relative cycle, its boundary lying in A , so by Lemma 2.7

$$f_*[K, \partial K] = \sum_s \varepsilon_s [f \circ \sigma_s] = \sum_s \varepsilon_s h(\kappa_A(f \circ \sigma_s)) = h\left(\sum_s \varepsilon_s \kappa_A(f \circ \sigma_s)\right). \quad \square$$

Lemma 2.9. *Let $n \geq 2$, let X and A be path connected, and suppose $\pi_j(X, A, x_0) = 0$ for $1 \leq j \leq n-1$. Let K be a finite simplicial complex of dimension at most n , let L be a subcomplex, and let $f: (|K|, |L|) \rightarrow (X, A)$ be a map of pairs. Then f is homotopic, through maps of pairs, to a map carrying $|K^{n-1}|$ into A .*

Proof. Apply Lemma 2.3 with $m = n-1$ to the complex K^{n-1} , its subcomplex $L \cap K^{n-1}$ and the restriction of f ; the simplices of L of dimension at most $n-1$ are carried into A , so the homotopy keeps them there. Extending it over $|K|$ by Lemma 6.4 gives a homotopy of f through maps of pairs, after which f carries $|K^{n-1}|$ into A . $\square$

Proof of surjectivity in Theorem 2.2(ii). Let $\alpha \in H_n(X, A)$. By Theorem 5.1 of *Pseudomanifolds* there are an oriented n -dimensional pseudomanifold with boundary K and a map of pairs $f: (|K|, |\partial K|) \rightarrow (X, A)$ with $f_*[K, \partial K] = \alpha$. By Lemma 2.9, applied with $L = \partial K$, we may deform f through maps of pairs so that it carries $|K^{n-1}|$ into A ; this does not change $f_*[K, \partial K]$. Lemma 2.8 now gives $\alpha = h(\sum_s \varepsilon_s \kappa_A(f \circ \sigma_s))$, so α lies in the image of h . $\square$

2.1. Preliminaries for injectivity. Injectivity rests on one identity, Lemma 2.15 below, the relative analogue of Lemma 1.7. Like that lemma it assumes nothing of the kind hypothesized in Theorem 2.2: no relative homotopy group is required to vanish, and all that is asked of A is the connectivity under which Definition 2.4 defines κ_A . Its proof is not the analogue of the Schlegel argument of §1. Instead we prove the identity in a universal example, where the absolute Hurewicz theorem is available for the two auxiliary spaces that arise. This subsection assembles what is needed.

Lemma 2.10. *Let K be a finite simplicial complex and $k \geq 0$. The inclusion induces an isomorphism $H_j(|K^k|) \rightarrow H_j(|K|)$ for $j < k$, and $H_j(|K^k|) = 0$ for $j > k$.*

Proof. In oriented simplicial chains, $C_j^{\text{or}}(K^k) = C_j^{\text{or}}(K)$ for $j \leq k$, both being free on the simplices of dimension j , and $C_j^{\text{or}}(K^k) = 0$ for $j > k$. Hence $H_j^{\text{or}}(K^k) = 0$ for $j > k$; and for $j < k$ the cycles in degree j agree, since the chain groups in degrees j and $j-1$ do, and so do the boundaries, since the chain groups in degree $j+1 \leq k$ agree. So the inclusion induces an isomorphism $H_j^{\text{or}}(K^k) \rightarrow H_j^{\text{or}}(K)$ for $j < k$. Now Θ_λ commutes with the inclusion of a subcomplex, as noted in §3 of *Pseudomanifolds*, and induces isomorphisms to singular homology by Theorem 5.5 of *Further Properties*. $\square$

Lemma 2.11. *Let $k \geq 2$ and $m \geq 0$. Then $|(\Delta^m)^k|$ is path connected and simply connected.*

Proof. It is path connected because it contains every vertex and every edge of Δ^m . We prove simple connectivity by induction on m . For $m \leq k$ we have $(\Delta^m)^k = \Delta^m$, whose realization is convex and hence simply connected.

Let $m > k$ and, for a point y of Δ^m , write y_m for its last barycentric coordinate. Put

$$U = \{y \in |(\Delta^m)^k| : y_m < \tfrac{1}{2}\}, \quad V = \{y \in |(\Delta^m)^k| : y_m > 0\},$$

both open, with $U \cup V = |(\Delta^m)^k|$. A point with $y_m > 0$ lies in a simplex s of $(\Delta^m)^k$ containing e_m , and then $s = e_m * s'$ with s' a simplex of $(\Delta^{m-1})^{k-1}$, the complex on the vertices $e_0, \dots, e_{m-1}$.

