

HOMOTOPY THEORY OF CW COMPLEXES

These notes continue *CW complexes* and *The homotopy category, and the homotopy groups*. They have three tasks. The first is the mapping cylinder of a cellular map, which keeps that construction inside the category of CW complexes. The second is that every path connected pointed space is, from the point of view of homotopy groups, a CW complex: given (Y, y_0) with Y path connected there are a CW complex X with a single 0-cell, taken as its base point x_0 , and a pointed map $(X, x_0) \rightarrow (Y, y_0)$ inducing isomorphisms on all homotopy groups. The third is Whitehead's theorem: a map of CW complexes inducing isomorphisms on all homotopy groups is a homotopy equivalence, not merely a weak one. References to the companions are given as *Theorem such-and-such of CW complexes*, and similarly for the others; all other references are internal. Throughout, $I = [0, 1]$ and all maps of spaces are continuous.

1. THE MAPPING CYLINDER OF A CELLULAR MAP

The mapping cylinder itself is in §8 of *The homotopy category, and the homotopy groups*: for a map $f: X \rightarrow Y$ it is the space

$$M_f = (X \times I \sqcup Y) / ((x, 1) \sim f(x))$$

of Definition 8.1 there, and by Lemma 8.2 the map $\iota: X \rightarrow M_f$, $\iota(x) = [(x, 0)]$, is a closed embedding, Y is a strong deformation retract of M_f with retraction r , and $r \circ \iota = f$. So every map is the inclusion of a closed subspace followed by a homotopy equivalence, and by Remark 8.4 the pair (M_f, X) measures the failure of f_* to be an isomorphism on homotopy groups.

The same reading holds in homology, and is recorded here rather than there because *The homotopy category, and the homotopy groups* makes no use of homology. Identify X with $\iota(X)$ and let $M = M_f$. Since r is a homotopy equivalence it induces isomorphisms $H_n(M) \rightarrow H_n(Y)$, by Corollary 4.2 of *Singular Homology*, and under them the maps induced by the inclusion $X \subseteq M$ become those induced by f , because $r \circ \iota = f$. So in the long exact sequence of the pair,

$$\cdots \rightarrow H_n(X) \rightarrow H_n(M) \rightarrow H_n(M, X) \rightarrow H_{n-1}(X) \rightarrow \cdots,$$

Theorem 6.4 of *Singular Homology*, the map $H_n(X) \rightarrow H_n(M)$ is f_* up to the isomorphism r_* ; hence f_* is an isomorphism on H_n for every n exactly when $H_n(M, X) = 0$ for every n .

What is added here is that the construction stays inside the category of CW complexes.

Proposition 1.1. *Let X and Y be CW complexes and let $f: X \rightarrow Y$ be cellular. Then M_f is a CW complex in which $\iota(X) = X \times \{0\}$ and Y are subcomplexes, its cells being those of Y together with, for each cell e of X , a cell $e \times \{0\}$ of the same dimension and a cell $e \times (0, 1)$ of one dimension more. Its skeleta are*

$$M_f^n = Y^n \cup (X^n \times \{0\}) \cup (X^{n-1} \times (0, 1)),$$

and $\dim M_f = \max(\dim X + 1, \dim Y)$.

Proof. Give I the CW structure with 0-cells $\{0\}$ and $\{1\}$ and one 1-cell $(0, 1)$. Then $X \times I$ is a CW complex by Example 3.6 of *CW complexes*, the caveat there being inapplicable because I is a finite complex and so locally compact; its cells are the products of a cell e of X with one of $\{0\}$, $(0, 1)$, $\{1\}$, of dimensions $\dim e$, $\dim e + 1$, $\dim e$. Both $X \times \{0\}$ and $X \times \{1\}$ are unions of cells containing the closure of each, hence subcomplexes by Proposition 2.10 of *CW complexes*, and each is carried to X by the projection cellularly and homeomorphically.

Now M_f is exactly $Y \cup_f (X \times I)$ in the sense of Proposition 2.12 of *CW complexes*, formed with $Z = X \times I$, the subcomplex $A = X \times \{1\}$, and the map $A \rightarrow Y$ which is f under the identification $A \cong X$; that map is cellular because f is. The proposition therefore makes M_f a CW complex with Y as a subcomplex and with the cells of Z not in A as its remaining cells — that is, the $e \times \{0\}$ and the $e \times (0, 1)$ — and with the stated skeleta, the cell $e \times (0, 1)$ having dimension $\dim e + 1$. Finally $X \times \{0\}$ is a subcomplex of $X \times I$ disjoint from A , so its image is a subcomplex of M_f , and ι is the composite of $x \mapsto (x, 0)$ with q , a homeomorphism onto it. The dimension count is read off from the list of cells. $\square$

Remark 1.2. So for a cellular map of CW complexes the pair (M_f, X) of Remark 8.4 of *The homotopy category, and the homotopy groups* is a CW pair, and the cellular machinery of §4 of *CW complexes* applies to it. This is what will be used for Whitehead's theorem.

