

FURTHER PROPERTIES OF SINGULAR HOMOLOGY

This lecture is divided into three parts. The first part (Sections 1–4) proves that $H_n(\Delta^n, \partial\Delta^n)$, $H_n(S^n)$, $H_n(I^n, \partial I^n)$ and $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are all infinite cyclic, and shows how an orientation of each of these manifolds determines a generator of the group. The second part (Section 5) compares the two homology theories we have: it constructs a chain map from the oriented simplicial chains of a simplicial complex K to the singular chains of $|K|$ and proves that it induces an isomorphism on homology, natural for simplicial maps: by induction on the number of simplices when K is finite, and then by passage to the direct limit over the finite subcomplexes. This discharges the promise of Remark 9.6 of *Homology* and is what allows the simplicial computations made there to be read as computations of singular homology. The third part (Section 6) proves the Künneth theorem for singular homology. It uses arguments from combinatorial topology as well as the method of acyclic models to reduce the computation of the singular homology of $X \times Y$ to algebraic operations on the homology of X and Y .

1. QUOTIENTS, AND THE CLASS OF THE SIMPLEX

Excision, Theorem 8.1 of *Singular Homology*, is what makes relative homology computable from quotient spaces.

Definition 1.1. Let A be a non-empty closed subspace of X . We write X/A for the quotient space obtained by collapsing A to a point, $c: X \rightarrow X/A$ for the quotient map, and $* = c(A)$ for the point so obtained. The pair (X, A) is a *good pair* if A is a deformation retract of some open neighbourhood of itself in X .

Proposition 1.2. *Let (X, A) be a good pair. Then the quotient map induces isomorphisms*

$$c_*: H_n(X, A) \xrightarrow{\cong} H_n(X/A, *) \cong \tilde{H}_n(X/A) \quad \text{for every } n.$$

Proof. Let V be an open neighbourhood of A having A as a deformation retract, and put $V/A = c(V)$, which is open in X/A because $c^{-1}(c(V)) = V$. Consider the diagram

$$\begin{array}{ccccc} H_n(X, A) & \longrightarrow & H_n(X, V) & \longleftarrow & H_n(X \setminus A, V \setminus A) \\ c_* \downarrow & & c_* \downarrow & & c_* \downarrow \\ H_n(X/A, *) & \longrightarrow & H_n(X/A, V/A) & \longleftarrow & H_n(X/A \setminus \{*\}, V/A \setminus \{*\}) \end{array}$$

in which the horizontal maps are induced by inclusions of pairs; it commutes because every map in it is induced by a map of pairs, and the two squares of maps of pairs commute already at the level of spaces.

The left-hand horizontal maps are isomorphisms. The inclusion $A \hookrightarrow V$ is a homotopy equivalence, so induces isomorphisms $H_n(A) \rightarrow H_n(V)$ by Corollary 4.2; compare the long exact sequences of (X, A) and of (X, V) of Theorem 6.4, in which the maps $H_n(X) \rightarrow H_n(X)$ are identities, and apply the five lemma (Exercise 4.6 of *Homology*). For the bottom row the same argument applies: a deformation retraction of V onto A carries A

into A at every time, so it descends to a deformation retraction of V/A onto $*$, continuous because $V \times I \rightarrow (V/A) \times I$ is a quotient map, I being locally compact.

The right-hand horizontal maps are isomorphisms by excision, Corollary 8.2. On the top row excise $Z = A$, which is legitimate because $\overline{A} = A \subseteq V = \text{int } V$; on the bottom row excise $Z = \{*\}$, which is closed and contained in the open set V/A .

Finally c restricts to a homeomorphism of $X \setminus A$ onto $X/A \setminus \{*\}$, carrying $V \setminus A$ onto $V/A \setminus \{*\}$: it is a continuous bijection between those sets, and it is a quotient map because $X \setminus A$ is open and saturated. So the right-hand vertical map is an isomorphism, and therefore so is the left-hand one. The last identification is Corollary 6.8. $\square$

The pairs we meet are good ones, by pushing outwards along rays.

