

HOMOLOGY

These notes introduce chain complexes and their homology and study their basic properties, including chain homotopy. We also discuss various algebraic operations on chain complexes, the mapping cone, tensor products, homology and cohomology with coefficients, and give structural results about these algebraic operations, including the Künneth theorem and the universal coefficient theorems. The last section gives a “geometric” example, the oriented simplicial homology of a simplicial complex.

1. EXACT SEQUENCES OF R -MODULES

The language of exact sequences is used throughout what follows. We state it for R -modules, where R denotes a ring with 1, not necessarily commutative, and modules are left R -modules; taking $R = \mathbb{Z}$ gives the case of abelian groups.

Definition 1.1. A *sequence* of R -modules is a family of R -modules and R -linear maps

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots,$$

finite or infinite. It is *exact at A_n* if

$$\operatorname{im} f_{n+1} = \ker f_n,$$

which presupposes that there is a map into A_n and a map out of it. The sequence is *exact* if it is exact at every such A_n .

Proposition 1.2. Let $f: A \rightarrow B$ be a map of R -modules and write 0 for the zero module.

- (i) $0 \rightarrow A \xrightarrow{f} B$ is exact at A if and only if f is injective.
- (ii) $A \xrightarrow{f} B \rightarrow 0$ is exact at B if and only if f is surjective.
- (iii) $0 \rightarrow A \xrightarrow{f} B \rightarrow 0$ is exact if and only if f is an isomorphism.

Proof. In (i) the image of the map $0 \rightarrow A$ is 0, so exactness at A says $\ker f = 0$, which is injectivity. In (ii) the kernel of the map $B \rightarrow 0$ is all of B , so exactness at B says $\operatorname{im} f = B$, which is surjectivity. Part (iii) is (i) and (ii) together, an injective and surjective R -linear map being an isomorphism of R -modules, its set-theoretic inverse being again R -linear. $\square$

Definition 1.3. A *short exact sequence* of R -modules is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Written out, the three conditions are: f is injective, g is surjective, and $\operatorname{im} f = \ker g$.

Proposition 1.4. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be short exact. Then f maps A isomorphically onto the submodule $\ker g \subseteq B$, and g induces an isomorphism

$$\bar{g}: B/f(A) \xrightarrow{\cong} C.$$

Conversely, if $A \subseteq B$ is a submodule then

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{q} B/A \longrightarrow 0,$$

with ι the inclusion and q the quotient map, is short exact. So a short exact sequence is no more and no less than a submodule together with its quotient.

Proof. The map f is injective and has image $\ker g$, so it is an isomorphism onto $\ker g$. Since g is surjective with kernel $f(A)$, the first isomorphism theorem gives the isomorphism $\bar{g}$, $\bar{g}(b + f(A)) = g(b)$. For the converse, ι is injective, q is surjective, and $\ker q = A = \operatorname{im} \iota$. $\square$

Example 1.5. For $m \geq 1$,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m \longrightarrow 0$$

is a short exact sequence of abelian groups: multiplication by m is injective, the quotient map is surjective, and the image $m\mathbb{Z}$ of the first map is the kernel of the second.

Example 1.6. For any R -modules A and C the sequence

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus C \xrightarrow{p_C} C \longrightarrow 0,$$

with $\iota_A(a) = (a, 0)$ and $p_C(a, c) = c$, is short exact. A short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is called *split* if it is isomorphic to one of this form: that is, if there is an isomorphism $h: B \rightarrow A \oplus C$ with $h \circ f = \iota_A$ and $p_C \circ h = g$, so that the diagram formed by the two sequences, with the identity at either end, commutes. Example 1.5 is not split for $m \geq 2$, since $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/m$, the latter having elements of finite order.

Exercise 1.7. Show that $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split if and only if f admits a retraction $r: B \rightarrow A$ with $r \circ f = \operatorname{id}_A$, and if and only if g admits a section $s: C \rightarrow B$ with $g \circ s = \operatorname{id}_C$.

Exercise 1.8. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of abelian groups. Show that B is finitely generated if A and C are.

2. CHAIN COMPLEXES

Homology assigns abelian groups to a space by a two-step process: first one manufactures out of the space an algebraic object, a chain complex, and then one extracts from that object its homology. The second step is pure algebra and is the same whatever the source of the complex. This section sets up that algebra: the objects, the maps between them, and the category they form.

2.1. Complexes of R -modules. Throughout, R is a ring with 1 and modules are left R -modules, as in §1; the case $R = \mathbb{Z}$, that of abelian groups, is the one that will arise from spaces.

Definition 2.1. A *chain complex of R -modules* C_* is a family of R -modules C_n , one for each $n \in \mathbb{Z}$, together with R -linear maps

$$\partial_n: C_n \longrightarrow C_{n-1} \quad (n \in \mathbb{Z})$$

satisfying

$$\partial_n \circ \partial_{n+1} = 0 \quad \text{for all } n.$$

The ∂_n are the *differentials*, or *boundary maps*, of the complex; an element of C_n is a *chain of degree n* . We write (C_*, ∂) , or simply C_* , and drop the subscript on ∂ when it is clear from the context, so that the defining condition reads $\partial \circ \partial = 0$. Thus a complex is a diagram

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

in which any two consecutive maps compose to zero.

Definition 2.2. Let C_* be a chain complex. The submodules

$$Z_n(C) = \ker(\partial_n: C_n \rightarrow C_{n-1}), \quad B_n(C) = \operatorname{im}(\partial_{n+1}: C_{n+1} \rightarrow C_n)$$

of C_n are the modules of *n -cycles* and of *n -boundaries*. The condition $\partial\partial = 0$ says exactly that $B_n(C) \subseteq Z_n(C)$, and the quotient module

$$H_n(C) = Z_n(C)/B_n(C)$$

is the *n -th homology* of C_* . The class of a cycle $z \in Z_n(C)$ in $H_n(C)$ is written $[z]$, and two cycles with the same class are said to be *homologous*. The complex is *exact* at n , or *acyclic* at n , if $H_n(C) = 0$, that is, if $B_n(C) = Z_n(C)$, which is exactness at C_n in the sense of Definition 1.1; it is *exact* if it is exact at every n .

Example 2.3. Any R -module A gives a complex concentrated in degree 0: take $C_0 = A$ and $C_n = 0$ for $n \neq 0$, all differentials being zero. Then $H_0 = A$ and $H_n = 0$ otherwise. More generally a family of modules A_n with all differentials zero is a complex, and $H_n = A_n$: a complex with zero differentials is its own homology.

Example 2.4. Take $R = \mathbb{Z}$, let $m \geq 1$ and let C_* be the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

with the two copies of $\mathbb{Z}$ in degrees 1 and 0 and the map multiplication by m . The condition $\partial\partial = 0$ holds trivially, there being no two consecutive non-zero maps. Here $Z_1 = 0$, so $H_1 = 0$; and $Z_0 = \mathbb{Z}$ while $B_0 = m\mathbb{Z}$, so $H_0 = \mathbb{Z}/m$. The homology measures the failure of exactness, and here it detects the cokernel of multiplication by m .

Definition 2.5. Let (C_*, ∂^C) and (D_*, ∂^D) be chain complexes of R -modules. A *chain map* $f: C_* \rightarrow D_*$ is a family of R -linear maps $f_n: C_n \rightarrow D_n$, one for each n , such that

$$\partial_n^D \circ f_n = f_{n-1} \circ \partial_n^C \quad \text{for all } n,$$

that is, such that every square

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n^C} & C_{n-1} \\ f_n \downarrow & & \downarrow f_{n-1} \\ D_n & \xrightarrow{\partial_n^D} & D_{n-1} \end{array}$$

commutes. One says that f *commutes with the differentials*.

Proposition 2.6. Chain complexes of R -modules and chain maps between them form a category, which we denote $\mathbf{Ch}(\mathbf{Mod}_R)$.

Proof. For each complex C_* the family $(\text{id}_{C_n})_n$ is a chain map id_{C_*} , since $\partial \circ \text{id} = \partial = \text{id} \circ \partial$. If $f: C_* \rightarrow D_*$ and $g: D_* \rightarrow E_*$ are chain maps, define $g \circ f$ degreewise by $(g \circ f)_n = g_n \circ f_n$; each $g_n \circ f_n$ is R -linear, and $g \circ f$ is a chain map, because

$$\partial_n^E \circ g_n \circ f_n = g_{n-1} \circ \partial_n^D \circ f_n = g_{n-1} \circ f_{n-1} \circ \partial_n^C.$$

Composition is associative and unital because it is so in each degree, the identities and the composition being those of $\mathbf{Mod}_R$. $\square$

Remark 2.7. The category $\mathbf{Ch}(\mathbf{Mod}_R)$ carries more structure, all of it inherited degreewise from $\mathbf{Mod}_R$.