2. CW APPROXIMATION

The theorem of this section is that every path connected pointed space has the homotopy groups of a CW complex, and of one with a single 0-cell. The complex is built one dimension at a time. At each stage the map already constructed is replaced by an inclusion, using its mapping cylinder, and cells are attached along the boundaries of a generating set for the relative homotopy group of that inclusion; this single family of cells supplies at once what the map lacked in surjectivity and kills what it lacked in injectivity. Three lemmas about attaching cells prepare the way.

The first says that a cell of dimension d cannot be filled by a cube of dimension less than d , in the precise form that the pair consisting of the cell and the cell with a point removed is $(d - 1)$ -connected.

Lemma 2.1. *Let e be a space homeomorphic to $\mathbb{R}^d$ with $d \geq 1$, let $x \in e$, and let $b \in e \setminus \{x\}$. Then $\pi_i(e, e \setminus \{x\}, b) = 0$ for $1 \leq i < d$; and $e \setminus \{x\}$ is path connected when $d \geq 2$.*

Proof. Transport the question to $\mathbb{R}^d$ with $x = 0$. As $\mathbb{R}^d$ is convex all its homotopy groups vanish, so the homotopy long exact sequence of the pair, Definition 4.14 of *The homotopy category, and the homotopy groups*, gives

$$\pi_i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}, b) \cong \pi_{i-1}(\mathbb{R}^d \setminus \{0\}, b).$$

The sphere S^{d-1} is a deformation retract of $\mathbb{R}^d \setminus \{0\}$, so the right-hand side is $\pi_{i-1}(S^{d-1})$, and S^{d-1} is path connected for $d \geq 2$. It remains to see that $\pi_{i-1}(S^{d-1}) = 0$ for $i-1 < d-1$. For $d = 2$ the only case is $i-1 = 0$, and S^1 is path connected. For $d \geq 3$ the sphere S^{d-1} is path connected and simply connected by Example 6.20 of *The homotopy category, and the homotopy groups*, and $\tilde{H}_r(S^{d-1}) = 0$ for $r < d-1$ by Theorem 2.5 of *Further Properties*

of *Singular Homology*; so Corollary 1.8 of *The Hurewicz theorems* gives $\pi_r(S^{d-1}) = 0$ for $r < d - 1$. $\square$

The second lemma is the deformation itself. Its proof pushes the cube off the cells one at a time, beginning with those of largest dimension, and uses the homotopy extension property together with its bound on tracks, Lemma 6.4 and Remark 6.5 of *The homotopy category, and the homotopy groups*, to hold in place what has already been arranged.

Lemma 2.2. *Let Z be a CW complex and $B \subseteq Z$ a subcomplex such that every cell of Z not lying in B has dimension at least m , and let $j < m$. Then every map $g: I^j \rightarrow Z$ is homotopic, by a homotopy stationary on $g^{-1}(B)$, to a map with image in B .*

Proof. The image of g is compact, so by Proposition 2.9 of *CW complexes* it meets only finitely many cells. We argue by downward induction on the largest dimension d of a cell of $Z \setminus B$ that the image meets. If there is no such cell the image lies in B and there is nothing to prove; otherwise $d \geq m > j$.

Let Z_d be the union of B with the cells of $Z \setminus B$ of dimension at most d . It is a subcomplex, the closure of such a cell meeting only B and cells of smaller dimension, and it contains the image of g . A cell e of $Z \setminus B$ of dimension d is of top dimension in Z_d , so it is open there, its complement $Z_d \setminus e$ being a subcomplex and hence closed. Write $x_e = \Phi_e(0)$.