Lemma 1.3. *Let $C \subseteq \mathbb{R}^m$ be compact and convex with non-empty interior and let $b \in \text{int } C$. Then radial projection from b makes ∂C a deformation retract of $C \setminus \{b\}$, and also of the open neighbourhood $V = \{b + t(y - b) : y \in \partial C, \frac{1}{2} < t < \frac{3}{2}\} \cap C$ of ∂C in C ; in particular $(C, \partial C)$ is a good pair.*

Proof. Every $x \in C \setminus \{b\}$ is $b + t(y - b)$ for a unique $y \in \partial C$ and a unique $t \in (0, 1]$, and y and t depend continuously on x . The formula $h_s(x) = b + ((1 - s)t + s)(y - b)$ is a homotopy from the identity to the radial projection onto ∂C , stationary on ∂C , where $t = 1$. Its parameter $(1 - s)t + s$ lies in $(0, 1]$, so h carries $C \setminus \{b\}$ into itself; and it lies between t and 1, so h carries V into itself as well. $\square$

Proposition 1.4. *Let $n \geq 1$ and let ι_n be the identity map of Δ^n regarded as a singular n -simplex, a relative cycle of $(\Delta^n, \partial\Delta^n)$. Then:*

- (i) $H_n(\Delta^n, \partial\Delta^n)$ is infinite cyclic, generated by $[\iota_n]$;
- (ii) $\partial[\iota_n] = [\sum_{i=0}^n (-1)^i (\delta^i)_{\#} \iota_{n-1}]$ generates $\tilde{H}_{n-1}(\partial\Delta^n) \cong \mathbb{Z}$.

Proof. Since Δ^n is convex, hence contractible, the reduced sequence of Proposition 6.7 gives an isomorphism

$$\partial: H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} \tilde{H}_{n-1}(\partial\Delta^n), \quad \partial[\iota_n] = [\partial\iota_n],$$

by Theorem 6.4. So (i) and (ii) are equivalent — each says that the group in question is infinite cyclic on the class named — and we prove them by induction on n . The induction yields the infinite cyclicity as well, so the computation of $\tilde{H}_*(S^n)$ is not presupposed.

For $n = 1$ the space $\partial\Delta^1$ is the two points e_0, e_1 and $\partial\iota_1 = \langle e_1 \rangle - \langle e_0 \rangle$, which generates $\tilde{H}_0(\partial\Delta^1)$ by Proposition 3.2 and Definition 5.1.

Let $n \geq 2$, write f_i for the face of Δ^n opposite e_i , so that δ^i is an affine homeomorphism of pairs $(\Delta^{n-1}, \partial\Delta^{n-1}) \rightarrow (f_i, \partial f_i)$, and put

$$\Lambda = f_0 \cup \cdots \cup f_{n-1},$$

the union of the facets of Δ^n that contain e_n , so that $\partial\Delta^n = \Lambda \cup f_n$ and $\Lambda \cap f_n = \partial f_n$. Now Λ is star-shaped about e_n : if $x \in \Lambda$ has $x_i = 0$ then so does every point of the segment from x to e_n . Hence Λ is contractible, $\tilde{H}_*(\Lambda) = 0$, and the reduced sequence of the pair $(\partial\Delta^n, \Lambda)$ shows that

$$\tilde{H}_{n-1}(\partial\Delta^n) \longrightarrow H_{n-1}(\partial\Delta^n, \Lambda)$$

is an isomorphism.

The inclusion $(f_n, \partial f_n) \hookrightarrow (\partial \Delta^n, \Lambda)$ also induces an isomorphism on H_{n-1} . Both are good pairs: $(f_n, \partial f_n)$ by Lemma 1.3 applied inside the hyperplane of f_n , and $(\partial \Delta^n, \Lambda)$ because the neighbourhood $U = \Lambda \cup W$ of Λ , with W the collar of ∂f_n in f_n provided by that lemma, deformation retracts onto Λ . Here U is open, its complement being the compact set $f_n \setminus W$, and the homotopy is the retraction of W onto ∂f_n together with the identity of Λ ; these agree on the overlap $\Lambda \cap W = \partial f_n$, and Λ and $W = U \cap f_n$ are both closed in U , so the two fit together continuously. The inclusion induces a continuous bijection $f_n/\partial f_n \rightarrow \partial \Delta^n/\Lambda$, hence a homeomorphism, the spaces being compact and Hausdorff; so the claim follows from Proposition 1.2 applied to both pairs and the naturality of c_* .

It remains to follow the generator. In $H_{n-1}(\partial \Delta^n, \Lambda)$ the terms $\delta^0, \dots, \delta^{n-1}$ of $\partial \iota_n$ are chains in Λ , so the image of $[\partial \iota_n]$ is $(-1)^n$ times the relative class of δ^n , which is the image of $[\iota_{n-1}]$ under the two isomorphisms

$$H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1}) \xrightarrow{\delta_*^n} H_{n-1}(f_n, \partial f_n) \longrightarrow H_{n-1}(\partial \Delta^n, \Lambda).$$

By the inductive hypothesis, in the form (i) in degree $n-1$, the group $H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1})$ is infinite cyclic on $[\iota_{n-1}]$; carrying this along the three isomorphisms above, $\tilde{H}_{n-1}(\partial \Delta^n)$ is infinite cyclic on $[\partial \iota_n]$, which is (ii) for n . $\square$

Corollary 1.5. *Let $n \geq 1$ and let τ be the affine automorphism of Δ^n transposing two vertices and fixing the rest. Then $\tau_*[\iota_n] = -[\iota_n]$ in $H_n(\Delta^n, \partial \Delta^n)$, and the automorphism induced by a permutation π of the vertices acts as multiplication by $\text{sgn } \pi$.*

Proof. The group is infinite cyclic by part (i), so every automorphism of it is ± 1 ; conjugate automorphisms therefore agree, and as any two transpositions are conjugate in the symmetric group it is enough to treat $\tau = (e_0 e_1)$. The statement for a general π then follows, transpositions generating the symmetric group and $\pi \mapsto \pi_*$ being multiplicative.