- (i) If $f, g: C_* \rightarrow D_*$ are chain maps then so is $f + g$, defined by $(f + g)_n = f_n + g_n$; with this addition $\text{Hom}_{\mathbf{Ch}(\mathbf{Mod}_R)}(C_*, D_*)$ is an abelian group, and composition is biadditive. The zero complex is a zero object: it is both an initial and a terminal object in the category.
- (ii) Direct sums are formed degreewise: $(C_* \oplus D_*)_n = C_n \oplus D_n$ with differential $\partial^C \oplus \partial^D$, and hence the category has all finite direct sums.
- (iii) So are kernels and cokernels. If $f: C_* \rightarrow D_*$ is a chain map then ∂^C carries $\ker f_n$ into $\ker f_{n-1}$, because $f_{n-1} \partial^C = \partial^D f_n$ vanishes on $\ker f_n$; the resulting complex $(\ker f_*, \partial^C)$, whose terms are submodules of the C_n , is the kernel of f in $\mathbf{Ch}(\mathbf{Mod}_R)$. Likewise ∂^D descends to the quotient modules $D_n / \text{im } f_n$ and makes them a complex, the cokernel of f .

Thus $\mathbf{Ch}(\mathbf{Mod}_R)$ is an additive category¹, and in fact an abelian one.

Proposition 2.8. *Let $f: C_* \rightarrow D_*$ be a chain map. Then f_n carries $Z_n(C)$ into $Z_n(D)$ and $B_n(C)$ into $B_n(D)$, so that*

$$H_n(f): H_n(C) \longrightarrow H_n(D), \quad H_n(f)[z] = [f_n(z)],$$

is a well defined R -linear map. With this, $H_n: \mathbf{Ch}(\mathbf{Mod}_R) \rightarrow \mathbf{Mod}_R$ is a functor for every $n \in \mathbb{Z}$, and it is additive: $H_n(f + g) = H_n(f) + H_n(g)$.

Proof. If $z \in Z_n(C)$ then $\partial^D f_n(z) = f_{n-1} \partial^C(z) = 0$, so $f_n(z) \in Z_n(D)$. If $b = \partial^C(c)$ with $c \in C_{n+1}$ then $f_n(b) = f_n \partial^C(c) = \partial^D f_{n+1}(c)$, which lies in $B_n(D)$. Hence f_n carries $Z_n(C)$ to $Z_n(D)$ and $B_n(C)$ to $B_n(D)$, and so induces a map of the quotients, which is the stated formula; it is well defined precisely because $B_n(C)$ is carried into $B_n(D)$, and it is R -linear because f_n is.

Functoriality is immediate from the formula: $H_n(\text{id})[z] = [z]$, and

$$H_n(g \circ f)[z] = [g_n f_n(z)] = H_n(g)[f_n(z)] = H_n(g)H_n(f)[z].$$

Additivity is equally immediate, since $[(f_n + g_n)(z)] = [f_n(z)] + [g_n(z)]$. $\square$

We form the category of graded R -modules, whose objects are collections $\mathbf{G} = \{G_n\}_{n \in \mathbb{Z}}$ of R -modules and whose morphisms $\mathbf{f}: \mathbf{G} \rightarrow \mathbf{G}'$ are collections $\{f_n\}_{n \in \mathbb{Z}}$ with $f_n: G_n \rightarrow G'_n$ an R -linear map for each n , composition and identities being formed degreewise. Then homology is a functor from chain complexes to the category of graded R -modules, carrying C_* to $H_*(C) = \{H_n(C)\}_{n \in \mathbb{Z}}$ and a chain map f to $H_*(f) = \{H_n(f)\}_{n \in \mathbb{Z}}$.

¹An *additive category* is one that satisfies the first two conditions in this remark.

Remark 2.9. A *cochain complex* is the same notion with the differentials raising degree: modules C^n and maps $d^n: C^n \rightarrow C^{n+1}$ with $d^{n+1} \circ d^n = 0$. The two notions are carried into one another by the relabelling $C^n = C_{-n}$, so nothing new is involved; the cohomology modules are $H^n(C) = \ker d^n / \operatorname{im} d^{n-1}$. We use whichever indexing is convenient.

Remark 2.10. A complex is *non-negative* if $C_n = 0$ for $n < 0$, *bounded below* if $C_n = 0$ for $n \ll 0$, *bounded above* if $C_n = 0$ for $n \gg 0$, and *bounded* if $C_n = 0$ for all but finitely many n . Each of these conditions is stable under the constructions of Remark 2.7, and the corresponding full subcategories of $\mathbf{Ch}(\mathbf{Mod}_R)$ are written $\mathbf{Ch}_{\geq 0}(\mathbf{Mod}_R)$, and so on. The complexes arising from spaces will be non-negative.

Remark 2.11. Taking $R = \mathbb{Z}$ gives the case of abelian groups: a $\mathbb{Z}$ -module is precisely an abelian group and a $\mathbb{Z}$ -linear map precisely a homomorphism, so that $\mathbf{Mod}_{\mathbb{Z}} = \mathbf{Ab}$, and we write

$$\mathbf{Ch}(\mathbf{Ab}) = \mathbf{Ch}(\mathbf{Mod}_{\mathbb{Z}})$$

for the category of chain complexes of abelian groups, and likewise $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ for the non-negative ones. This is the case in which the complexes coming from spaces will first be met. At the other extreme, for $R = k$ a field a complex of R -modules is a complex of vector spaces, and its homology is a family of vector spaces. A ring homomorphism $\varphi: R \rightarrow S$ makes every S -module an R -module and so gives a functor $\mathbf{Ch}(\mathbf{Mod}_S) \rightarrow \mathbf{Ch}(\mathbf{Mod}_R)$ commuting with H_n .

2.2. Direct limits. Let

$$C_*^0 \xrightarrow{g^0} C_*^1 \xrightarrow{g^1} C_*^2 \longrightarrow \dots$$

be a direct system of complexes of R -modules and chain maps, and write $g^{ij}: C_*^i \rightarrow C_*^j$ for the composite of the transition maps, $i \leq j$. Its *direct limit* is the complex with

$$\left(\varinjlim_i C^i \right)_n = \varinjlim_i C_n^i,$$

the direct limit of the system of modules in degree n , with the differential induced by the ∂^{C^i} ; these are compatible with the transition maps, the g^i being chain maps, so the induced differential exists, and $\partial\partial = 0$ holds because it holds at each stage. We use the two standard properties of a direct limit: every element of $\varinjlim_i C_n^i$ is $\iota_i(x)$ for some i and some $x \in C_n^i$, where ι_i is the canonical map; and $\iota_i(x) = 0$ if and only if $g^{ij}(x) = 0$ for some $j \geq i$.

Proposition 2.12. *Homology commutes with direct limits: for every n the canonical map*

$$\varinjlim_i H_n(C^i) \longrightarrow H_n\left(\varinjlim_i C^i\right)$$

is an isomorphism of R -modules.

Proof. The maps $H_n(\iota_i)$ are compatible with the transition maps, so the canonical map exists.

Surjectivity. Let $z \in Z_n(\varinjlim_i C^i)$. Write $z = \iota_i(x)$ with $x \in C_n^i$. Then $\iota_i(\partial x) = \partial z = 0$, so $g^{ij}(\partial x) = 0$ for some $j \geq i$, that is, $\partial(g^{ij}(x)) = 0$. So $y = g^{ij}(x)$ is a cycle in C_n^j with $\iota_j(y) = \iota_i(x) = z$, and $[z]$ is the image of $[y]$.

Injectivity. An element of the direct limit is the image of some $[y]$ with $y \in Z_n(C^i)$, and it maps to $[\iota_i(y)]$. Suppose this vanishes, say $\iota_i(y) = \partial w$ with $w \in (\varinjlim_i C^i)_{n+1}$. Write $w = \iota_j(u)$ with $u \in C_{n+1}^j$ and, replacing j by a larger index, assume $j \geq i$. Then

$$\iota_j(g^{ij}(y) - \partial u) = \iota_i(y) - \partial w = 0,$$

so $g^{jk}(g^{ij}(y) - \partial u) = 0$ for some $k \geq j$, that is, $g^{ik}(y) = \partial(g^{jk}(u))$. Hence $[g^{ik}(y)] = 0$ in $H_n(C^k)$, and the image of $[y]$ in $\varinjlim_i H_n(C^i)$ is zero. $\square$

Remark 2.13. The proof uses only the two properties displayed above, so it applies verbatim to direct limits over any directed index set. In particular, a direct sum $\bigoplus_{i \geq 0} C_*^i$ is the direct limit of its partial sums $C_*^0 \oplus \cdots \oplus C_*^N$, so

$$H_n\left(\bigoplus_i C_*^i\right) \cong \bigoplus_i H_n(C_*^i).$$

3. CHAIN HOMOTOPIES AND THE HOMOTOPY CATEGORY

Two chain maps can be different and yet induce the same map on homology. One way this happens is through a relation, called chain homotopy, modeled on homotopy of maps of spaces. As with spaces, one defines a category — the homotopy category — in which the homotopy relation has been divided out and in which homology is still defined.

Throughout this section, complexes are complexes of R -modules.