Missing one point. Fix such a cell e . The sets e and $Z_d \setminus \{x_e\}$ are open and cover Z_d , so their preimages cover I^j . By Lemma 6.2(ii) and (iii) of *The homotopy category, and the homotopy groups* there is a triangulation of I^j — any one, subdivided barycentrically often enough — in which every closed simplex has image in e or in $Z_d \setminus \{x_e\}$. Let K be that complex, $L \subseteq K$ the subcomplex of simplices carried into $Z_d \setminus \{x_e\}$ and $M \subseteq K$ the subcomplex of those carried into e ; a face of a simplex of either kind is again of that kind, so both are subcomplexes, and $K = L \cup M$. Note that $g^{-1}(B) \subseteq |L|$, since a simplex whose image lies in e contains no point of $g^{-1}(B)$, the cell e being disjoint from B .

Run through the simplices of M not in L in order of increasing dimension. Let τ be one of them, of dimension $i \leq j < d$, and suppose those of smaller dimension have already been arranged to miss x_e . Then g carries $|\partial\tau|$ into $e \setminus \{x_e\}$: each proper face lies in L , or lies in M and has been arranged. So $g|_\tau$ is a map of pairs $(\tau, \partial\tau) \rightarrow (e, e \setminus \{x_e\})$. For $i \geq 1$, Lemma 2.1 makes $e \setminus \{x_e\}$ path connected and $\pi_i(e, e \setminus \{x_e\}) = 0$, so Corollary 7.5 of *The homotopy category, and the homotopy groups* — the pair $(\tau, \partial\tau)$ being homeomorphic to (D^i, S^{i-1}) — makes $g|_\tau$ homotopic rel $\partial\tau$ to a map into $e \setminus \{x_e\}$. For $i = 0$ the same holds trivially, e being path connected and $e \setminus \{x_e\}$ non-empty.

Prescribe on $|K^i \cup L| \times I$ the homotopy that is this one on τ and is stationary everywhere else; the two prescriptions agree on $|\partial\tau|$, where the first is stationary. Extend it over $|K| \times I$ by Lemma 6.4 of *The homotopy category, and the homotopy groups*. The extension leaves $|L|$ where it was, the prescription being stationary there. It also keeps $|M|$ inside e : the map carries $|M|$ into e , and the prescribed homotopy carries $(|M| \cap |K^i \cup L|) \times I$ into e , being the compression of $g|_\tau$ there or else stationary at values already in e ; so Remark 6.5 of *The homotopy category, and the homotopy groups* puts the whole track of $|M|$ inside e . After the homotopy, therefore, g still carries $|M|$ into e , it has not moved $|L|$, the simplices already arranged have not moved either, being of dimension at most i , and τ

now misses x_e . Running through the simplices of M in this way gives a homotopy of g , stationary on $|L|$ and hence on $g^{-1}(B)$, after which the image misses x_e .

Lowering d . Carry this out for each of the finitely many d -cells of $Z \setminus B$ that the image meets; each homotopy is supported in its own cell, and distinct open cells are disjoint, so they do not disturb one another. The image now misses x_e for every d -cell e of $Z \setminus B$, met or not. Let S be the set of those points; it is closed and discrete by Lemma 2.8 of *CW complexes*, and

$$R(v, t) = v \quad (v \in Z_{d-1}), \quad R(\Phi_e(z), t) = \Phi_e\left((1-t)z + t \frac{z}{|z|}\right) \quad (z \neq 0)$$

is a deformation retraction of $Z_d \setminus S$ onto Z_{d-1} , by the argument used for Lemma 4.1 of *CW complexes*: the two prescriptions agree where $|z| = 1$, and R is continuous because $(Z_d \setminus S) \times I$ carries the weak topology determined by $Z_{d-1} \times I$ and the sets $(D_e \setminus \{0\}) \times I$. Composing g with R gives a further homotopy, stationary on Z_{d-1} and hence on $g^{-1}(B)$, after which the image lies in Z_{d-1} and so meets no cell of $Z \setminus B$ of dimension d or more. The induction continues, and stops because every cell of $Z \setminus B$ has dimension at least m . $\square$

Lemma 2.3. *Let Z be a CW complex, $B \subseteq Z$ a subcomplex such that every cell of Z not lying in B has dimension at least m , and let $b_0 \in B$. Then the inclusion induces*

$$\pi_k(B, b_0) \longrightarrow \pi_k(Z, b_0)$$

an isomorphism for $k \leq m-2$ and a surjection for $k = m-1$.

Proof. Surjectivity for $k \leq m-1$. Let $\sigma: (I^k, \partial I^k) \rightarrow (Z, b_0)$ represent a class of $\pi_k(Z, b_0)$. Since $k < m$, Lemma 2.2 deforms σ to a map into B by a homotopy stationary on $\sigma^{-1}(B)$, hence on ∂I^k ; so that homotopy is rel ∂I^k , and the class lies in the image.