Induct on n . For $n = 1$ the isomorphism $\partial: H_1(\Delta^1, \partial \Delta^1) \rightarrow \tilde{H}_0(\partial \Delta^1)$ carries $[\iota_1]$ to $\langle e_1 \rangle - \langle e_0 \rangle$ and is natural for maps of pairs; τ interchanges e_0 and e_1 , so τ_* carries that class to its negative.

Let $n \geq 2$ and keep the notation Λ, f_n of the proof of Proposition 1.4. Since τ fixes e_n it carries each facet through e_n to a facet through e_n , hence maps Λ onto Λ , f_n onto f_n and ∂f_n onto ∂f_n . The isomorphisms

$$\begin{aligned} H_n(\Delta^n, \partial \Delta^n) &\xrightarrow{\partial} \tilde{H}_{n-1}(\partial \Delta^n) \longrightarrow H_{n-1}(\partial \Delta^n, \Lambda), \\ H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1}) &\xrightarrow{\delta_*^n} H_{n-1}(f_n, \partial f_n) \longrightarrow H_{n-1}(\partial \Delta^n, \Lambda) \end{aligned}$$

used there are induced by inclusions and by δ^n , hence natural for τ ; and δ^n carries e_j to e_j for $j \leq n-1$, so it intertwines the transposition $\tau' = (e_0 e_1)$ of Δ^{n-1} with τ . Therefore τ_* on $H_n(\Delta^n, \partial \Delta^n)$ is τ'_* on $H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1})$, which is -1 by induction. $\square$

Remark 1.6. Part (ii) says that the boundary cycle $\sum_i (-1)^i (\delta^i)_{\#} \iota_{n-1}$ of the simplex, which is the obvious geometric cycle carried by $\partial \Delta^n \cong S^{n-1}$, is not merely a cycle but a generator of the homology of that sphere.

Exercise 1.7. Show that (X, A) is a good pair whenever X is a finite simplicial complex and A a subcomplex, and deduce that $H_n(X, A) \cong \tilde{H}_n(X/A)$ in that case.

2. LOCAL HOMOLOGY AND THE LOCAL CLASSES o_x

Throughout this section $\mathbb{R}^n$ carries its standard orientation and $n \geq 1$. By an *affine embedding* $A: \Delta^n \rightarrow \mathbb{R}^n$ we mean the restriction of an affine isomorphism of the affine span of Δ^n onto $\mathbb{R}^n$; identifying that span with $\mathbb{R}^n$ by the basis $e_1 - e_0, \dots, e_n - e_0$ gives A a linear part $DA \in GL_n(\mathbb{R})$, and A is *orientation preserving* when $\det DA > 0$. Recall from Proposition 1.4 that $H_n(\Delta^n, \partial\Delta^n)$ is infinite cyclic on the class of the identity simplex ι_n .

Lemma 2.1. *For $x \in \mathbb{R}^n$ let E_x be the set of orientation-preserving affine embeddings $A: \Delta^n \rightarrow \mathbb{R}^n$ with $x \in \text{int } A(\Delta^n)$, topologized by the coefficients. Then $A \mapsto (DA, A^{-1}(x))$ is a homeomorphism $E_x \rightarrow GL_n^+(\mathbb{R}) \times \text{int } \Delta^n$. In particular E_x is path connected.*

Proof. An affine embedding is determined by its linear part together with the image of one point, so the map is a bijection: given (L, y) there is exactly one affine A with $DA = L$ and $A(y) = x$, and x then lies in $\text{int } A(\Delta^n)$ because $y \in \text{int } \Delta^n$ and A is a homeomorphism onto its image. Both the map and its inverse are continuous in the coefficients. Finally $GL_n^+(\mathbb{R})$ is path connected and $\text{int } \Delta^n$ is convex. $\square$

Definition 2.2. For $x \in \mathbb{R}^n$ let

$$o_x = A_*[\iota_n] \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}), \quad A \in E_x,$$

the image of the class of the identity simplex under A , regarded as a map of pairs $(\Delta^n, \partial\Delta^n) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$.