Definition 3.1. Let $f, g: C_* \rightarrow D_*$ be chain maps. A *chain homotopy* from f to g is a family of R -linear maps

$$s_n: C_n \longrightarrow D_{n+1} \quad (n \in \mathbb{Z})$$

— of degree +1, and not required to be a chain map — such that

$$f_n - g_n = \partial_{n+1}^D \circ s_n + s_{n-1} \circ \partial_n^C \quad \text{for all } n,$$

an identity abbreviated $f - g = \partial s + s \partial$. If such an s exists we say f and g are *chain homotopic* and write $f \simeq g$. A chain map homotopic to 0 is *null-homotopic*, and a complex C_* is *chain contractible* if $\text{id}_{C_*} \simeq 0$; a chain homotopy s from id to 0 is then called a *contracting homotopy*. The word *contractible*, unqualified, is reserved for spaces; and a chain contractible complex is acyclic, by Theorem 3.3 below, but not conversely: the complex $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$, in degrees 2, 1 and 0, is exact and so acyclic, but a contracting homotopy for it would split the short exact sequence $0 \rightarrow 2\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$, which by Example 1.6 is not split.

Proposition 3.2. *Chain homotopy is an equivalence relation on the set of chain maps $C_* \rightarrow D_*$. It is compatible with addition and with composition: if $f \simeq g: C_* \rightarrow D_*$ and $f' \simeq g': C_* \rightarrow D_*$ then $f + f' \simeq g + g'$; and if moreover $h: D_* \rightarrow E_*$ and $k: B_* \rightarrow C_*$ are chain maps then*

$$h \circ f \simeq h \circ g \quad \text{and} \quad f \circ k \simeq g \circ k.$$

Equivalently, the null-homotopic maps form a subgroup of $\text{Hom}_{\mathbf{Ch}(\mathbf{Mod}_R)}(C_, D_*)$, and $f \simeq g$ if and only if $f - g$ lies in it.*

Proof. The zero family is a homotopy from f to f ; if s is a homotopy from f to g then $-s$ is one from g to f ; and if s goes from f to g and s' from g to h then $s + s'$ goes from f to h , since

$$\partial(s + s') + (s + s')\partial = (f - g) + (g - h) = f - h.$$

The same computation with $f' - g'$ in place of $g - h$ gives compatibility with addition. For the composites, let s be a homotopy from f to g . Then $h \circ s$ is a homotopy from $h \circ f$ to $h \circ g$, because h commutes with the differentials:

$$\partial^E(hs) + (hs)\partial^C = h\partial^D s + hs\partial^C = h(\partial^D s + s\partial^C) = h(f - g).$$

Likewise $s \circ k$ is a homotopy from $f \circ k$ to $g \circ k$:

$$\partial^D(sk) + (sk)\partial^B = \partial^D sk + s\partial^C k = (\partial^D s + s\partial^C)k = (f - g)k,$$

using that k commutes with the differentials. The last sentence restates the first two, a homotopy from f to g being the same thing as a homotopy from $f - g$ to 0. $\square$

Theorem 3.3. *If $f \simeq g: C_* \rightarrow D_*$ then $H_n(f) = H_n(g)$ for every n . In particular a null-homotopic map induces the zero map on homology, and a chain contractible complex is acyclic: $H_n(C) = 0$ for all n .*

Proof. Let s be a chain homotopy from f to g and let $z \in Z_n(C)$. Then

$$f_n(z) - g_n(z) = \partial^D s_n(z) + s_{n-1}(\partial^C z) = \partial^D s_n(z),$$

because $\partial^C z = 0$. So $f_n(z)$ and $g_n(z)$ differ by an element of $B_n(D)$ and have the same class in $H_n(D)$; that is, $H_n(f)[z] = H_n(g)[z]$. Taking $g = 0$ gives the second statement, and taking $C_* = D_*$, $f = \text{id}$, $g = 0$ gives $\text{id} = 0$ on $H_n(C)$, whence $H_n(C) = 0$. $\square$

Definition 3.4. The *homotopy category of chain complexes* $\mathbf{K}(\mathbf{Mod}_R)$ has as objects the chain complexes of R -modules, and as morphisms $C_* \rightarrow D_*$ the chain homotopy classes of chain maps:

$$\text{Hom}_{\mathbf{K}(\mathbf{Mod}_R)}(C_*, D_*) = \text{Hom}_{\mathbf{Ch}(\mathbf{Mod}_R)}(C_*, D_*) / \{\text{null-homotopic maps}\}.$$

Composition is induced by composition of chain maps. We write $\mathbf{K}(\mathbf{Ab})$ for $\mathbf{K}(\mathbf{Mod}_{\mathbb{Z}})$.

Proposition 3.5. *$\mathbf{K}(\mathbf{Mod}_R)$ is an additive category, and the passage $f \mapsto [f]$ defines a functor*

$$\mathbf{Ch}(\mathbf{Mod}_R) \longrightarrow \mathbf{K}(\mathbf{Mod}_R)$$

which is the identity on objects. Moreover H_n factors through it: there is a unique functor $H_n: \mathbf{K}(\mathbf{Mod}_R) \rightarrow \mathbf{Mod}_R$ whose composite with $\mathbf{Ch}(\mathbf{Mod}_R) \rightarrow \mathbf{K}(\mathbf{Mod}_R)$ is the homology functor of Proposition 2.8.

Proof. By Proposition 3.2 the null-homotopic maps form a subgroup, so the quotient groups of the definition make sense; and the compatibility of homotopy with composition says exactly that $[h][f] = [h \circ f]$ is well defined, that is, independent of the representatives. Associativity and the identities are inherited from $\mathbf{Ch}(\mathbf{Mod}_R)$, so $\mathbf{K}(\mathbf{Mod}_R)$ is a category and $f \mapsto [f]$ is a functor. Composition remains biadditive on the quotients, and direct sums are unchanged, so $\mathbf{K}(\mathbf{Mod}_R)$ is additive. Finally $H_n(f)$ depends only on $[f]$, by Theorem 3.3, so $H_n[f] = H_n(f)$ is a well defined functor on $\mathbf{K}(\mathbf{Mod}_R)$, and it is the only one with the stated property, since every morphism of $\mathbf{K}(\mathbf{Mod}_R)$ is of the form $[f]$. $\square$

Definition 3.6. A chain map $f: C_* \rightarrow D_*$ is a *chain homotopy equivalence* if there is a chain map $g: D_* \rightarrow C_*$ with

$$g \circ f \simeq \text{id}_{C_*} \quad \text{and} \quad f \circ g \simeq \text{id}_{D_*};$$

such a g is a *homotopy inverse* of f . Equivalently, $[f]$ is an isomorphism in $\mathbf{K}(\mathbf{Mod}_R)$. A chain map f is a *quasi-isomorphism* if $H_n(f)$ is an isomorphism for every n .

Corollary 3.7. A chain homotopy equivalence is a quasi-isomorphism. In particular homotopy equivalent complexes have isomorphic homology.

Proof. By Definition 3.6 the class $[f]$ is an isomorphism in $\mathbf{K}(\mathbf{Mod}_R)$, on which H_n is defined by Proposition 3.5; and functors carry isomorphisms to isomorphisms. Concretely, $g \circ f \simeq \text{id}$ gives $H_n(g)H_n(f) = \text{id}$ and $f \circ g \simeq \text{id}$ gives $H_n(f)H_n(g) = \text{id}$, by Theorem 3.3. $\square$

Example 3.8. The converse of Corollary 3.7 fails. Fix $m \geq 2$, let C_* be the complex $\mathbb{Z} \xrightarrow{m} \mathbb{Z}$ of Example 2.4, in degrees 1 and 0, and let D_* be $\mathbb{Z}/m$ concentrated in degree 0. The quotient map in degree 0, together with 0 in all other degrees, is a chain map $f: C_* \rightarrow D_*$, and it is a quasi-isomorphism: $H_0(C) = \mathbb{Z}/m = H_0(D)$ and all other homology vanishes. But f is not a chain homotopy equivalence. For any chain map $g: D_* \rightarrow C_*$ has $g_0: \mathbb{Z}/m \rightarrow \mathbb{Z}$ equal to zero, there being no non-zero homomorphism from a finite group to $\mathbb{Z}$; so $f \circ g = 0$, and $H_0(f \circ g) = 0$ is not the identity of $\mathbb{Z}/m$, whereas $f \circ g \simeq \text{id}_{D_*}$ would force it to be.

Remark 3.9. The name *chain homotopy* is more than analogy. To a space X one associates, functorially, a chain complex, the singular chain complex; and in *Properties of Singular Homology* we shall show that homotopic maps $f, g: X \rightarrow Y$ induce chain homotopic maps $f_\#$ and $g_\#$. Granting this, Theorem 3.3 says that homotopic maps of spaces induce the same map on homology, and Corollary 3.7 that a homotopy equivalence of spaces induces isomorphisms on homology. Homology is thus a functor on the homotopy category of spaces, which is what makes it computable.

Exercise 3.10. Show that a contracting homotopy for the complex $\cdots \rightarrow 0 \rightarrow A \xrightarrow{\text{id}} A \rightarrow 0 \rightarrow \cdots$, with the two copies of A in degrees 1 and 0, is given by $s_0 = \text{id}_A$ and $s_n = 0$ otherwise. Deduce that this complex is chain contractible, and check directly that its homology vanishes.

Exercise 3.11. Show that if $f \simeq g$ then f is a quasi-isomorphism if and only if g is, and f is a chain homotopy equivalence if and only if g is. Give an example of a chain homotopy s which is not itself a chain map, and one which is.

Exercise 3.12. Let $0 \rightarrow C_* \rightarrow D_* \rightarrow E_* \rightarrow 0$ be a short exact sequence of complexes which is split in each degree. Show that a splitting need not make D_* isomorphic to $C_* \oplus E_*$ as a complex, and find the extra condition on the splittings under which it does.