For $0 \leq c < 1$ and $y_m < 1$ set

$$\theta_c(y) = c e_m + \frac{1-c}{1-y_m} (y - y_m e_m),$$

the point whose last barycentric coordinate is c and whose remaining ones are those of y rescaled to sum to $1-c$. It depends continuously on (c, y) , lies in the same simplex of $(\Delta^m)^k$ as y , and $\theta_{y_m}(y) = y$. Then

$$(y, t) \longmapsto \theta_{(1-t)y_m}(y)$$

is a deformation retraction of U onto $|(\Delta^{m-1})^k|$, fixing that subspace pointwise and keeping the last coordinate at most $y_m < \frac{1}{2}$; and

$$(y, t) \longmapsto \theta_{(1-t)y_m+t/4}(y)$$

is a deformation retraction of $U \cap V = \{0 < y_m < \frac{1}{2}\}$ onto the slice $\{y_m = \frac{1}{4}\}$, the last coordinate lying between y_m and $\frac{1}{4}$ throughout and so staying in $(0, \frac{1}{2})$. That slice is carried homeomorphically onto $|(\Delta^{m-1})^{k-1}|$ by $y \mapsto (y - y_m e_m)/(1 - y_m)$, and that space is path connected because $k-1 \geq 1$. Finally V is contractible: the straight line

$(y, t) \mapsto (1 - t)y + te_m$ stays in the simplex s , which contains e_m , and its last coordinate $(1 - t)y_m + t$ is positive.

Van Kampen's theorem, Theorem 6.12 of *The homotopy category, and the homotopy groups*, now presents $\pi_1(|(\Delta^m)^k|)$ as the pushout of $\pi_1(U) \leftarrow \pi_1(U \cap V) \rightarrow \pi_1(V)$. Here $\pi_1(V) = 1$ and $\pi_1(U) \cong \pi_1(|(\Delta^{m-1})^k|) = 1$ by the inductive hypothesis, so the pushout is trivial. $\square$

Corollary 2.12. *Let $n \geq 3$. Then*

- (a) $|(\Delta^{n+1})^{n-1}|$ is $(n - 2)$ -connected and $H_n(|(\Delta^{n+1})^{n-1}|) = 0$;
- (b) $|(\Delta^{n+1})^n| = \partial\Delta^{n+1}$ is $(n - 1)$ -connected, and $H_{n-1}(\partial\Delta^{n+1}) = 0$.

Proof. Both complexes are simply connected by Lemma 2.11, the skeleta being taken in dimensions $n - 1 \geq 2$ and $n \geq 2$. By Lemma 2.10 the homology of the k -skeleton agrees with that of Δ^{n+1} , which is convex, in degrees below k , and vanishes above k . So the reduced homology of $|(\Delta^{n+1})^{n-1}|$ vanishes in degrees below $n - 1$ and in degree n , and that of $\partial\Delta^{n+1}$ vanishes in degrees below n , in particular in degree $n - 1$. Corollary 1.8 now gives (a) with $m = n - 1$ and (b) with $m = n$. $\square$

Lemma 2.13. *Let $n \geq 2$, let A be path connected and simply connected, and let $u: \Delta^n \rightarrow X$ be a map with $u(\partial\Delta^n) = \{x_0\}$. Then $\kappa_A(u)$ is the image of $\kappa(u) \in \pi_n(X, x_0)$ under the map $\pi_n(X, x_0) \rightarrow \pi_n(X, A, x_0)$.*

Proof. It suffices to treat $A = \{x_0\}$; the general case then follows by applying the map of triples $(X, \{x_0\}, x_0) \rightarrow (X, A, x_0)$, which carries $\kappa_{\{x_0\}}(u)$ to $\kappa_A(u)$, the bijection of Proposition 7.3 being natural for such maps.