Lemma 2.3. *The class o_x does not depend on the choice of $A \in E_x$, and it generates $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$. Moreover $L_*o_0 = (\text{sgn } \det L) o_0$ for every $L \in GL_n(\mathbb{R})$, and $(T_v)_*o_x = o_{x+v}$ for every translation T_v . More generally $B_*[\iota_n] = -o_x$ for an orientation-reversing affine embedding B with $x \in \text{int } B(\Delta^n)$.*

Proof. If A_t is a path in E_x then $x \notin A_t(\partial\Delta^n)$ for every t , so $(u, t) \mapsto A_t(u)$ is a homotopy of maps of pairs $(\Delta^n, \partial\Delta^n) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$; hence $(A_0)_* = (A_1)_*$ on homology, and Lemma 2.1 makes E_x path connected. That o_x generates follows from Proposition 1.4 together with the excision $H_n(A(\Delta^n), \partial A(\Delta^n)) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$, both groups being infinite cyclic and the map carrying a generator to a generator. For the next two statements, when $\det L > 0$ the embeddings $L \circ A$ and $T_v \circ A$ lie in E_0 and E_{x+v} respectively for $A \in E_0$, $A \in E_x$.

For an orientation-reversing B with $x \in \text{int } B(\Delta^n)$, let τ be the affine automorphism of Δ^n transposing e_0 and e_1 . In the basis $e_1 - e_0, \dots, e_n - e_0$ its linear part carries $e_1 - e_0$ to $-(e_1 - e_0)$ and $e_i - e_0$ to $(e_i - e_0) - (e_1 - e_0)$ for $i \geq 2$, so $\det D\tau = -1$. Hence $D(B \circ \tau) = DB \cdot D\tau$ has positive determinant, and $B \circ \tau(\Delta^n) = B(\Delta^n)$ still contains x in its interior; so $B \circ \tau \in E_x$ and

$$o_x = (B \circ \tau)_*[\iota_n] = B_*(\tau_*[\iota_n]) = -B_*[\iota_n]$$

by Corollary 1.5. Taking $B = L \circ A$ with $\det L < 0$ and $A \in E_0$ gives $L_*o_0 = -o_0$, which completes the statement about $GL_n(\mathbb{R})$. $\square$

Lemma 2.4 (straightening). *Let $U, V \subseteq \mathbb{R}^n$ be open, $0 \in U$, and let $f: U \rightarrow V$ be a diffeomorphism onto its image with $f(0) = 0$ and $\det df_0 > 0$. Then $f_*o_0 = o_0$ in $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, the class o_0 being computed in U and in V through the excisions $H_n(U, U \setminus \{0\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.*

Proof. Choose $r > 0$ with $\overline{B(0, r)} \subseteq U$ and put, for $0 < t \leq 1$,

$$f_t(u) = \frac{f(tu)}{t} \quad (u \in B(0, r)), \quad f_0 = df_0.$$

Each f_t is defined and continuous on $B(0, r)$, carries 0 to 0 and nothing else to 0, and $f_t \rightarrow df_0$ uniformly on $B(0, r)$ as $t \rightarrow 0$ by differentiability at 0; so $(u, t) \mapsto f_t(u)$ is a homotopy of maps of pairs $(B(0, r), B(0, r) \setminus \{0\}) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ from df_0 to $f|_{B(0, r)}$. Hence $f_* = (df_0)_*$ on H_n , and $(df_0)_*o_0 = o_0$ by Lemma 2.3, $\det df_0$ being positive. $\square$

Theorem 2.5. *For $n \geq 0$,*

$$\tilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z}, & i = n, \\ 0, & i \neq n. \end{cases}$$

Proof. For $n = 0$ the space S^0 is two points, so $H_0 = \mathbb{Z}^2$ and $\tilde{H}_0 = \mathbb{Z}$, the other groups vanishing.

Let $n \geq 1$ and fix $x_0 \in S^n$. The complement $S^n \setminus \{x_0\}$ is homeomorphic to $\mathbb{R}^n$, hence contractible, so the reduced sequence of the pair $(S^n, S^n \setminus \{x_0\})$ gives an isomorphism

$$\tilde{H}_i(S^n) \xrightarrow{\cong} H_i(S^n, S^n \setminus \{x_0\}) \quad \text{for every } i.$$