4. THE LONG EXACT SEQUENCE OF HOMOLOGY

Definition 4.1. A sequence of chain complexes of R -modules and chain maps

$$0 \longrightarrow C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \longrightarrow 0$$

is a *short exact sequence of complexes* if for every n the sequence of R -modules

$$0 \longrightarrow C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \longrightarrow 0$$

is short exact. Exactness of a sequence of complexes is thus checked degreewise, as are the kernels and cokernels of Remark 2.7; the differentials play no part in the condition, though of course f and g are required to commute with them.

Homology does not carry short exact sequences of complexes to short exact sequences of modules. What it does carry them to is a long exact sequence, in which the discrepancy is measured by a connecting homomorphism lowering degree by one. This is one of the principal computational tools of the subject. Throughout the section

$$0 \longrightarrow C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \longrightarrow 0$$

is a short exact sequence of complexes as in Definition 4.1.

Lemma 4.2. *There is a well defined R -linear map*

$$\delta_n: H_n(E) \longrightarrow H_{n-1}(C)$$

given as follows. Let $e \in Z_n(E)$. Choose $d \in D_n$ with $g_n(d) = e$; then $\partial^D d$ lies in the image of f_{n-1} , say $\partial^D d = f_{n-1}(c)$ with $c \in C_{n-1}$ unique, and c is a cycle. Set $\delta_n[e] = [c]$.

Proof. Such a d exists because g_n is surjective. Then

$$g_{n-1}(\partial^D d) = \partial^E g_n(d) = \partial^E e = 0,$$

so $\partial^D d \in \ker g_{n-1} = \text{im } f_{n-1}$, and c with $f_{n-1}(c) = \partial^D d$ is unique because f_{n-1} is injective. It is a cycle: $f_{n-2}(\partial^C c) = \partial^D f_{n-1}(c) = \partial^D \partial^D d = 0$, and f_{n-2} is injective, so $\partial^C c = 0$.

The class $[c]$ does not depend on the choice of d . If $g_n(d') = e$ as well then $d - d' \in \ker g_n = \text{im } f_n$, say $d - d' = f_n(x)$ with $x \in C_n$; then

$$f_{n-1}(c - c') = \partial^D d - \partial^D d' = \partial^D f_n(x) = f_{n-1}(\partial^C x),$$

so $c - c' = \partial^C x$ by injectivity, and $[c] = [c']$.

Nor does it depend on the representative of the class of e . If $e' = e + \partial^E(y)$ with $y \in E_{n+1}$, choose $z \in D_{n+1}$ with $g_{n+1}(z) = y$; then $d + \partial^D z$ is a lift of e' , and $\partial^D(d + \partial^D z) = \partial^D d$, so the same c results.

Finally δ_n is R -linear: if d lifts e and d' lifts e' then $d + d'$ lifts $e + e'$ and rd lifts re , and the corresponding elements of C_{n-1} are $c + c'$ and rc , again by the injectivity of f_{n-1} . $\square$

Theorem 4.3. *For a short exact sequence of complexes $0 \rightarrow C_* \rightarrow D_* \rightarrow E_* \rightarrow 0$ the sequence*

$$\cdots \longrightarrow H_n(C) \xrightarrow{H_n(f)} H_n(D) \xrightarrow{H_n(g)} H_n(E) \xrightarrow{\delta_n} H_{n-1}(C) \xrightarrow{H_{n-1}(f)} H_{n-1}(D) \longrightarrow \cdots$$

is exact.

Proof. There are three places to check exactness.

Exactness at $H_n(D)$. Since $g \circ f = 0$ we have $H_n(g)H_n(f) = 0$, so $\text{im } H_n(f) \subseteq \ker H_n(g)$. Conversely let $d \in Z_n(D)$ with $H_n(g)[d] = 0$, say $g_n(d) = \partial^E(y)$ with $y \in E_{n+1}$, and choose $z \in D_{n+1}$ with $g_{n+1}(z) = y$. Then

$$g_n(d - \partial^D z) = g_n(d) - \partial^E g_{n+1}(z) = 0,$$

so $d - \partial^D z = f_n(c)$ for a unique $c \in C_n$. It is a cycle, because $f_{n-1}(\partial^C c) = \partial^D(d - \partial^D z) = 0$ and f_{n-1} is injective; and $H_n(f)[c] = [d - \partial^D z] = [d]$.

Exactness at $H_n(E)$. If $e = g_n(d)$ with $d \in Z_n(D)$, then d may be used as the lift in Lemma 4.2, and $\partial^D d = 0 = f_{n-1}(0)$ gives $\delta_n[e] = 0$. Hence $\text{im } H_n(g) \subseteq \ker \delta_n$. Conversely suppose $\delta_n[e] = 0$, and let d and c be as in the construction, so that $c = \partial^C(x)$ for some $x \in C_n$. Then $d - f_n(x)$ satisfies

$$g_n(d - f_n(x)) = e, \quad \partial^D(d - f_n(x)) = \partial^D d - f_{n-1}(\partial^C x) = f_{n-1}(c) - f_{n-1}(c) = 0,$$

so $d - f_n(x)$ is a cycle and $H_n(g)[d - f_n(x)] = [e]$.

Exactness at $H_{n-1}(C)$. In the notation of the construction above, $H_{n-1}(f)\delta_n[e] = [f_{n-1}(c)] = [\partial^D d] = 0$, so $\text{im } \delta_n \subseteq \ker H_{n-1}(f)$. Conversely let $c \in Z_{n-1}(C)$ with $H_{n-1}(f)[c] = 0$, say $f_{n-1}(c) = \partial^D(d)$ with $d \in D_n$, and put $e = g_n(d)$. Then

$$\partial^E e = g_{n-1}(\partial^D d) = g_{n-1}f_{n-1}(c) = 0,$$

so $e \in Z_n(E)$; and d is a lift of e whose boundary is $f_{n-1}(c)$, so by construction $\delta_n[e] = [c]$. $\square$

Proposition 4.4 (naturality). *Let*

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_* & \xrightarrow{f} & D_* & \xrightarrow{g} & E_* \longrightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & C'_* & \xrightarrow{f'} & D'_* & \xrightarrow{g'} & E'_* \longrightarrow 0 \end{array}$$

be a commutative diagram of complexes and chain maps whose rows are short exact. Then the diagram of long exact sequences commutes; that is, in addition to $H_n(f')H_n(\alpha) = H_n(\beta)H_n(f)$ and $H_n(g')H_n(\beta) = H_n(\gamma)H_n(g)$, which hold by functoriality, one has

$$\delta'_n \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta_n.$$

Proof. Let $e \in Z_n(E)$ and choose $d \in D_n$ with $g_n(d) = e$ and $c \in C_{n-1}$ with $f_{n-1}(c) = \partial^D d$, so that $\delta_n[e] = [c]$. Then $g'_n(\beta_n(d)) = \gamma_n g_n(d) = \gamma_n(e)$, so $\beta_n(d)$ is an admissible lift of the cycle $\gamma_n(e)$; and

$$\partial^{D'} \beta_n(d) = \beta_{n-1}(\partial^D d) = \beta_{n-1}f_{n-1}(c) = f'_{n-1}(\alpha_{n-1}(c)).$$

Hence $\delta'_n[\gamma_n(e)] = [\alpha_{n-1}(c)] = H_{n-1}(\alpha)\delta_n[e]$. $\square$

Corollary 4.5. *If E_* is acyclic then f is a quasi-isomorphism; if C_* is acyclic then g is. More generally, if two of the three complexes are acyclic then so is the third.*

Proof. If $H_n(E) = 0$ for all n then in the long exact sequence $H_n(f)$ has $H_{n+1}(E) = 0$ to its left, along δ_{n+1} , and $H_n(E) = 0$ to its right; exactness at $H_n(C)$ makes $H_n(f)$ injective and exactness at $H_n(D)$ makes it surjective. The second statement is the same argument, and the third follows since a portion of an exact sequence with zero groups on both sides of a term forces that term to vanish. $\square$

Exercise 4.6 (the five lemma). Let

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & \varphi_4 \downarrow & & \varphi_5 \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram of R -modules with exact rows. Show that if φ_2 and φ_4 are isomorphisms, φ_1 is surjective and φ_5 is injective, then φ_3 is an isomorphism. (Two chases. For injectivity one needs only that φ_1 is surjective and that φ_2 and φ_4 are injective; for surjectivity, that φ_5 is injective and that φ_2 and φ_4 are surjective.) Deduce, using Proposition 4.4, that in a map of short exact sequences of complexes as there, if two of α, β, γ are quasi-isomorphisms then so is the third.

Exercise 4.7. Let $0 \rightarrow C_* \rightarrow D_* \rightarrow E_* \rightarrow 0$ be short exact with D_* acyclic. Show that $\delta_n: H_n(E) \rightarrow H_{n-1}(C)$ is an isomorphism for every n .

Exercise 4.8. Apply Theorem 4.3 to the short exact sequence of complexes concentrated in degree 0 determined by a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of R -modules, and check that the long exact sequence reduces to that sequence.