For $A = \{x_0\}$ a map of pairs $(I^n, \partial I^n) \rightarrow (X, \{x_0\})$ is already a map of triples, and homotopies of pairs are homotopies of triples, so Proposition 7.3 is there the identity and $\kappa_{\{x_0\}}(u)$ is the class of $\alpha_A(u, T)$ in $\pi_n(X, x_0)$, for an admissible T . But u carries $\partial\Delta^n$ to x_0 , so $\alpha_A(u, T)$ is the map $\alpha(u, T)$, whose class is $\kappa(u)$ by Definition 9.8 of *The homotopy category, and the homotopy groups*. Hence $\kappa_{\{x_0\}}(u) = \kappa(u)$. $\square$

Lemma 2.14 (the Hurewicz ladder). *Let $n \geq 2$, let $B \subseteq Z$ with $z_0 \in B$, and consider*

$$\begin{array}{ccccccccc} \pi_n(B) & \longrightarrow & \pi_n(Z) & \longrightarrow & \pi_n(Z, B) & \xrightarrow{\partial} & \pi_{n-1}(B) & \longrightarrow & \pi_{n-1}(Z) \\ h \downarrow & & h \downarrow & & h \downarrow & & h \downarrow & & h \downarrow \\ H_n(B) & \longrightarrow & H_n(Z) & \longrightarrow & H_n(Z, B) & \xrightarrow{\partial} & H_{n-1}(B) & \longrightarrow & H_{n-1}(Z) \end{array}$$

with base points suppressed, the outer vertical maps the absolute Hurewicz maps of Definition 1.1 and the middle one that of Definition 2.1. The first, second and fourth squares commute, and the third commutes up to a sign $\varepsilon_n = \pm 1$ depending only on n .

Proof. The first and fourth squares are the naturality of the absolute Hurewicz map. For the second, $\pi_n(Z, z_0) = \pi_n(Z, \{z_0\}, z_0)$ by Remark 4.10 of *The homotopy category, and the homotopy groups*, the map to $\pi_n(Z, B, z_0)$ is induced by the map of triples $(Z, \{z_0\}, z_0) \rightarrow (Z, B, z_0)$, and h of Definition 2.1 is natural for maps of triples and reduces on $\pi_n(Z, \{z_0\}, z_0)$ to j_*h .

For the third, let φ be a map of triples $(I^n, \partial I^n, J^{n-1}) \rightarrow (Z, B, z_0)$ and $\beta: I^{n-1} \rightarrow \partial I^n$ the bottom face, so that $\partial[\varphi] = [\varphi \circ \beta]$. By naturality of the connecting homomorphism

for the map of pairs $\varphi: (I^n, \partial I^n) \rightarrow (Z, B)$,

$$\partial(h[\varphi]) = \partial(\varphi_*\mu_n) = (\varphi|_{\partial I^n})_*(\partial\mu_n).$$

Since $\varphi(J^{n-1}) = \{z_0\}$, the map $\varphi|_{\partial I^n}$ is also a map of pairs $(\partial I^n, J^{n-1}) \rightarrow (B, \{z_0\})$, so the right-hand side, viewed in $H_{n-1}(B, \{z_0\})$, is the image of the class of $\partial\mu_n$ in $H_{n-1}(\partial I^n, J^{n-1})$. Now β is a map of pairs $(I^{n-1}, \partial I^{n-1}) \rightarrow (\partial I^n, J^{n-1})$ carrying the quotient $I^{n-1}/\partial I^{n-1}$ homeomorphically onto $\partial I^n/J^{n-1}$, so by Proposition 1.2 of *Further Properties* it induces an isomorphism on H_{n-1} ; and both groups are infinite cyclic, generated by μ_{n-1} and by the class of $\partial\mu_n$, the latter because ∂ is an isomorphism $H_n(I^n, \partial I^n) \rightarrow \tilde{H}_{n-1}(\partial I^n)$, the cube being contractible. Hence that class is $\varepsilon_n \beta_* \mu_{n-1}$ for a sign ε_n depending only on n , and applying φ_* gives $\partial h[\varphi] = \varepsilon_n j_* h([\varphi \circ \beta]) = \varepsilon_n j_* h(\partial[\varphi])$, with j_* the isomorphism $\tilde{H}_{n-1}(B) \rightarrow H_{n-1}(B, \{z_0\})$. $\square$

The sign is harmless: replacing the two right-hand vertical maps by $\varepsilon_n h$ makes every square commute without changing which of them are isomorphisms, so the five lemma, Exercise 4.6 of *Homology*, may be applied to the ladder as it stands. For $n \geq 3$ all ten groups are abelian and all the maps are homomorphisms, so the exercise applies.