Let $\varphi: W \rightarrow U \subseteq \mathbb{R}^n$ be a chart at x_0 with $\varphi(x_0) = 0$. Excising the closed set $S^n \setminus W$, which lies in the open set $S^n \setminus \{x_0\}$, and then excising $\mathbb{R}^n \setminus U$ from $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, gives isomorphisms

$$(2.1) \quad H_i(S^n, S^n \setminus \{x_0\}) \xleftarrow{\cong} H_i(W, W \setminus \{x_0\}) \xrightarrow{\varphi_*} H_i(U, U \setminus \{0\}) \xrightarrow{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}),$$

by Corollary 8.2. Finally $\mathbb{R}^n$ is contractible, so the reduced sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ gives

$$H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \tilde{H}_{i-1}(\mathbb{R}^n \setminus \{0\}) \cong \tilde{H}_{i-1}(S^{n-1}),$$

the second isomorphism because S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$. So $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$ for every i , and the result follows by induction on n . $\square$

Corollary 2.6. *There is no retraction $D^n \rightarrow S^{n-1}$ for $n \geq 1$, and every map $D^n \rightarrow D^n$ has a fixed point.*

Proof. A retraction r would give $\text{id} = (r \circ i)_* = r_* i_*$ on $\tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$, with i_* factoring through $\tilde{H}_{n-1}(D^n) = 0$ by Corollary 4.4, which is impossible. If $f: D^n \rightarrow D^n$ had no fixed point, then sending x to the point where the ray from $f(x)$ through x meets S^{n-1} would define such a retraction, continuously. $\square$

Corollary 2.7. *Let $n \geq 1$. An orientation of S^n determines a generator of $H_n(S^n) \cong \tilde{H}_n(S^n)$, and the opposite orientation determines the opposite generator.*

Proof. Take $i = n$ in the proof above and let the chart φ at x_0 belong to the given orientation. The last group of (2.1) is infinite cyclic on the class o_0 , which by Definition 2.2 and Lemma 2.3 is determined by the standard orientation of $\mathbb{R}^n$; carrying o_0 back along those isomorphisms produces a generator of $H_n(S^n)$.

The generator does not depend on the chart: two charts at x_0 from the same orientation differ by a diffeomorphism between neighbourhoods of 0 in $\mathbb{R}^n$ fixing 0 and having positive Jacobian there, which carries o_0 to o_0 by Lemma 2.4.

Nor does it depend on x_0 . Given a second point x_1 , choose $R \in SO(n+1)$ with $R(x_0) = x_1$. Then R is an orientation-preserving diffeomorphism of S^n , and $SO(n+1)$ is path connected, so R is homotopic to the identity and R_* is the identity of $H_n(S^n)$. If φ is a positively oriented chart at x_0 then $\varphi \circ R^{-1}$ is one at x_1 , and R carries the whole diagram at x_0 to the corresponding diagram at x_1 ; so the two generators agree.

Finally, a chart at x_0 positively oriented for the opposite orientation is $r \circ \varphi$ with r a reflection of $\mathbb{R}^n$, and $r_* o_0 = -o_0$ by Lemma 2.3, $\det r$ being negative. So the class produced by $r \circ \varphi$ is the negative of the one produced by φ . $\square$

3. THE FUNDAMENTAL CLASS OF THE CUBE

Throughout, $n \geq 1$ and $I^n = [0, 1]^n \subseteq \mathbb{R}^n$, which carries the standard orientation; o_x is the local class of Definition 2.2 and ι_n the identity simplex.

Lemma 3.1. *Let $K \subseteq I^n$ be compact and convex and let $x \in \text{int } K$; the case $K = I^n$ is allowed, and then the first map below is the identity. The inclusions of pairs induce isomorphisms*

$$H_n(I^n, \partial I^n) \longrightarrow H_n(I^n, I^n \setminus \text{int } K) \longrightarrow H_n(I^n, I^n \setminus \{x\}) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}),$$

the last by excision. All four groups are infinite cyclic, the last on o_x .

Proof. Radial projection from x is a deformation retraction of $I^n \setminus \{x\}$ onto ∂I^n , and it restricts to one of $I^n \setminus \text{int } K$ onto ∂I^n : if $z \in I^n \setminus \text{int } K$ and $r(z)$ is the point where the ray from x through z meets ∂I^n , then no point w of the segment $[z, r(z)]$ lies in $\text{int } K$: otherwise the segment $[x, w]$ would lie in $\text{int } K$, that set being convex and containing x , and z lies on $[x, w]$, contrary to $z \notin \text{int } K$. So in the long exact sequences of the three pairs, Theorem 6.4 of *Singular Homology*, the maps of the subspaces are isomorphisms on homology and the maps of $H_*(I^n)$ are identities; the five lemma, Exercise 4.6 of *Homology*, gives the first two isomorphisms. For the last, excise $Z = \mathbb{R}^n \setminus I^n$ from $(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$: $\bar{Z} = \mathbb{R}^n \setminus \text{int } I^n$ misses x , so Corollary 8.2 of *Singular Homology* applies. The last group is infinite cyclic on o_x by Lemma 2.3. $\square$