Injectivity for $k \leq m-2$. Let $\sigma_0, \sigma_1: (I^k, \partial I^k) \rightarrow (B, b_0)$ and let $H: I^k \times I \rightarrow Z$ be a homotopy rel ∂I^k between them. Identify $I^k \times I$ with I^{k+1} ; then H carries ∂I^{k+1} into B , the two ends by σ_0 and σ_1 and the remaining faces to b_0 . Since $k+1 < m$, Lemma 2.2 deforms H to a map into B by a homotopy stationary on $H^{-1}(B)$, hence on ∂I^{k+1} . What results is a homotopy rel ∂I^k from σ_0 to σ_1 inside B , so the two represent the same class of $\pi_k(B, b_0)$. $\square$

Theorem 2.4 (CW approximation). *Let Y be a path connected space with base point y_0 . Then there are a CW complex X with a single 0-cell x_0 and a map $f: (X, x_0) \rightarrow (Y, y_0)$ such that*

$$f_*: \pi_k(X, x_0) \longrightarrow \pi_k(Y, y_0)$$

is an isomorphism for every $k \geq 1$.

Proof. We construct CW complexes $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$, each a subcomplex of the next, with X_n of dimension at most n and with a single 0-cell x_0 , together with maps $f_n: X_n \rightarrow Y$ with $f_{n+1}|_{X_n} = f_n$ and $f_n(x_0) = y_0$, such that

$$(*)_n \quad (f_n)_*: \pi_k(X_n, x_0) \rightarrow \pi_k(Y, y_0) \text{ is onto for } k \leq n \text{ and injective for } k < n.$$

The start. Let $X_0 = \{x_0\}$ and $f_0(x_0) = y_0$. Then $(*)_0$ holds: in degree 0 both sets are a point, Y being path connected, and there is nothing to check below.

The step. Suppose X_n and f_n are given, with $(*)_n$. Let M_n be the mapping cylinder of f_n , Definition 8.1 of *The homotopy category*, and the homotopy groups, so that X_n is

a closed subspace of M_n and the retraction $r: M_n \rightarrow Y$ is a homotopy equivalence with $r|_{X_n} = f_n$.

Choose elements $g_1, g_2, \dots$ generating $\pi_{n+1}(M_n, X_n, x_0)$, a group because $n+1 \geq 1$ and, when $n = 0$, because X_0 is a point, so that the relative set is then the group $\pi_1(M_0, x_0)$. For each i choose a representative

$$\varphi_i: (I^{n+1}, \partial I^{n+1}, J^n) \longrightarrow (M_n, X_n, x_0).$$

Identify $(I^{n+1}, \partial I^{n+1})$ with (D^{n+1}, S^n) , each being a compact convex body with its boundary, and attach to X_n an $(n+1)$ -cell along $\varphi_i|_{\partial I^{n+1}}$, read through that identification as a map $S^n \rightarrow X_n$. Extend f_n over the cell by $r \circ \varphi_i$; this agrees with f_n on the attaching sphere, since φ_i carries ∂I^{n+1} into X_n , where r is f_n . Call the result X_{n+1} and the extended map f_{n+1} . Then X_n is a subcomplex of X_{n+1} , every cell of X_{n+1} not in X_n has dimension $n+1$, and $\dim X_{n+1} \leq n+1$.

Two consequences of the construction drive everything. First, the maps φ_i and the inclusion of X_n fit together into a map

$$F: X_{n+1} \longrightarrow M_n$$

which is the inclusion on X_n and φ_i on the i -th cell, and $r \circ F = f_{n+1}$; as r_* is an isomorphism, $(f_{n+1})_*$ and F_* have the same kernels and the same images. Second, writing Φ_i for the characteristic map of the i -th cell, read on $(I^{n+1}, \partial I^{n+1})$, its restriction to ∂I^{n+1} is the attaching map and so carries J^n to x_0 ; thus Φ_i is a map of triples into (X_{n+1}, X_n, x_0) , and $F \circ \Phi_i = \varphi_i$. Regarding F as a map of pairs $(X_{n+1}, X_n) \rightarrow (M_n, X_n)$, which is the identity on X_n , this says that

$$F_*: \pi_{n+1}(X_{n+1}, X_n, x_0) \longrightarrow \pi_{n+1}(M_n, X_n, x_0)$$

carries $[\Phi_i]$ to g_i ; its image is therefore a subgroup containing a generating set, and F_* is onto.