Lemma 2.15. *Let $n \geq 2$, let (X, A) be a pair of spaces with $x_0 \in A$ and with A path connected and simply connected, and let $G: \Delta^{n+1} \rightarrow X$ be a map carrying $|(\Delta^{n+1})^{n-1}|$ into A . Then each $G \circ \delta^i$ is a map of pairs $(\Delta^n, \partial \Delta^n) \rightarrow (X, A)$ and*

$$\sum_{i=0}^{n+1} (-1)^i \kappa_A(G \circ \delta^i) = 0 \quad \text{in } \pi_n(X, A, x_0).$$

Proof. Each δ^i carries $\partial \Delta^n$ into the $(n-1)$ -skeleton, so $G \circ \delta^i$ is a map of pairs.

We may assume that G carries $|(\Delta^{n+1})^1|$ to x_0 . That set is a graph, and G carries it into A , the 1-skeleton lying in the $(n-1)$ -skeleton because $n \geq 2$. Move its vertices to x_0 along paths in A and then contract its edges, which have become loops at x_0 and are null-homotopic since A is simply connected: this is a homotopy of G restricted to the 1-skeleton, inside A , ending at the constant map at x_0 . Extend it over the $(n-1)$ -skeleton inside A by Lemma 6.4, and then over Δ^{n+1} inside X by Lemma 6.4 again. Throughout, the $(n-1)$ -skeleton stays in A , so at every time each $G_t \circ \delta^i$ is a map of pairs, and $\kappa_A(G \circ \delta^i)$ is therefore unchanged, by Lemma 2.5.

Suppose first $n = 2$. Then the $(n-1)$ -skeleton is the 1-skeleton, so G is constant with value x_0 on it, and Lemma 1.7 gives $\sum_i (-1)^i \kappa(G \circ \delta^i) = 0$ in $\pi_2(X, x_0)$. Each $G \circ \delta^i$ carries $\partial \Delta^2$ to x_0 , so Lemma 2.13 applies to each term, and the image of the relation in $\pi_2(X, A, x_0)$ is the one asserted.

Let $n \geq 3$. Write $Z^k = |(\Delta^{n+1})^k|$ for the k -skeleton, so that $Z^n = \partial \Delta^{n+1}$, and take e_0 as base point; being a vertex it lies in every skeleton. By Corollary 2.12 the space Z^{n-1} is $(n-2)$ -connected with $H_n(Z^{n-1}) = 0$, and Z^n is $(n-1)$ -connected with $H_{n-1}(Z^n) = 0$. Since $n-1 \geq 2$, the space Z^{n-1} is in particular path connected and simply connected, so the read-off $\kappa_{Z^{n-1}}$ of Definition 2.4 is defined.

The comparison. Apply Lemma 2.14 to the pair (Z^n, Z^{n-1}) . The first vertical map lands in $H_n(Z^{n-1}) = 0$ and so is surjective; the second is an isomorphism by Theorem 1.2 applied to the $(n-1)$ -connected space Z^n ; the fourth is an isomorphism by Theorem 1.2

applied to the $(n-2)$ -connected space Z^{n-1} , which is legitimate because $n-1 \geq 2$; and the fifth is injective, its source $\pi_{n-1}(Z^n)$ being 0. By the five lemma,

$$h: \pi_n(Z^n, Z^{n-1}, e_0) \longrightarrow H_n(Z^n, Z^{n-1})$$

is an isomorphism. Observe that $\pi_n(Z^{n-1})$, the one group here that is not known, occurs only at the surjective end, where the target vanishes.

The universal relation. Each $\delta^i: \Delta^n \rightarrow Z^n$ carries $\partial\Delta^n$ into Z^{n-1} , so it is a map of pairs $(\Delta^n, \partial\Delta^n) \rightarrow (Z^n, Z^{n-1})$; write $[t_i] = \kappa_{Z^{n-1}}(\delta^i) \in \pi_n(Z^n, Z^{n-1}, e_0)$. What will be shown is that the alternating sum of the $[t_i]$ comes from $\pi_n(Z^n, e_0)$.