Definition 3.2. For $x \in \text{int } I^n$ let $\mu_n(x) \in H_n(I^n, \partial I^n)$ be the class carried to o_x by the isomorphisms of Lemma 3.1 taken with $K = I^n$, that is, by

$$H_n(I^n, \partial I^n) \longrightarrow H_n(I^n, I^n \setminus \{x\}) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}).$$

Lemma 3.3. *$\mu_n(x)$ is independent of $x \in \text{int } I^n$. Moreover, if $K \subseteq I^n$ is compact and convex and A is an affine embedding of positive determinant with $A(\Delta^n) = K$, then $\mu_n(x)$ and $A_*[\iota_n]$ have the same image in $H_n(I^n, I^n \setminus \text{int } K)$ for every $x \in \text{int } K$.*

Proof. Let $x, y \in \text{int } I^n$. Since $\text{int } I^n$ is open and convex it contains a compact convex neighbourhood of the segment $[x, y]$, and hence the image $K = A(\Delta^n)$ of an affine embedding A of positive determinant with $x, y \in \text{int } K$. Then A lies in E_x and in E_y , so by Definition 2.2 the class $A_*[\iota_n]$ of $H_n(K, \partial K)$ has image o_x in $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ and image o_y in $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$. Both images are computed through $H_n(I^n, I^n \setminus \text{int } K)$, since

$$(K, \partial K) \longrightarrow (I^n, I^n \setminus \text{int } K) \longrightarrow (I^n, I^n \setminus \{x\}) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

are maps of pairs whose composite is the one defining o_x , and likewise for y . Write w for the image of $A_*[\iota_n]$ in $H_n(I^n, I^n \setminus \text{int } K)$. By Lemma 3.1 the map $H_n(I^n, \partial I^n) \rightarrow$

$H_n(I^n, I^n \setminus \text{int } K)$ is an isomorphism, and its inverse carries w to $\mu_n(x)$ and to $\mu_n(y)$; so these agree. The same computation, for a given K and A and any $x \in \text{int } K$, is the second assertion. $\square$

Definition 3.4. The *fundamental class of the cube* is the class $\mu_n = \mu_n(x) \in H_n(I^n, \partial I^n)$ of Definition 3.2, attached to the standard orientation of $\mathbb{R}^n$.

Definition 3.5. Let $K \subseteq \mathbb{R}^n$ be compact and convex and let $p \in \text{int } K$. The *gauge* of K about p , also called its Minkowski functional, is

$$g_K(z) = \inf \{ t > 0 : p + t^{-1}(z - p) \in K \}.$$

It is finite and continuous, vanishes exactly at p , and is positively homogeneous about p : $g_K(p + s(z - p)) = s g_K(z)$ for $s \geq 0$. Consequently

$$K = \{g_K \leq 1\}, \quad \text{int } K = \{g_K < 1\}, \quad \partial K = \{g_K = 1\},$$

and $K \subseteq L$ implies $g_K \geq g_L$ when both are taken about p . The gauge is a convex function, and conversely a continuous convex function that is positively homogeneous about p and vanishes only there is the gauge about p of the compact convex set on which it is at most 1.

Lemma 3.6 (radial expansion). *Let A be an affine embedding of positive determinant with $K = A(\Delta^n) \subseteq I^n$, let $p \in \text{int } K$, let g and g_K be the gauges of I^n and of K about p as in Definition 3.5, and let*

$$\rho(z) = p + \frac{g_K(z)}{g(z)}(z - p) \quad (z \neq p), \quad \rho(p) = p.$$

Then ρ is a homeomorphism of pairs $(K, \partial K) \rightarrow (I^n, \partial I^n)$ and $\rho_ A_*[\iota_n] = \mu_n$.*

Proof. The gauges are continuous and positive away from p , and $g(\rho(z)) = g_K(z)$, so ρ carries $\{g_K \leq 1\} = K$ onto $\{g \leq 1\} = I^n$ and ∂K onto ∂I^n ; the map $z \mapsto p + (g(z)/g_K(z))(z - p)$ is a continuous inverse.

Both gauges are piecewise linear. Indeed K and I^n are polytopes, say $K = \{z : \lambda_i(z - p) \leq 1\}$ and $I^n = \{z : \nu_j(z - p) \leq 1\}$ for finitely many linear forms λ_i, ν_j , and then $g_K(z) = \max_i \lambda_i(z - p)$ and $g(z) = \max_j \nu_j(z - p)$. The set of z at which both maxima are attained at a single index is open and dense, its complement being contained in finitely many hyperplanes through p ; choose z_0 in it with $z_0 \in \text{int } K$. Near z_0 we have $g(z) = \ell(z - p)$ and $g_K(z) = \ell_K(z - p)$ for linear forms ℓ, ℓ_K , so with $w = z - p$,

$$\rho(z) = p + \frac{\ell_K(w)}{\ell(w)} w,$$

which is smooth there. Writing $c = \ell_K(w)/\ell(w) > 0$ and u for the differential of ℓ_K/ℓ at w , the differential of ρ is $v \mapsto cv + u(v)w$, of determinant $c^{n-1}(c + u(w)) = c^n$, since ℓ_K/ℓ is homogeneous of degree 0 and so $u(w) = 0$. Thus ρ is an orientation-preserving diffeomorphism near z_0 .