Exercise 4.9. Show that the connecting homomorphism cannot be dispensed with: give a short exact sequence of complexes for which $H_n(g)$ fails to be surjective for some n , so that $0 \rightarrow H_n(C) \rightarrow H_n(D) \rightarrow H_n(E) \rightarrow 0$ is not exact.

5. THE MAPPING CONE

Definition 5.1. For a complex C_* the *shift* $C_*[1]$ is the complex with $(C[1])_n = C_{n-1}$ and differential $-\partial^C$.

Definition 5.2. Let $f: C_* \rightarrow D_*$ be a chain map. Its *mapping cone* is the complex $\text{cone}(f)$ with

$$\text{cone}(f)_n = D_n \oplus C_{n-1}, \quad \partial(d, c) = (\partial^D d + f_{n-1}(c), -\partial^C c).$$

This is a complex: since f commutes with the differentials,

$$\partial \partial(d, c) = \partial(\partial^D d + f(c), -\partial^C c) = (\partial^D f(c) - f(\partial^C c), \partial^C \partial^C c) = 0.$$

Proposition 5.3. For a chain map $f: C_* \rightarrow D_*$ the maps $d \mapsto (d, 0)$ and $(d, c) \mapsto c$ form a short exact sequence of complexes

$$0 \longrightarrow D_* \longrightarrow \text{cone}(f) \longrightarrow C_*[1] \longrightarrow 0,$$

whose connecting homomorphism $H_n(C[1]) = H_{n-1}(C) \rightarrow H_{n-1}(D)$ is $H_{n-1}(f)$. Consequently f is a quasi-isomorphism if and only if $\text{cone}(f)$ is acyclic.

Proof. Both maps are chain maps by the definitions of the two differentials, and in each degree the sequence is $0 \rightarrow D_n \rightarrow D_n \oplus C_{n-1} \rightarrow C_{n-1} \rightarrow 0$, which is exact. For the connecting homomorphism, let $c \in C_{n-1}$ be a cycle of $C[1]$ in degree n , that is, $\partial^C c = 0$. It lifts to $(0, c) \in \text{cone}(f)_n$, and

$$\partial(0, c) = (f_{n-1}(c), 0),$$

which is the image of $f_{n-1}(c) \in D_{n-1}$; so $\delta_n[c] = [f_{n-1}(c)] = H_{n-1}(f)[c]$.

Now apply Theorem 4.3. If $\text{cone}(f)$ is acyclic then exactness of

$$0 = H_n(\text{cone}(f)) \longrightarrow H_{n-1}(C) \xrightarrow{H_{n-1}(f)} H_{n-1}(D) \longrightarrow H_{n-1}(\text{cone}(f)) = 0$$

makes every $H_{n-1}(f)$ an isomorphism. Conversely suppose every $H_n(f)$ is an isomorphism. Then $H_n(\text{cone}(f)) \rightarrow H_{n-1}(C)$ has image $\ker H_{n-1}(f) = 0$, so $H_n(D) \rightarrow H_n(\text{cone}(f))$ is surjective; and its kernel is the image of the surjection $H_n(f)$, so it is also zero. Hence $H_n(\text{cone}(f)) = 0$. $\square$

N.B. If C is a cochain complex, then $C[1]^n = C^{n+1}$, with differential $d[1] = -d$. This is Definition 5.1 read through the relabelling $C^n = C_{-n}$ of Remark 2.9.

6. TENSOR PRODUCTS OF CHAIN COMPLEXES

In this section R is a commutative ring and all complexes are complexes of R -modules; a reader interested only in abelian groups may take $R = \mathbb{Z}$ throughout. The construction is needed whenever two spaces, or two chain-level constructions, are to be combined, and it is the algebra underlying the cup product in cohomology.

Definition 6.1. Let C_* and D_* be chain complexes. Their *tensor product* $C_* \otimes D_*$ is the graded R -module

$$(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q$$

with the differential determined on generators by

$$(6.1) \quad \partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y \quad (x \in C_p, y \in D_q).$$

Lemma 6.2. The map (6.1) satisfies $\partial\partial = 0$, so $C_* \otimes D_*$ is a chain complex.

Proof. For $x \in C_p$ and $y \in D_q$,

$$\partial\partial(x \otimes y) = \partial\partial x \otimes y + (-1)^{p-1}\partial x \otimes \partial y + (-1)^p\partial x \otimes \partial y + (-1)^p(-1)^p x \otimes \partial\partial y,$$

and the two middle terms cancel while the outer ones vanish. $\square$

Remark 6.3. The sign is not a convention that could be dropped: without it the two middle terms above would add rather than cancel, and one would get $\partial\partial(x \otimes y) = 2\partial x \otimes \partial y$, which is non-zero in general. The rule behind it is the Koszul sign rule: moving a symbol of degree -1 , here ∂ , past a symbol of degree p introduces the sign $(-1)^{(-1)p} = (-1)^p$.

Proposition 6.4. If $f: C_* \rightarrow C'_*$ and $g: D_* \rightarrow D'_*$ are chain maps, then $f \otimes g: C_* \otimes D_* \rightarrow C'_* \otimes D'_*$, defined by $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$, is a chain map, and $(f' \otimes g')(f \otimes g) = (f'f) \otimes (g'g)$, $\text{id} \otimes \text{id} = \text{id}$. Thus $\otimes$ is a functor $\mathbf{Ch}(\mathbf{Mod}_R) \times \mathbf{Ch}(\mathbf{Mod}_R) \rightarrow \mathbf{Ch}(\mathbf{Mod}_R)$.

Proof. For $x \in C_p$,

$$\begin{aligned} \partial(f \otimes g)(x \otimes y) &= \partial f(x) \otimes g(y) + (-1)^p f(x) \otimes \partial g(y) \\ &= f(\partial x) \otimes g(y) + (-1)^p f(x) \otimes g(\partial y) = (f \otimes g)\partial(x \otimes y), \end{aligned}$$

using that f and g commute with ∂ and preserve degree. The other two identities are immediate on generators. $\square$

Lemma 6.5. The map $\tau: C_* \otimes D_* \rightarrow D_* \otimes C_*$ determined by $\tau(x \otimes y) = (-1)^{pq} y \otimes x$ for $x \in C_p, y \in D_q$, is an isomorphism of chain complexes, and $\tau\tau = \text{id}$.

Proof. It is bijective with inverse of the same form, so only $\partial\tau = \tau\partial$ needs checking. On one side

$$\partial\tau(x \otimes y) = (-1)^{pq}(\partial y \otimes x + (-1)^q y \otimes \partial x),$$

and on the other

$$\tau\partial(x \otimes y) = \tau(\partial x \otimes y) + (-1)^p \tau(x \otimes \partial y) = (-1)^{(p-1)q} y \otimes \partial x + (-1)^{p+p(q-1)} \partial y \otimes x.$$

The coefficients agree: $(-1)^{(p-1)q} = (-1)^{pq+q}$ and $(-1)^{p+p(q-1)} = (-1)^{pq}$. $\square$

Proposition 6.6. *Let P be a chain homotopy from f to f' , so that $\partial P + P\partial = f - f'$, and let $g: D_* \rightarrow D'_*$ be a chain map. Then $P \otimes g$, defined by $(P \otimes g)(x \otimes y) = P(x) \otimes g(y)$, is a chain homotopy from $f \otimes g$ to $f' \otimes g$. The same holds in the second variable, by Lemma 6.5. In particular chain homotopic chain maps have chain homotopic tensor products, and $\otimes$ descends to the homotopy category $\mathbf{K}(\mathbf{Mod}_R)$ of Definition 3.4.*

Proof. For $x \in C_p$ and $y \in D_q$ one has $P(x) \in C_{p+1}$, so

$$\begin{aligned} \partial(P \otimes g)(x \otimes y) &= \partial P(x) \otimes g(y) + (-1)^{p+1} P(x) \otimes g(\partial y), \\ (P \otimes g)\partial(x \otimes y) &= P(\partial x) \otimes g(y) + (-1)^p P(x) \otimes g(\partial y). \end{aligned}$$

Adding, the two terms involving ∂y cancel and what is left is $(\partial P + P\partial)(x) \otimes g(y) = ((f - f') \otimes g)(x \otimes y)$. $\square$

Remark 6.7. The signs interact with dualization as follows. Let $C^p = \text{Hom}_R(C_p, R)$ and $D^q = \text{Hom}_R(D_q, R)$ be the dual complexes, and give the tensor product of the duals the differential $\delta(\alpha \otimes \beta) = \delta\alpha \otimes \beta + (-1)^{|\alpha|} \alpha \otimes \delta\beta$ analogous to (6.1). If the duals carry the graded differential $(\delta\alpha)(x) = (-1)^{|\alpha|+1} \alpha(\partial x)$, then the pairing compatible with these differentials is

$$(\alpha \otimes \beta)(x \otimes y) = (-1)^{|\beta||x|} \alpha(x) \beta(y),$$

the sign being the one produced by moving β past x . If instead one dualizes without a sign, $(\delta\alpha)(x) = \alpha(\partial x)$, then compatibility forces the unsigned pairing $(\alpha \otimes \beta)(x \otimes y) = \alpha(x) \beta(y)$. The two prescriptions correspond under the rescaling $\alpha \mapsto \varepsilon_p \alpha$ of C^p , where $\varepsilon_p = (-1)^{p(p+1)/2}$, because $\varepsilon_p \varepsilon_q (-1)^{pq} = \varepsilon_{p+q}$. Our convention is that the dual is the unsigned one.