Write p_* for the maps induced by inclusion, $H_n(Z^n) \rightarrow H_n(Z^n, Z^{n-1})$ on homology and $\pi_n(Z^n, e_0) \rightarrow \pi_n(Z^n, Z^{n-1}, e_0)$ on homotopy. Since Z^n is $(n-1)$ -connected, Theorem 1.2 provides a $g \in \pi_n(Z^n, e_0)$ with $h(g) = [\partial\Delta^{n+1}]$, the fundamental class of Example 2.5 of *Pseudomanifolds*. That class is represented by the cycle $\sum_i (-1)^i \delta^i$, so

$$p_* h(g) = \sum_{i=0}^{n+1} (-1)^i [\delta^i] \quad \text{in } H_n(Z^n, Z^{n-1}).$$

Each $[\delta^i]$ is $h([t_i])$, by Lemma 2.7 applied to the pair (Z^n, Z^{n-1}) , and h is a homomorphism; so the right-hand side is $h(\sum_i (-1)^i [t_i])$. The left-hand side is $h(p_* g)$: the relative Hurewicz map is natural for maps of triples, as noted after Definition 2.1, and on $\pi_n(Z^n, e_0) = \pi_n(Z^n, \{e_0\}, e_0)$ it is $j_* h$, so applying it to the inclusion of triples $(Z^n, \{e_0\}, e_0) \rightarrow (Z^n, Z^{n-1}, e_0)$ gives $h(p_* g) = p_* h(g)$. Hence

$$h(p_* g) = h\left(\sum_{i=0}^{n+1} (-1)^i [t_i]\right),$$

and h is injective on $\pi_n(Z^n, Z^{n-1}, e_0)$ by the comparison just established, so

$$p_* g = \sum_{i=0}^{n+1} (-1)^i [t_i].$$

Let $\iota_*: \pi_n(Z^n, Z^{n-1}, e_0) \rightarrow \pi_n(\Delta^{n+1}, Z^{n-1}, e_0)$ be induced by the inclusion $Z^n \subseteq \Delta^{n+1}$. In the square

$$\begin{array}{ccc} \pi_n(Z^n, e_0) & \xrightarrow{p_*} & \pi_n(Z^n, Z^{n-1}, e_0) \\ \downarrow & & \downarrow \iota_* \\ \pi_n(\Delta^{n+1}, e_0) & \longrightarrow & \pi_n(\Delta^{n+1}, Z^{n-1}, e_0) \end{array}$$

every map is induced by an inclusion of triples, so the square commutes; and the lower left corner vanishes, Δ^{n+1} being convex. Hence

$$\iota_*\left(\sum_i (-1)^i [t_i]\right) = \iota_* p_* g = 0.$$

Descent to (X, A) . The map G carries Z^{n-1} into A and e_0 to x_0 , so it is a map of triples $(\Delta^{n+1}, Z^{n-1}, e_0) \rightarrow (X, A, x_0)$. The read-off is natural for such a map: $\alpha_A(G \circ u, T) =$

$G \circ \alpha_A(u, T)$ straight from its definition, and the bijection of Proposition 7.3 is natural as well. Applying G_* ,

$$\sum_{i=0}^{n+1} (-1)^i \kappa_A(G \circ \delta^i) = G_* \iota_* \left(\sum_i (-1)^i [t_i] \right) = 0.$$

Of the normalization the case $n \geq 3$ used only the single value $G(e_0) = x_0$. It is the case $n = 2$ that needs all of it: there the argument runs through Lemma 1.7, whose read-offs $\kappa(G \circ \delta^i)$ are the absolute ones and so require each $G \circ \delta^i$ to carry $\partial\Delta^2$ to x_0 . $\square$

2.2. Injectivity.

Proof of injectivity in Theorem 2.2(ii). Let $\xi \in \pi_n(X, A, x_0)$ with $h(\xi) = 0$.

A representative. By Lemma 2.5 we may write $\xi = \kappa_A(u)$ for a map of pairs u from $(\Delta^n, \partial\Delta^n)$ to (X, A) . Nothing need be arranged at the vertices: Lemma 2.15, where the argument is going, asks only that the $(n-1)$ -skeleton be carried into A .

If $u(\Delta^n) \subseteq A$ then $\xi = \kappa_A(u) = 0$ by Lemma 2.5. Assume from now on that it is not.