Now compare local classes at z_0 . The class $A_*[\iota_n]$ has image o_{z_0} , by Definition 2.2, and ρ is a map of pairs $(K, K \setminus \{z_0\}) \rightarrow (I^n, I^n \setminus \{\rho(z_0)\})$, so by the straightening lemma 2.4, together with the translations of Lemma 2.3, it carries o_{z_0} to $o_{\rho(z_0)}$. By Definition 3.2 the class μ_n has that same image, and by Lemma 3.1 the map $H_n(I^n, \partial I^n) \rightarrow H_n(I^n, I^n \setminus \{\rho(z_0)\})$ is injective. Hence $\rho_* A_*[\iota_n] = \mu_n$. $\square$

Remark 3.7. Polytopes entered only because $K = A(\Delta^n)$ is one; what the argument uses is the differentiability of g_K/g near a single point. If $z_0 \neq p$ lies in $\text{int } K$ and the ray from p through z_0 meets ∂K and ∂I^n at points where those boundaries are C^1 , then g_K and g are C^1 near z_0 and the computation applies word for word: the differential of ρ at z_0 is $v \mapsto cv + u(v)w$ with $c = (g_K/g)(z_0) > 0$, $w = z_0 - p$ and u the differential of g_K/g , which annihilates w because g_K/g is homogeneous of degree 0; so the determinant is $c^n > 0$. It is thus enough that some non-empty open subset of ∂K be C^1 : the rays from p through it form an open cone, while the rays meeting ∂I^n outside the interiors of its facets lie in finitely many hyperplanes, so some ray does both.

4. THE FUNDAMENTAL CLASS OF S^n AND PARAMETRIZATIONS

Definition 4.1. Give $S^n \subseteq \mathbb{R}^{n+1}$, $n \geq 1$, the orientation determined by the standard orientation of $\mathbb{R}^{n+1}$ and the outward unit normal $\nu(x) = x$, taken first: a basis $v_1, \dots, v_n$ of $T_x S^n$ is positively oriented exactly when $\nu(x), v_1, \dots, v_n$ is a positively oriented basis of $\mathbb{R}^{n+1}$. Let

$$[S^n] \in \tilde{H}_n(S^n)$$

be the generator that this orientation determines, by Corollary 2.7. For $n = 0$ write $S^0 = \{\pm 1\} \subseteq \mathbb{R}$ and put $[S^0] = [\langle 1 \rangle - \langle -1 \rangle]$, the class attached in the same way to the outward normals ± 1 at the two points.

Proposition 4.2. Give $S^n \subseteq \mathbb{R}^{n+1}$, $n \geq 1$, the orientation determined by the standard orientation of $\mathbb{R}^{n+1}$ and the outward unit normal $\nu(x) = x$: a basis $v_1, \dots, v_n$ of $T_x S^n$ is positively oriented exactly when $\nu(x), v_1, \dots, v_n$ is a positively oriented basis of $\mathbb{R}^{n+1}$. Give $\text{int } \Delta^n$ the orientation for which $e_1 - e_0, \dots, e_n - e_0$ is positively oriented. Let $x_0 \in S^n$ and let $f: \Delta^n \rightarrow S^n$ carry $\partial \Delta^n$ to x_0 and $\text{int } \Delta^n$ bijectively onto $S^n \setminus \{x_0\}$, and suppose that on some non-empty open subset U of $\text{int } \Delta^n$ the map f is a smooth orientation-preserving diffeomorphism onto its image. Then f is a relative cycle of $(S^n, \{x_0\})$ and its class is

$$[f] = [S^n] \in H_n(S^n, \{x_0\}) \cong \tilde{H}_n(S^n),$$

the isomorphism being that of Corollary 6.8.

Proof. Since f carries $\partial \Delta^n$ to x_0 it is a map of pairs $(\Delta^n, \partial \Delta^n) \rightarrow (S^n, \{x_0\})$; so ∂f is a chain of $\{x_0\}$, f is a relative cycle, and $[f] = f_*[\iota_n]$. We identify that class by comparing local classes at a single point, which is why the smooth hypothesis is needed only on U .