We check $(*)_{n+1}$, using throughout the long exact sequences of the pairs (X_{n+1}, X_n) and (M_n, X_n) and the naturality of those sequences for F , Remark 4.15 of *The homotopy category, and the homotopy groups*.

Onto for $k \leq n+1$. It is enough to treat F_* on absolute groups. For $k \leq n$ the composite $\pi_k(X_n, x_0) \rightarrow \pi_k(X_{n+1}, x_0) \rightarrow \pi_k(M_n, x_0)$ is $(f_n)_*$ up to r_* , which is onto by $(*)_n$; so F_* is onto in those degrees. Let $k = n+1$ and let $\tilde{c} \in \pi_{n+1}(M_n, x_0)$, with image $\bar{c}$ in $\pi_{n+1}(M_n, X_n, x_0)$. Choose u with $F_*u = \bar{c}$. Since $\bar{c}$ comes from $\pi_{n+1}(M_n, x_0)$ we have $\partial \bar{c} = 0$, so $F_*\partial u = \partial F_*u = 0$; and F_* is the identity on $\pi_n(X_n, x_0)$, so $\partial u = 0$ and u is the image of some $v \in \pi_{n+1}(X_{n+1}, x_0)$. Now F_*v and $\tilde{c}$ have the same image $\bar{c}$ in the relative group, so by exactness they differ by the image of some $w \in \pi_{n+1}(X_n, x_0)$; and that image is F_* of the image of w in $\pi_{n+1}(X_{n+1}, x_0)$. Hence $\tilde{c}$ lies in the image of F_* .

Injective for $k \leq n$. Again it is enough to treat F_* . Let $a \in \pi_k(X_{n+1}, x_0)$ have $F_*a = 0$, with $k \leq n$. By Lemma 2.3 with $m = n+1$ the map $\pi_k(X_n, x_0) \rightarrow \pi_k(X_{n+1}, x_0)$ is onto for $k \leq n$, so a is the image of some $b \in \pi_k(X_n, x_0)$, and $(f_n)_*b = 0$. For $k < n$ this gives $b = 0$ by $(*)_n$, hence $a = 0$. For $k = n$ exactness puts b in the image of $\partial: \pi_{n+1}(M_n, X_n, x_0) \rightarrow \pi_n(X_n, x_0)$, which is generated by the ∂g_i . By naturality, $\partial g_i = \partial F_*[\Phi_i] = F_*\partial[\Phi_i] = \partial[\Phi_i]$, the last because F_* is the identity on $\pi_n(X_n, x_0)$; and $\partial[\Phi_i]$ dies in $\pi_n(X_{n+1}, x_0)$, the composite $\pi_{n+1}(X_{n+1}, X_n, x_0) \rightarrow \pi_n(X_n, x_0) \rightarrow \pi_n(X_{n+1}, x_0)$ being zero by exactness. So b dies in $\pi_n(X_{n+1}, x_0)$ and $a = 0$.

This is $(*)_{n+1}$, and the construction proceeds.

The limit. Let $X = \bigcup_n X_n$, topologized so that a subset is closed exactly when its intersection with each X_n is closed. Since X_n has dimension at most n and X_{n+1} is obtained from it by attaching $(n+1)$ -cells, the X_n are the skeleta of a CW structure on X in the sense of Definition 1.2 of *CW complexes*: $X_0 = \{x_0\}$ is discrete, clause (ii) is the construction, and clause (iii) is the topology just imposed. In particular X has the single 0-cell x_0 . The maps f_n agree on overlaps, so they define $f: X \rightarrow Y$, continuous because its restriction to each X_n is.

Let $k \geq 1$. For surjectivity, $(f_k)_*$ is onto in degree k by $(*)_k$, and it is the composite of $\pi_k(X_k, x_0) \rightarrow \pi_k(X, x_0)$ with f_* ; so f_* is onto. For injectivity, let $a \in \pi_k(X, x_0)$ have $f_*a = 0$ and represent a by $\sigma: (I^k, \partial I^k) \rightarrow (X, x_0)$. The image of σ is compact, so by Theorem 2.11 of *CW complexes* it lies in a finite subcomplex. Each cell of X was attached at some stage of the construction, so a finite subcomplex lies in X_N for N the largest of the finitely many stages concerned; and we may enlarge N so that $N > k$. Thus a is the image of a class $a_N \in \pi_k(X_N, x_0)$, and $(f_N)_*a_N = f_*a = 0$. Since $k < N$, $(*)_N$ gives $a_N = 0$, whence $a = 0$. $\square$