A filling. By Lemma 2.7, $0 = h(\xi) = [u]$ in $H_n(X, A)$, so $u = \partial d + e$ with $d \in C_{n+1}(X)$ and $e \in C_n(A)$. Every singular simplex occurring in e has image in A , so u does not occur in e ; hence in

$$\partial d = u - e$$

the coefficient of u is 1, and every other singular n -simplex occurring has image in A . In particular $d \neq 0$. Write $d = \sum_{j=1}^N \rho_j \sigma_j$ with $\rho_j = \pm 1$, call the pairs (j, i) with $0 \leq i \leq n+1$ the facets, and attach to (j, i) the singular simplex $\sigma_j \circ \delta^i$ and the sign $\eta_{j,i} = \rho_j (-1)^i$, as in Step 2 of the proof of Theorem 5.1 of *Pseudomanifolds*. Grouping the facets by the singular simplices they carry and pairing off opposite signs as far as the signs allow leaves, at each singular simplex, as many unpaired facets as the absolute value of its coefficient in ∂d , all of one sign. So there is exactly one unpaired facet (j_0, i_0) carrying u , its sign being $\eta_{j_0, i_0} = +1$, and every other unpaired facet carries a singular simplex with image in A .

By Lemma 4.6 of those notes, gluing N copies $P_1, \dots, P_N$ of $\text{sd}^2 \Delta^{n+1}$ along the paired facets produces a finite simplicial complex W with quotient map $q: \coprod_j \Delta_j^{n+1} \rightarrow |W|$; write q_j for its restriction to the j -th copy. The maps σ_j agree on identified points, so there is a unique $F: |W| \rightarrow X$ with $F \circ q_j = \sigma_j$.

Three subcomplexes. Let $Z \subseteq W$ be the union of the images under q of the $(n-1)$ -skeleta of the copies. By Corollary 4.2 there the image of a face of a copy is sd^2 of that face, so Z is a subcomplex of dimension $n-1$, and it contains the image of the boundary of every facet. Let $B \subseteq W$ be the union of the images of the unpaired facets other than (j_0, i_0) ; it is a subcomplex of dimension n and $F(|B|) \subseteq A$. Put $V = Z \cup B$. Every simplex of V not carried into A by F lies in Z , so has dimension at most $n-1$. Note also that F carries the image of $\partial\Delta^n$ under $q_{j_0} \circ \delta^{i_0}$ into A , that map being u on $\partial\Delta^n$.

Deforming F . Lemma 2.3, applied to V with $m = n-1$, deforms $F|_V$ through maps of pairs to a map with image in A , by a homotopy that keeps in A every simplex already carried there — in particular the simplices of B and those of the image of $\partial\Delta^n$ under $q_{j_0} \circ \delta^{i_0}$. Extending it over $|W|$ by Lemma 6.4 gives a homotopy of F to a map F_1 with $F_1(|V|) \subseteq A$.

Summing. Fix j and put $G_j = F_1 \circ q_j: \Delta^{n+1} \rightarrow X$. The image under q_j of the $(n-1)$ -skeleton lies in $|Z| \subseteq |V|$, where F_1 has image in A , so G_j satisfies the hypothesis of Lemma 2.15 and

$$\sum_{i=0}^{n+1} (-1)^i \kappa_A(G_j \circ \delta^i) = 0.$$

Multiplying by ρ_j and summing over j ,

$$(2.1) \quad \sum_{j,i} \eta_{j,i} \kappa_A(G_j \circ \delta^i) = 0.$$

If the facets (j, i) and (j', i') are paired then $q_j \circ \delta^i = q_{j'} \circ \delta^{i'}$, the gluing being by $\delta^{i'} \circ (\delta^i)^{-1}$, so $G_j \circ \delta^i = G_{j'} \circ \delta^{i'}$ while $\eta_{j,i} = -\eta_{j',i'}$, and those terms cancel in pairs. If (j, i) is an unpaired facet other than (j_0, i_0) , then its image lies in $B \subseteq V$, where F_1 has image in A ; so $G_j \circ \delta^i$ has image in A and $\kappa_A(G_j \circ \delta^i) = 0$ by Lemma 2.5.