Remark 6.8. The evident maps give natural isomorphisms $(C_* \otimes D_*) \otimes E_* \cong C_* \otimes (D_* \otimes E_*)$ and $R \otimes C_* \cong C_*$, where R is regarded as a complex concentrated in degree 0; so up to natural isomorphism $\otimes$ is associative with unit R . More generally, for an R -module M viewed as a complex concentrated in degree 0, the complex $C_* \otimes M$ has $(C_* \otimes M)_n = C_n \otimes_R M$ and differential $\partial \otimes \text{id}$; no signs intervene.

Remark 6.9. This last case is how coefficients are changed: the complex C_* with coefficients in M is $C_* \otimes M$, and its homology is written $H_n(C; M)$. The relation between $H_*(C \otimes D)$ and $H_*(C) \otimes H_*(D)$, and in particular between $H_n(C; M)$ and $H_n(C) \otimes M$, is the content of the Künneth and universal coefficient theorems of §7 and §8; the two are not equal in general, because $\otimes$ is not exact.

Exercise 6.10. Show that if C_* and D_* are non-negative complexes then so is $C_* \otimes D_*$, and that $(C_* \otimes D_*)_0 = C_0 \otimes_R D_0$. Show also that the tensor product of two bounded complexes is bounded.

Exercise 6.11. Verify the computation of Remark 6.3: with the unsigned differential $\partial(x \otimes y) = \partial x \otimes y + x \otimes \partial y$ one has $\partial\partial(x \otimes y) = 2\partial x \otimes \partial y$. Give complexes C_* and D_* of abelian groups for which this is non-zero.

Exercise 6.12. Let I_* be the complex $\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}$ concentrated in degrees 1 and 0. Show that $C_* \otimes I_*$ is isomorphic to the mapping cone of id_{C_*} of Definition 5.2, keeping track of the signs, and show that a chain homotopy between chain maps $f, g: C_* \rightarrow D_*$ is the same thing as a chain map $C_* \otimes I_* \rightarrow D_*$ restricting to f and g on the two copies of C_* .

7. TOR AND THE ALGEBRAIC KÜNNETH THEOREM

Our goal is to understand the homology of a product. Two ingredients are needed. The first is algebraic and is the subject of this section: the homology of a tensor product of chain complexes, in terms of the homologies of the factors. The second is geometric — the singular chain complex of $X \times Y$ is chain homotopy equivalent to $C_*(X) \otimes C_*(Y)$, which is the Eilenberg–Zilber theorem — and is the subject of the last section of *Further Properties of Singular Homology*, where the two are combined. Tensor products of complexes are those of §6.

Throughout this section R is a principal ideal domain, and all tensor products and Tor groups are taken over R ; the reader may take $R = \mathbb{Z}$ and read “abelian group” for “ R -module”. What is used about R is that a submodule of a free module is free.

7.1. The functor Tor.

Definition 7.1. Let A be an R -module. A *free resolution* of A of length one is a short exact sequence $0 \rightarrow F_2 \rightarrow F_1 \rightarrow A \rightarrow 0$ with F_1 and F_2 free; such a sequence exists, since one may take F_1 free on a set of generators of A and F_2 the kernel, which is free because R is a principal ideal domain. For an R -module M put

$$\text{Tor}(A, M) = \ker(F_2 \otimes M \rightarrow F_1 \otimes M).$$

Lemma 7.2. $\text{Tor}(A, M)$ is independent of the chosen resolution, up to a canonical isomorphism, and is a functor of A and of M .

Proof. Let $0 \rightarrow F_2 \rightarrow F_1 \rightarrow A \rightarrow 0$ and $0 \rightarrow F'_2 \rightarrow F'_1 \rightarrow A \rightarrow 0$ be two such resolutions. Since F_1 is free, the map $F_1 \rightarrow A$ lifts to $u_1: F_1 \rightarrow F'_1$, and u_1 carries F_2 into F'_2 ; write u_2 for the restriction. Interchanging the roles gives v_1, v_2 in the other direction, and Tor inherits maps u_*, v_* between the two candidate groups.

It remains to see that v_*u_* is the identity, and for that it is enough to show that two lifts of the same map induce the same homomorphism. So let $u = (u_2, u_1)$ and $w = (w_2, w_1)$ both lift id_A . Then $u_1 - w_1$ maps F_1 into $\ker(F'_1 \rightarrow A) = F'_2$; call this map s . Since the inclusion $F'_2 \subseteq F'_1$ is injective and the squares commute, $u_2 - w_2 = s|_{F_2}$. Now if $\xi \in \ker(F_2 \otimes M \rightarrow F_1 \otimes M)$ then

$$(u_2 - w_2) \otimes \text{id}(\xi) = (s \otimes \text{id})((\text{incl} \otimes \text{id})(\xi)) = 0,$$

so $u_* = w_*$ on Tor. Functoriality in M is clear from the definition, and functoriality in A follows by lifting maps to resolutions, as above. $\square$

Proposition 7.3. Let A and M be R -modules.

(i) For every free resolution $0 \rightarrow F_2 \rightarrow F_1 \rightarrow A \rightarrow 0$ the sequence

$$0 \longrightarrow \operatorname{Tor}(A, M) \longrightarrow F_2 \otimes M \longrightarrow F_1 \otimes M \longrightarrow A \otimes M \longrightarrow 0$$

is exact.

- (ii) $\operatorname{Tor}(A, M) = 0$ if A is free; in particular if A is finitely generated and torsion free.
- (iii) $\operatorname{Tor}(\bigoplus_i A_i, M) = \bigoplus_i \operatorname{Tor}(A_i, M)$.
- (iv) For $R = \mathbb{Z}$ and $A = \mathbb{Z}/m$ one has $\operatorname{Tor}(\mathbb{Z}/m, M) = \{x \in M : mx = 0\}$.
- (v) $\operatorname{Tor}(A, M) = 0$ if M is free.

Proof. (i) Exactness at $F_1 \otimes M$ and at $A \otimes M$ is the right exactness of $\otimes$, and exactness at $F_2 \otimes M$ is the definition of Tor .

(ii) If A is free take $F_2 = 0$, and a finitely generated torsion-free module over a principal ideal domain is free.

(iii) A direct sum of resolutions resolves the direct sum, and $\otimes$ and $\ker$ commute with direct sums.

(iv) Use the resolution $0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$; tensoring with M gives $M \xrightarrow{m} M$, whose kernel is the m -torsion of M .

(v) If M is free then $F_2 \otimes M \rightarrow F_1 \otimes M$ is a direct sum of copies of the injection $F_2 \rightarrow F_1$, hence injective. $\square$

Remark 7.4. In fact $\operatorname{Tor}(A, M) = 0$ for every torsion-free A , since such an A is the directed union of its finitely generated submodules, which are free by (ii), and Tor commutes with direct limits. We shall not need this.

Exercise 7.5. Show that $\operatorname{Tor}(A, M) \cong \operatorname{Tor}(M, A)$ for finitely generated abelian groups, using the structure theorem and Proposition 7.3.

7.2. The algebraic Künneth theorem.

Theorem 7.6. Let C_* be a chain complex of free R -modules and let D_* be any chain complex of R -modules. For every n there is a short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \longrightarrow H_n(C \otimes D) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(C), H_q(D)) \longrightarrow 0,$$

natural in C_* and D_* , in which the first map sends $[x] \otimes [y]$ to $[x \otimes y]$. If moreover D_* admits a chain map to $H_*(D)$ inducing the identity on homology — for instance if D_* is free, or is concentrated in a single degree — then the sequence splits, although not naturally.

Proof. Write $Z_p \subseteq C_p$ for the cycles and $B_p \subseteq C_p$ for the boundaries; both are free, being submodules of free modules. Regard Z_* as a complex with zero differential, and let B'_* be the complex with $B'_p = B_{p-1}$ and zero differential. Then

$$0 \longrightarrow Z_* \longrightarrow C_* \xrightarrow{\partial} B'_* \longrightarrow 0$$

is a short exact sequence of complexes, and it is split in each degree because B'_p is free. Hence it stays exact after $- \otimes D_*$, and the long exact sequence of Theorem 4.3 gives

$$\cdots \longrightarrow H_n(Z_* \otimes D) \longrightarrow H_n(C \otimes D) \longrightarrow H_n(B'_* \otimes D) \xrightarrow{\lambda_n} H_{n-1}(Z_* \otimes D) \longrightarrow \cdots$$

Here, and only in this proof, the connecting homomorphism of Lemma 4.2 is written λ_n rather than δ_n , to match the notation used for it in the topological Künneth theorem of *Further Properties of Singular Homology*, where δ^i denotes the face maps. Since Z_p is free, tensoring with Z_p sends exact sequences to exact sequences. Also, Z_* has zero differential. Thus $H_n(Z_* \otimes D) = \bigoplus_{p+q=n} Z_p \otimes H_q(D)$, and likewise $H_n(B'_* \otimes D) = \bigoplus_{p+q=n-1} B_p \otimes H_q(D)$. Unwinding the definition of the connecting homomorphism in Lemma 4.2, λ_n is, up to a sign that will not matter, induced by the inclusions $B_p \subseteq Z_p$.