Corollary 2.5. *Let Y be any space. Then there are a CW complex X , with one 0-cell in each of its path components, and a map $f: X \rightarrow Y$ inducing a bijection $\pi_0(X) \rightarrow \pi_0(Y)$ and an isomorphism $\pi_k(X, x) \rightarrow \pi_k(Y, f(x))$ for every $k \geq 1$ and every $x \in X$.*

Proof. Let $\{Y_j\}$ be the path components of Y and choose $y_j \in Y_j$. Each Y_j is path connected, so Theorem 2.4 provides a CW complex X_j with a single 0-cell x_j and a map $f_j: (X_j, x_j) \rightarrow (Y_j, y_j)$ inducing isomorphisms on all homotopy groups. Put

$$X = \coprod_j X_j, \quad f = \coprod_j f_j: X \longrightarrow Y,$$

the second composed with the inclusions $Y_j \subseteq Y$. Filtering X by $X^n = \coprod_j X_j^n$ exhibits it as a CW complex, each clause of Definition 1.2 of *CW complexes* holding summand by summand, and f is continuous because each f_j is.

A CW complex with a single 0-cell is path connected, every cell having its closure meeting the 0-skeleton; so the path components of X are the X_j , and $j \mapsto [X_j]$ matches them bijectively with the Y_j . That is the statement about π_0 . For $k \geq 1$ and $x \in X_j$ we have $\pi_k(X, x) = \pi_k(X_j, x)$ and $\pi_k(Y, f(x)) = \pi_k(Y_j, f(x))$, homotopy groups depending only on the path component; and the change-of-base-point isomorphisms along a path, Remark 4.8 of *The homotopy category, and the homotopy groups*, are natural for maps, so the square they form with $(f_j)_*$ at x and at x_j commutes. Since $(f_j)_*$ is an isomorphism at x_j , it is one at x . $\square$

Lemma 2.2 has a second consequence, recorded here because §3 needs it: every map of CW complexes may be pushed into cellular form.

Proposition 2.6 (cellular approximation). *Let X and Y be CW complexes and $f: X \rightarrow Y$ a map. Then f is homotopic to a cellular map. If $A \subseteq X$ is a subcomplex on which f is already cellular, that is $f(A \cap X^k) \subseteq Y^k$ for every k , the homotopy may be taken rel A .*

Proof. Put $t_n = 1 - 2^{-n}$. We construct a homotopy of f in stages, the n -th occupying $[t_n, t_{n+1}]$, so that after the n -th stage the map carries X^k into Y^k for every $k \leq n$, and so that the n -th stage is stationary on $X^{n-1} \cup A$.

Suppose the stages before the n -th have been performed, and write f again for the resulting map, which carries X^k into Y^k for $k \leq n-1$. Let Φ be the characteristic map of an n -cell of X not lying in A . Every cell of Y not lying in Y^n has dimension at least $n+1$, so Lemma 2.2, applied with $Z = Y$, $B = Y^n$, $m = n+1$ and $j = n$ — and with D^n identified with I^n — deforms $f \circ \Phi$ to a map with image in Y^n by a homotopy stationary on $(f\Phi)^{-1}(Y^n)$. That set contains S^{n-1} , since $\Phi(S^{n-1}) \subseteq X^{n-1}$ and f already carries X^{n-1} into Y^{n-1} ; so the homotopy is rel S^{n-1} . For an n -cell lying in A take the stationary homotopy, which is legitimate because f is cellular on A and so already carries that cell into Y^n .

Being rel the attaching spheres, these homotopies agree on overlaps and fit together with the stationary homotopy of f on $X^{n-1} \cup A$ to give a homotopy of f on $X^n \cup A$, stationary on $X^{n-1} \cup A$ and ending at a map that carries X^n into Y^n . Extend it over X by Proposition 2.13 of *CW complexes*, applied to the CW pair $(X, X^n \cup A)$. The extension agrees with the prescription on $X^n \cup A$, so it is stationary on the whole of $X^{n-1} \cup A$ — this is why the subcomplex extended over is $X^n \cup A$ and not X^n , the extension being free to move the cells of A of dimension greater than n otherwise — and it does not disturb the condition $f(X^k) \subseteq Y^k$ for $k < n$. This is the n -th stage.