So (2.1) reduces to $\eta_{j_0, i_0} \kappa_A(G_{j_0} \circ \delta^{i_0}) = 0$, and $\eta_{j_0, i_0} = +1$. On the other hand the deformation kept the image of $\partial\Delta^n$ under $q_{j_0} \circ \delta^{i_0}$ inside A ; so $G_{j_0} \circ \delta^{i_0}$ is obtained from u by a homotopy through maps of pairs and

$$\kappa_A(G_{j_0} \circ \delta^{i_0}) = \kappa_A(u) = \xi.$$

Hence $\xi = 0$, and h is injective. □

Remark 2.16. Beyond making $\pi_2(X, A, x_0)$ abelian, simple connectivity of A enters in two places. It is what makes the read-off κ_A of Definition 2.4 well defined, by Proposition 7.3, so it is present wherever that read-off is; and it is what lets the 1-skeleton be contracted inside A at the start of the proof of Lemma 2.15. Part (i) uses neither, needing only Lemma 2.3, which asks no more than that A be path connected.

The hypothesis cannot be dropped from the injectivity. Let A be path connected with $\pi_1(A, x_0)$ non-trivial and let $X = CA$ be the cone on A , so that X is contractible and $\pi_1(X, A, x_0)$ is trivial: the hypotheses hold with $n = 2$ but for the simple connectivity of A . The exact sequence of the pair gives $\partial: \pi_2(X, A, x_0) \cong \pi_1(A, x_0)$, while $H_2(X, A) \cong \tilde{H}_1(A)$ is its abelianization, and h is the abelianization map: surjective, and injective only when $\pi_1(A, x_0)$ is abelian. In general the correct statement is that h identifies $H_n(X, A)$ with the quotient of $\pi_n(X, A, x_0)$ by the action of $\pi_1(A, x_0)$, of which this is the case $n = 2$, the action there being by conjugation.

3. A HOMOLOGY CRITERION FOR MAPS

The relative theorem applies to a map rather than a pair once the map is replaced by an inclusion. The tool is the mapping cylinder, Definition 8.1 of *The homotopy category, and the homotopy groups*: for a map $f: X \rightarrow Y$ it is the space

$$M_f = (X \times [0, 1] \sqcup Y) / ((x, 1) \sim f(x)),$$

with $\iota: X \rightarrow M_f$, $\iota(x) = [(x, 0)]$, and $r: M_f \rightarrow Y$ given by $r[(x, t)] = f(x)$ and $r[y] = y$. By Lemma 8.2 of those notes ι is a closed embedding, Y is a strong deformation retract of M_f with retraction r , and $r \circ \iota = f$; so f is the inclusion of X in M_f followed by a homotopy equivalence.

Theorem 3.1. *Let $n \geq 2$, let X and Y be non-empty, path-connected, simply connected spaces, let $f: X \rightarrow Y$ be a map, and let $x_0 \in X$. Then*

$f_: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for $i < n$ and surjective for $i = n$*

if and only if

$f_: \pi_i(X, x_0) \rightarrow \pi_i(Y, f(x_0))$ is an isomorphism for $i < n$ and surjective for $i = n$.*

Proof. Identify X with its image in $M = M_f$ under ι and regard x_0 as a point of $X \subseteq M$. By Lemma 8.2 of those notes the retraction r is a homotopy equivalence carrying M to Y , so it induces isomorphisms on all homotopy and homology groups, and under them the maps induced by the inclusion $X \subseteq M$ become those induced by f . Also M is path connected and simply connected, being homotopy equivalent to Y .

Two translations. Consider the long exact sequence of the pair (M, X) in homotopy,

$$\pi_{i+1}(M, X, x_0) \rightarrow \pi_i(X, x_0) \rightarrow \pi_i(M, x_0) \rightarrow \pi_i(M, X, x_0) \xrightarrow{\partial} \pi_{i-1}(X, x_0) \rightarrow \pi_{i-1}(M, x_0),$$

of Definition 4.14 of those notes, in which $\pi_i(X, x_0) \rightarrow \pi_i(M, x_0)$ is f_* up to the isomorphism r_* . If $\pi_i(M, X, x_0) = 0$ for $1 \leq i \leq n$, then for $i < n$ the terms on either side of that map vanish and it is an isomorphism, while for $i = n$ the term on its right vanishes and it is onto. Conversely, suppose it is an isomorphism for $i < n$ and onto for $i = n$, and let $1 \leq i \leq n$. Being onto, its image is all of $\pi_i(M, x_0)$, so $\pi_i(M, x_0) \rightarrow \pi_i(M, X, x_0)$ is trivial and ∂ is injective; and $\pi_{i-1}(X, x_0) \rightarrow \pi_{i-1}(M, x_0)$ is injective, so the image of ∂ is trivial. Hence $\pi_i(M, X, x_0) = 0$. The second condition of the theorem is therefore equivalent to

$$\pi_i(M, X, x_0) = 0 \quad \text{for } 1 \leq i \leq n.$$

The same argument in the homology sequence of the pair, where every term is a group, shows that the first condition is equivalent to

$$H_i(M, X) = 0 \quad \text{for } 1 \leq i \leq n;$$

and $H_0(M, X) = 0$ in any case, X being non-empty and M path connected.