A long exact sequence breaks into short exact sequences $0 \rightarrow \text{coker } \lambda_{n+1} \rightarrow H_n(C \otimes D) \rightarrow \ker \lambda_n \rightarrow 0$. Now $0 \rightarrow B_p \rightarrow Z_p \rightarrow H_p(C) \rightarrow 0$ is a free resolution of $H_p(C)$, so by right exactness of $\otimes$

$$\text{coker}(B_p \otimes H_q(D) \rightarrow Z_p \otimes H_q(D)) = H_p(C) \otimes H_q(D),$$

while by Definition 7.1 and Lemma 7.2

$$\ker(B_p \otimes H_q(D) \rightarrow Z_p \otimes H_q(D)) = \text{Tor}(H_p(C), H_q(D)).$$

Summing over $p + q = n$ and $p + q = n - 1$ respectively gives the asserted sequence; the description of the first map and its naturality are read off from the construction.

For the splitting, choose in each degree a splitting $C_n \cong Z_n \oplus B_{n-1}$ of $\partial: C_n \rightarrow B_{n-1}$, and let $\pi: C_* \rightarrow H_*(C)$ be the projection to Z_n followed by the quotient map, where $H_*(C)$ is regarded as a complex with zero differential. Then π is a chain map inducing the identity on homology, since a boundary already lies in Z_n and dies in $H_n(C)$. Let $\pi': D_* \rightarrow H_*(D)$ be a chain map inducing the identity, as provided by the additional hypothesis: the same construction serves when D_* is free, and the identity serves when D_* is concentrated in a single degree. The composite

$$H_n(C \otimes D) \xrightarrow{(\pi \otimes \pi')^*} H_n(H_*(C) \otimes H_*(D)) = \bigoplus_{p+q=n} H_p(C) \otimes H_q(D)$$

restricts to the identity on the submodule $\bigoplus_{p+q=n} H_p(C) \otimes H_q(D)$, so it is a retraction and the sequence splits. The choices made are not natural, and neither is the splitting. $\square$

8. UNIVERSAL COEFFICIENTS

Fix a principal ideal domain R . Changing the coefficients of a free chain complex C_* of R -modules means forming either $C_* \otimes M$ or $\text{Hom}(C_*, M)$ for an R -module M , and in each case one wants the answer in terms of $H_*(C)$ alone. The two answers are the universal coefficient theorems of this section. They are not isomorphisms: a second term intervenes, built from Tor in the first case and from its dual Ext in the second.

8.1. Universal coefficients in homology.

Theorem 8.1. *Let C_* be a chain complex of free R -modules and M an R -module. For every n there is a short exact sequence*

$$0 \longrightarrow H_n(C) \otimes M \longrightarrow H_n(C \otimes M) \longrightarrow \text{Tor}(H_{n-1}(C), M) \longrightarrow 0,$$

natural in C_ and M , and it splits, although not naturally.*

Proof. Regard M as a chain complex concentrated in degree 0 and apply Theorem 7.6, whose exact sequence collapses to the one asserted because $H_q(M) = 0$ for $q \neq 0$. For the splitting, the chain map $C_* \rightarrow H_*(C)$ constructed there is available, and the complex M maps to its own homology by the identity. $\square$

8.2. The functor Ext .

Definition 8.2. Let A be an R -module and $0 \rightarrow F_2 \rightarrow F_1 \rightarrow A \rightarrow 0$ a free resolution as in Definition 7.1. For an R -module M put

$$\text{Ext}(A, M) = \text{coker}(\text{Hom}(F_1, M) \rightarrow \text{Hom}(F_2, M)).$$

Lemma 8.3. $\text{Ext}(A, M)$ is independent of the chosen resolution, up to a canonical isomorphism, and is a functor of A and of M .

Proof. As in Lemma 7.2 it suffices to show that two lifts $u = (u_2, u_1)$ and $w = (w_2, w_1)$ of id_A to a map of resolutions induce the same homomorphism. Again $s = u_1 - w_1$ maps F_1 into F'_2 and $u_2 - w_2 = s|_{F_2}$. Hence for $\varphi \in \text{Hom}(F'_2, M)$,

$$(u_2 - w_2)^* \varphi = \varphi \circ s|_{F_2} = (\varphi \circ s)|_{F_2},$$

which lies in the image of $\text{Hom}(F_1, M) \rightarrow \text{Hom}(F_2, M)$ because $\varphi \circ s \in \text{Hom}(F_1, M)$. So u^* and w^* agree on the cokernel. $\square$

Proposition 8.4. Let A and M be R -modules.

(i) For every free resolution $0 \rightarrow F_2 \rightarrow F_1 \rightarrow A \rightarrow 0$ the sequence

$$0 \rightarrow \text{Hom}(A, M) \rightarrow \text{Hom}(F_1, M) \rightarrow \text{Hom}(F_2, M) \rightarrow \text{Ext}(A, M) \rightarrow 0$$

is exact.

- (ii) $\text{Ext}(A, M) = 0$ if A is free; in particular if A is finitely generated and torsion free.
- (iii) $\text{Ext}(\bigoplus_i A_i, M) = \prod_i \text{Ext}(A_i, M)$.
- (iv) For $R = \mathbb{Z}$ and $A = \mathbb{Z}/m$ one has $\text{Ext}(\mathbb{Z}/m, M) = M/mM$; in particular $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m$ and $\text{Ext}(\mathbb{Z}/m, \mathbb{Q}) = 0$.

Proof. (i) The functor $\text{Hom}(-, M)$ is left exact, which gives exactness at $\text{Hom}(A, M)$ and at $\text{Hom}(F_1, M)$; exactness at $\text{Ext}(A, M)$ is the definition.

(ii) Take $F_2 = 0$; and a finitely generated torsion-free module over a principal ideal domain is free.

(iii) A direct sum of resolutions resolves the direct sum, and Hom carries direct sums in the first variable to direct products.

(iv) From $0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$ one gets $M \xrightarrow{m} M$, whose cokernel is M/mM . A divisible group such as $\mathbb{Q}$ satisfies $M = mM$. $\square$

Remark 8.5. Tor and Ext are built from the same resolution, one by taking the kernel after $-\otimes M$ and the other by taking the cokernel after $\text{Hom}(-, M)$. The two are genuinely different functors: $\text{Tor}(\mathbb{Z}/m, \mathbb{Z}) = 0$ while $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m$.

8.3. Universal coefficients in cohomology. For a chain complex C_* of R -modules and an R -module M , let $\text{Hom}(C_*, M)$ be the cochain complex with $\text{Hom}(C_n, M)$ in degree n and coboundary $\delta = \text{Hom}(\partial, M)$, as in *Singular Cohomology*; write $H^n(C; M)$ for its cohomology.

Theorem 8.6. *Let C_* be a chain complex of free R -modules and M an R -module. For every n there is a short exact sequence*

$$0 \longrightarrow \text{Ext}(H_{n-1}(C), M) \longrightarrow H^n(C; M) \longrightarrow \text{Hom}(H_n(C), M) \longrightarrow 0,$$

natural in C_ and M , in which the second map sends the class of a cocycle φ to the homomorphism $[z] \mapsto \varphi(z)$ on cycles. The sequence splits, although not naturally.*

Proof. Keep the notation of the proof of Theorem 7.6: Z_p and B_p are the cycles and boundaries, both free, and $0 \rightarrow Z_* \rightarrow C_* \xrightarrow{\partial} B'_* \rightarrow 0$ is split in each degree. Applying $\text{Hom}(-, M)$ to a degreewise split short exact sequence leaves it exact, so

$$0 \longrightarrow \text{Hom}(B'_*, M) \longrightarrow \text{Hom}(C_*, M) \longrightarrow \text{Hom}(Z_*, M) \longrightarrow 0$$

is a short exact sequence of cochain complexes. Since Z_* and B'_* have zero differentials, the cohomology of the two outer complexes is $\text{Hom}(Z_n, M)$ and $\text{Hom}(B_{n-1}, M)$ in degree n , and the long exact sequence reads

$$\begin{aligned} \cdots \longrightarrow \text{Hom}(B_{n-2}, M) \longrightarrow H^{n-1}(C; M) \longrightarrow \text{Hom}(Z_{n-1}, M) \\ \xrightarrow{\lambda_{n-1}} \text{Hom}(B_{n-1}, M) \longrightarrow H^n(C; M) \longrightarrow \cdots, \end{aligned}$$

where, here and only in this proof, the connecting homomorphism $\lambda_m: \text{Hom}(Z_m, M) \rightarrow \text{Hom}(B_m, M)$ is written λ rather than δ , to avoid confusion with the coboundary $\delta = \text{Hom}(\partial, M)$; up to a sign that will not matter it is restriction along $B_m \subseteq Z_m$. Breaking the sequence into short exact sequences gives

$$0 \longrightarrow \text{coker } \lambda_{n-1} \longrightarrow H^n(C; M) \longrightarrow \ker \lambda_n \longrightarrow 0.$$

The sequence $0 \rightarrow B_{n-1} \rightarrow Z_{n-1} \rightarrow H_{n-1}(C) \rightarrow 0$ is a free resolution, so $\text{coker } \lambda_{n-1} = \text{Ext}(H_{n-1}(C), M)$ by Definition 8.2, while $\ker \lambda_n$ is the group of homomorphisms $Z_n \rightarrow M$ vanishing on B_n , that is, $\text{Hom}(H_n(C), M)$. This is the asserted sequence, and the description of the second map is immediate from the construction.