Every stage is stationary on A , so the whole homotopy is rel A ; and a point of X^n is stationary after time t_{n+1} , the later stages being stationary on X^n , so the concatenation of the stages is defined at $t = 1$. Its restriction to $X^n \times I$ is built from finitely many continuous pieces on closed sets, and $X \times I$ carries the weak topology determined by the $X^n \times I$, the product of a quotient map with the identity of the locally compact space I being again a quotient map; so the concatenation is continuous. It is a homotopy rel A from f to a map carrying X^n into Y^n for every n , which is to say to a cellular map. $\square$

3. WHITEHEAD'S THEOREM

A map of spaces inducing isomorphisms on all homotopy groups need not have a homotopy inverse: Remark 3.3 of *The Hurewicz theorems* exhibits a homology isomorphism that is not one, and the same phenomenon occurs for homotopy groups outside the world of CW complexes. Inside it, however, a weak homotopy equivalence is a homotopy equivalence. That is Whitehead's theorem, and the whole of it is contained in the following statement about a pair.

Theorem 3.1. *Let X be a path connected CW complex and $A \subseteq X$ a path connected subcomplex such that the inclusion induces isomorphisms $\pi_i(A, a_0) \rightarrow \pi_i(X, a_0)$ for every $i \geq 1$ and some, hence any, $a_0 \in A$. Then A is a strong deformation retract of X in the sense of Definition 1.8 of The homotopy category, and the homotopy groups: there is a homotopy $F: X \times I \rightarrow X$ with $F_0 = \text{id}_X$, with $F_t|_A = \text{id}_A$ for every t , and with $F_1(X) \subseteq A$.*

Proof. First, $\pi_n(X, A, a_0) = 0$ for every $n \geq 1$. In the homotopy long exact sequence of the pair, Definition 4.14 of *The homotopy category, and the homotopy groups*,

$$\pi_n(A, a_0) \rightarrow \pi_n(X, a_0) \rightarrow \pi_n(X, A, a_0) \rightarrow \pi_{n-1}(A, a_0) \rightarrow \pi_{n-1}(X, a_0),$$

the first map is onto and the last injective — for $n \geq 2$ by hypothesis, and for $n = 1$ because $\pi_0(A, a_0)$ and $\pi_0(X, a_0)$ are both a point, A and X being path connected. Hence the middle term vanishes.

Put $t_n = 1 - 2^{-n}$, so that $t_0 = 0$ and $t_n \uparrow 1$, and write X^n for the skeleta of X . We construct homotopies

$$F^{(n)}: (A \cup X^n) \times [0, t_{n+1}] \longrightarrow X \quad (n \geq 0)$$

with the properties

- (a) $F_0^{(n)}$ is the inclusion and $F_s^{(n)}|_A = \text{id}_A$ for every s ;
- (b) $F^{(n)}$ restricted to $(A \cup X^{n-1}) \times [0, t_n]$ is $F^{(n-1)}$, and $F^{(n)}$ is stationary on $(A \cup X^{n-1}) \times [t_n, t_{n+1}]$;
- (c) $F_{t_{n+1}}^{(n)}$ carries $A \cup X^n$ into A .

For $n = 0$ read $A \cup X^{-1} = A$ and $F^{(-1)}$ as the stationary homotopy of the inclusion of A , so that the construction below applies unchanged.

Suppose $F^{(n-1)}$ is given. The pair $(A \cup X^n, A \cup X^{n-1})$ is a CW pair, so by Proposition 2.13 of *CW complexes* the homotopy $F^{(n-1)}$, prescribed on $(A \cup X^{n-1}) \times [0, t_n]$, extends to a homotopy

$$\tilde{F}: (A \cup X^n) \times [0, t_n] \longrightarrow X$$

of the inclusion; here the interval $[0, t_n]$ has been identified with I . By (c) for $n - 1$ the map $\tilde{F}_{t_n}$ carries $A \cup X^{n-1}$ into A , though it need not carry the n -cells there.

Let Φ be the characteristic map of an n -cell of X not lying in A . Then $\tilde{F}_{t_n} \circ \Phi$ carries S^{n-1} into A , being $\tilde{F}_{t_n}$ on a subset of X^{n-1} , so it is a map of pairs $(D^n, S^{n-1}) \rightarrow (X, A)$. Since A is path connected and $\pi_n(X, A, a_0) = 0$, Corollary 7.5 of *The homotopy category, and the homotopy groups* makes it homotopic rel S^{n-1} to a map with image in A . The homotopies belonging to different n -cells are stationary on the attaching spheres, so together with the stationary homotopy of $\tilde{F}_{t_n}$ on $A \cup X^{n-1}$ they define a homotopy of $\tilde{F}_{t_n}$ on $A \cup X^n$, which we run during $[t_n, t_{n+1}]$. It is stationary on $A \cup X^{n-1}$, and at time t_{n+1} it carries $A \cup X^n$ into A .