The two vanishings agree. In degree 1 both hold: $\pi_1(M, X, x_0)$ is trivial because $\pi_1(M, x_0) = 0$ and $\pi_0(X, x_0)$ is a point, and $H_1(M, X) = 0$ because $H_1(X)$ and $H_1(M)$ vanish, X and M being simply connected, while $H_0(X) \rightarrow H_0(M)$ is an isomorphism. Let $2 \leq m \leq n$ and suppose $\pi_j(M, X, x_0) = 0$ for $1 \leq j \leq m-1$. The pair (M, X) then satisfies the hypotheses of Theorem 2.2 in degree m : M is path connected, X is non-empty, path connected and simply connected, and the lower relative groups vanish. So

$$h: \pi_m(M, X, x_0) \longrightarrow H_m(M, X)$$

is an isomorphism, and in particular $\pi_m(M, X, x_0) = 0$ if and only if $H_m(M, X) = 0$. Running m from 2 to n — the hypothesis on the lower groups being available at each step, in the one direction because it has just been established and in the other because it is given — shows that $\pi_i(M, X, x_0) = 0$ for $1 \leq i \leq n$ if and only if $H_i(M, X) = 0$ for $1 \leq i \leq n$. With the two translations this is the theorem. $\square$

Corollary 3.2. *Let X and Y be non-empty, path-connected, simply connected spaces, let $f: X \rightarrow Y$ be a map and let $x_0 \in X$. Then f induces an isomorphism $H_i(X) \rightarrow H_i(Y)$ for every i if and only if it induces an isomorphism $\pi_i(X, x_0) \rightarrow \pi_i(Y, f(x_0))$ for every i .*

Proof. Apply Theorem 3.1 with n arbitrary: for a given i take any $n > i$. $\square$

A map $f: X \rightarrow Y$ is a *weak homotopy equivalence* if it induces a bijection on path components and an isomorphism $\pi_i(X, x) \rightarrow \pi_i(Y, f(x))$ for every $i \geq 1$ and every $x \in X$. Between simply connected spaces the base point is immaterial and there is one path component, so Corollary 3.2 says that such a map is a weak homotopy equivalence exactly when it is an isomorphism on homology. What the corollary does *not* say is that f has a homotopy inverse; that is Whitehead's theorem, and it needs X and Y to be CW complexes.

Remark 3.3. Simple connectivity is essential, and not only as a device in the proof. Let G be a non-abelian group with a finite presentation $\langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$, and let K be the presentation complex: a wedge of k circles with m two-cells attached along the words r_j . By Van Kampen's theorem, Theorem 6.12 of *The homotopy category, and the homotopy groups*, $\pi_1(K) \cong G$, and hence $H_1(K) \cong G^{\text{ab}}$. Let L be the presentation complex of

$$\langle x_1, \dots, x_k \mid r_1, \dots, r_m, [x_i, x_j] \ (i < j) \rangle,$$

obtained from K by attaching further two-cells along the commutators, so that $\pi_1(L) \cong G^{\text{ab}}$ and $H_1(L) \cong G^{\text{ab}}$. The inclusion $f: K \hookrightarrow L$ induces the abelianization $G \rightarrow G^{\text{ab}}$ on π_1 , which is not injective since G is not abelian, and the identity of G^{ab} on H_1 . So once the spaces are not simply connected an isomorphism on H_1 says nothing about π_1 .

That example is an isomorphism in degree 1 only, L having more two-cells than K . The simplest map that is a homology isomorphism in every degree, and so meets the hypothesis of the corollary in full, comes from Poincaré's homology sphere P : collapsing the complement of an open ball gives a map $P \rightarrow S^3$ of degree one, and since $\tilde{H}_i(P) = 0$ for $i < 3$ this induces an isomorphism on H_i for every i . Yet $\pi_1(P)$ is the binary icosahedral group, of order 120, while $\pi_1(S^3) = 1$.