For the splitting, choose in each degree a splitting $C_n \cong Z_n \oplus B_{n-1}$ of $\partial: C_n \rightarrow B_{n-1}$ and let $p: C_n \rightarrow Z_n$ be the resulting projection. Given $\varphi \in \text{Hom}(H_n(C), M)$, let ψ be the composite $C_n \xrightarrow{p} Z_n \rightarrow H_n(C) \xrightarrow{\varphi} M$. For $c \in C_{n+1}$ we have $\partial c \in B_n \subseteq Z_n$, so $p(\partial c) = \partial c$ and its class in $H_n(C)$ is zero; hence $\delta\psi = \psi \circ \partial = 0$ and ψ is a cocycle. Its image under the second map is φ , so $\varphi \mapsto [\psi]$ is the required splitting. $\square$

Exercise 8.7. Show that both $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}/n)$ and $\text{Tor}(\mathbb{Z}/m, \mathbb{Z}/n)$ are cyclic of order d , where $d = \gcd(m, n)$.

9. ORIENTED SIMPLICIAL HOMOLOGY

For a simplicial complex K , the singular complex of the polyhedron $|K|$, constructed in the next notes, *Singular Homology*, is enormous: it has one generator for every continuous map $\Delta^n \rightarrow |K|$. A finite complex, however, ought to have a homology computable from its finitely many simplices, and this section introduces the small model that does it. We

use the notation of the companion notes *Simplicial complexes*: K is an abstract simplicial complex with vertex set $V(K)$, its simplices are certain finite non-empty subsets of $V(K)$, and $|K|$ is its geometric realization, in which a vertex v is the point with barycentric coordinate 1 at v .

9.1. Oriented simplices.

Definition 9.1. Let σ be an n -simplex of K , with vertex set V . An *ordering* of σ is a bijection $\{0, 1, \dots, n\} \rightarrow V$, written $(v_0, \dots, v_n)$; two orderings are *equivalent* if they differ by an even permutation, and an *orientation* of σ is an equivalence class. For $n \geq 1$ there are exactly two orientations, and for $n = 0$ exactly one. We write $[v_0, \dots, v_n]$ for the orientation of the class of $(v_0, \dots, v_n)$.

Definition 9.2. The group $C_n^{\text{or}}(K)$ of *oriented n -chains* of K is the abelian group generated by the symbols $[v_0, \dots, v_n]$, one for each ordering of each n -simplex of K , subject to the relations

$$[v_{\pi(0)}, \dots, v_{\pi(n)}] = \text{sgn}(\pi) [v_0, \dots, v_n] \quad (\pi \in \Sigma_{n+1}),$$

and $C_n^{\text{or}}(K) = 0$ for $n < 0$. The boundary is defined on generators by

$$\partial[v_0, \dots, v_n] = \sum_{i=0}^n (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n].$$

Lemma 9.3. *Choosing an orientation for each n -simplex of K exhibits $C_n^{\text{or}}(K)$ as a free abelian group with those orientations as a basis; in particular its rank is the number of n -simplices of K . The boundary is well defined, and $\partial\partial = 0$, so that $(C_*^{\text{or}}(K), \partial)$ is a non-negative chain complex of abelian groups.*

Proof. The symbols for a fixed simplex σ are permuted transitively by Σ_{n+1} and the relations identify them, up to sign, with any one of them; so the subgroup they generate is infinite cyclic on either orientation of σ , and distinct simplices generate independent subgroups. This gives the basis.

For the boundary it is enough to check that the defining relations are respected, and since the transpositions of adjacent entries generate Σ_{n+1} it is enough to treat one of those. Let w be obtained from v by exchanging v_k and v_{k+1} . In the terms of $\partial[w]$ with $i \notin \{k, k+1\}$ the two exchanged vertices both survive, so by the relation in degree $n-1$ each such term is $-[v_0, \dots, \widehat{v_i}, \dots, v_n]$; the term $i = k$ deletes $w_k = v_{k+1}$ and the term $i = k+1$ deletes $w_{k+1} = v_k$, contributing

$$(-1)^k [v_0, \dots, \widehat{v_{k+1}}, \dots, v_n] + (-1)^{k+1} [v_0, \dots, \widehat{v_k}, \dots, v_n],$$

which is -1 times the corresponding two terms of $\partial[v]$. Hence $\partial[w] = -\partial[v]$, as required. Finally $\partial\partial = 0$ by cancellation in pairs: in $\partial\partial[v_0, \dots, v_n]$ the term deleting first v_j and then v_i , with $i < j$, is the term deleting first v_i and then v_j , with the opposite sign. $\square$

Definition 9.4. The *oriented homology* of K is $H_n^{\text{or}}(K) = H_n(C_*^{\text{or}}(K))$.

9.2. Functoriality.

Proposition 9.5. *Let $f: K \rightarrow L$ be a simplicial map. Then*

$$f_{\#}[v_0, \dots, v_n] = \begin{cases} [f(v_0), \dots, f(v_n)], & \text{if the } f(v_i) \text{ are distinct,} \\ 0, & \text{otherwise,} \end{cases}$$

defines a chain map $C_^{\text{or}}(K) \rightarrow C_*^{\text{or}}(L)$, and H_n^{or} is a functor from simplicial complexes and simplicial maps to abelian groups.*

Proof. The formula respects the sign relations, both cases being visibly equivariant for permutations, so $f_{\#}$ is defined. For $\partial f_{\#} = f_{\#} \partial$ there are three cases. If the $f(v_i)$ are distinct, both sides are the alternating sum of the faces of $[f(v_0), \dots, f(v_n)]$. If three or more of the $f(v_i)$ coincide, then every face of $[v_0, \dots, v_n]$ still has a repetition among its images, and both sides vanish. If exactly two coincide, say $f(v_k) = f(v_l)$ with $k < l$, then in $f_{\#} \partial[v_0, \dots, v_n]$ only the terms deleting v_k or v_l survive; the two resulting oriented simplices of L differ by the transposition moving one of the two equal vertices past the other, so they cancel, while $\partial f_{\#}[v_0, \dots, v_n] = 0$. Functoriality on generators is clear, and H_n^{or} is the composite with the homology functor of Proposition 2.8. $\square$

Remark 9.6. The two theories agree. We shall eventually establish that for every simplicial complex K there is an isomorphism

$$H_n^{\text{or}}(K) \cong H_n(|K|),$$

natural for simplicial maps, between the oriented simplicial homology of K and the singular homology of its realization, defined in the next notes. The comparison map is constructed, and the isomorphism proved, in the section of *Further Properties of Singular Homology* on oriented simplicial homology and singular homology: a total order on the vertices sends the orientation of a simplex to the corresponding affine singular simplex, and the resulting chain map is shown to be a quasi-isomorphism by induction on the number of simplices. Until we reach that, the two theories are unrelated here, and what is computed below is H_*^{or} .

9.3. A first computation.

Example 9.7. Let $K = \partial\Delta^2$, the boundary of a 2-simplex on $\{0, 1, 2\}$, so that $|K| \cong S^1$. Here $C_1^{\text{or}}(K) = \mathbb{Z}^3$ on $[0, 1], [1, 2], [0, 2]$ and $C_0^{\text{or}}(K) = \mathbb{Z}^3$ on $[0], [1], [2]$, with

$$\partial[0, 1] = [1] - [0], \quad \partial[1, 2] = [2] - [1], \quad \partial[0, 2] = [2] - [0].$$

The image of ∂ is the rank-two subgroup of elements of coefficient sum zero, so $H_0^{\text{or}} = \mathbb{Z}$; and the kernel is infinite cyclic, generated by $[0, 1] + [1, 2] - [0, 2]$, so $H_1^{\text{or}} = \mathbb{Z}$. This agrees with the singular homology of S^1 , as Remark 9.6 promises.

Remark 9.8. One may instead take the free abelian group on *all* the ordered tuples $(v_0, \dots, v_n)$, repetitions allowed, whose vertices span a simplex of K , with the same boundary formula. This *ordered* complex is literally a subcomplex of $C_*(|K|)$, namely the one spanned by the affine singular simplices, and requires no choice of order; but it is enormous, non-zero in every positive degree as soon as K has a 1-simplex. The oriented complex is the efficient model. The ordered complex is nevertheless a bridge between the two theories: it maps to $C_*(|K|)$ by the inclusion just described, and to $C_*^{\text{or}}(K)$ by the

map sending a tuple with a repeated vertex to 0 and a tuple of distinct vertices to the corresponding oriented simplex, and both maps are quasi-isomorphisms. That gives one proof of the isomorphism of Remark 9.6; the proof given there instead compares the two complexes directly, one simplex at a time.

Exercise 9.9. Show that $H_0^{\text{or}}(K)$ is free abelian on the set of connected components of K , two vertices lying in the same component when they are joined by a chain of 1-simplices.

Exercise 9.10. Compute $H_*^{\text{or}}(\Delta^n)$ directly, and check that it agrees with $H_*(|\Delta^n|)$, the realization being convex.