Let $F^{(n)}$ be $\tilde{F}$ followed by that homotopy. Properties (a), (b) and (c) hold by construction.

Now define $F: X \times I \rightarrow X$ by letting F agree with $F^{(n)}$ on $(A \cup X^n) \times [0, t_{n+1}]$ and be stationary on $(A \cup X^n) \times [t_{n+1}, 1]$. By (b) the prescriptions for different n agree, and by (c) the stationary part is legitimate, a point of $A \cup X^n$ having been carried into A by time t_{n+1} and being fixed thereafter by (a). The restriction of F to $X^n \times I$ is continuous, being built from finitely many continuous pieces on closed sets, and $X \times I$ carries the weak topology determined by the $X^n \times I$, the product of a quotient map with the identity of the locally compact space I being again a quotient map; so F is continuous. Finally $F_0 = \text{id}_X$, each F_t is the identity on A , and $F_1(X) \subseteq A$, every point of X lying in some X^n . $\square$

Theorem 3.2 (Whitehead). *Let X and Y be path connected CW complexes and let $f: X \rightarrow Y$ be a map inducing isomorphisms $\pi_i(X, x_0) \rightarrow \pi_i(Y, f(x_0))$ for every $i \geq 1$. Then f is a homotopy equivalence.*

Proof. By Proposition 2.6 the map f is homotopic to a cellular map f' . Such a homotopy carries the base point along its track, and $(f')_*$ differs from f_* by the resulting change-of-base-point isomorphism, Remark 4.8 of *The homotopy category, and the homotopy*

groups; so f' again induces isomorphisms on all homotopy groups, and it is a homotopy equivalence exactly when f is. We may therefore assume f cellular.

Let $M = M_f$ be the mapping cylinder, with $\iota: X \rightarrow M$ and $r: M \rightarrow Y$ as in §8 of *The homotopy category, and the homotopy groups*. By Proposition 1.1, M is a CW complex in which $\iota(X)$ and Y are subcomplexes; we identify X with $\iota(X)$. It is path connected, being homotopy equivalent to Y by Lemma 8.2 of *The homotopy category, and the homotopy groups*.

That lemma also makes r a homotopy equivalence with $r \circ \iota = f$, so r_* is an isomorphism and $\iota_* = r_*^{-1} f_*$ is an isomorphism on π_i for every $i \geq 1$. Theorem 3.1, applied to the CW pair (M, X) , therefore makes X a strong deformation retract of M ; in particular ι is a homotopy equivalence, by Proposition 1.9 of *The homotopy category, and the homotopy groups*. Hence so is $f = r \circ \iota$, a composite of two of them. $\square$

Corollary 3.3. *Let X and Y be CW complexes, not necessarily connected, and let $f: X \rightarrow Y$ be a map inducing a bijection $\pi_0(X) \rightarrow \pi_0(Y)$ and isomorphisms $\pi_i(X, x) \rightarrow \pi_i(Y, f(x))$ for every $i \geq 1$ and every $x \in X$ — that is, a weak homotopy equivalence. Then f is a homotopy equivalence.*

Proof. The path components of a CW complex are subcomplexes. Indeed a closed cell is path connected, being a continuous image of a ball, so each cell lies with its closure in a single path component; a component is therefore a union of cells containing the closure of each, hence a subcomplex by Proposition 2.10 of *CW complexes*. Each component is then also open, its complement being the union of the others, itself a subcomplex and so closed; so X is the topological disjoint union of its components, and likewise Y . Since f_* is a bijection on π_0 , it carries the components X_j of X bijectively to the components Y_j of Y , and the restriction $f_j: X_j \rightarrow Y_j$ is a map of path connected CW complexes inducing isomorphisms on all homotopy groups. By Theorem 3.2 each f_j has a homotopy inverse h_j and homotopies $h_j f_j \simeq \text{id}$, $f_j h_j \simeq \text{id}$. Assembling the h_j and those homotopies over the disjoint unions gives a homotopy inverse of f . $\square$