Choose $x \in U$ and put $y = f(x)$. Identify the affine span of Δ^n with $\mathbb{R}^n$ by the basis $e_1 - e_0, \dots, e_n - e_0$, which is the identification defining the orientation of $\text{int } \Delta^n$; then the inclusion of Δ^n belongs to E_x , so by Definition 2.2 the image of $[\iota_n]$ under

$$H_n(\Delta^n, \partial \Delta^n) \longrightarrow H_n(\Delta^n, \Delta^n \setminus \{x\}) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

is o_x , the second map being the excision of $\mathbb{R}^n \setminus \Delta^n$.

Since f is injective on $\text{int } \Delta^n$ and carries $\partial\Delta^n$ to $x_0 \neq y$, we have $f^{-1}(y) = \{x\}$, so f is also a map of pairs $(\Delta^n, \Delta^n \setminus \{x\}) \rightarrow (S^n, S^n \setminus \{y\})$ and the square

$$\begin{array}{ccc} H_n(\Delta^n, \partial\Delta^n) & \xrightarrow{f_*} & H_n(S^n, \{x_0\}) \\ \downarrow & & \downarrow \\ H_n(\Delta^n, \Delta^n \setminus \{x\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus \{y\}) \end{array}$$

commutes, the vertical maps being induced by inclusions. The right-hand one is an isomorphism, and by the construction of Corollary 2.7 it carries $[S^n]$ to the class corresponding to o_0 under an orientation-preserving chart φ at y with $\varphi(y) = 0$.

Now $\varphi \circ f$ is defined on a neighbourhood of x inside U and is there an orientation-preserving diffeomorphism onto a neighbourhood of 0, both f and φ preserving orientation. Translating x to the origin, which carries o_x to o_0 by Lemma 2.3, the straightening lemma 2.4 gives $(\varphi \circ f)_* o_x = o_0$. So $f_*[\iota_n]$ and $[S^n]$ have the same image under the right-hand vertical isomorphism, and therefore $[f] = [S^n]$. $\square$

Remark 4.3. One might instead normalize these classes recursively. That is a different normalization: $[S^n]$ would be the class carried to $[S^{n-1}]$ by the Mayer–Vietoris isomorphism $\tilde{H}_n(S^n) \rightarrow \tilde{H}_{n-1}(S^{n-1})$ of the cover of S^n by the complements A, B of the two poles, whose intersection is a collar of the equator. The connecting homomorphism of that cover, computed on a cycle split as a chain of the lower cap plus a chain of the upper one, is the boundary-orientation map with the outward normal taken first; along the equator the outward normal of the lower cap is e_{n+1} , and moving it past the $n-1$ directions of the equator costs $(-1)^{n-1}$. So

$$\partial[S^n] = (-1)^{n-1}[S^{n-1}],$$

and the two normalizations differ by $(-1)^{\lfloor n/2 \rfloor}$, agreeing for $n \equiv 0, 1 \pmod{4}$. The sign is an artifact of taking the poles on the last coordinate axis: with them at $\pm e_1$ and the equator $S^n \cap \{x_1 = 0\}$, the normal to be inserted is $\mp e_1$, which stands next to the position vector instead of having to travel past the $n-1$ tangent directions of the equator, and the recursion holds with no sign at all. Nothing below uses the recursion.

5. ORIENTED SIMPLICIAL HOMOLOGY AND SINGULAR HOMOLOGY

Let K be a simplicial complex, with vertex set $V(K)$ and geometric realization $|K|$, and let $C_*^{\text{or}}(K)$ be its oriented simplicial chain complex, Definition 9.2 of *Homology*. Remark 9.6 there promised a comparison between $H_*^{\text{or}}(K)$ and the singular homology of $|K|$; we construct it now and prove that it is an isomorphism.

For points $p_0, \dots, p_k$ of a closed simplex $|s|$ of K , which is a convex subset of a Euclidean space, we write

$$\langle p_0, \dots, p_k \rangle: \Delta^k \longrightarrow |s| \subseteq |K|$$

for the affine singular simplex with $e_j \mapsto p_j$, regarded as a singular simplex of $|K|$; this is the map written $[p_0, \dots, p_k]$ in §4 of *Singular Homology*, and the change of brackets is to separate it from the orientation $[v_0, \dots, v_p]$ of Definition 9.1 of *Homology*. The passage from $C_*^{\text{or}}(K)$ to $C_*(|K|)$ depends on a choice of order, and the point of the second lemma below is that the map on homology does not.

Definition 5.1. Let λ be a total order on $V(K)$, written $<$. For a p -simplex s with vertices $v_0 < v_1 < \dots < v_p$ let

$$\sigma_s = \langle v_0, \dots, v_p \rangle: \Delta^p \longrightarrow |K|$$

be the affine singular simplex determined by the increasing ordering, and define

$$\Theta_\lambda: C_p^{\text{or}}(K) \longrightarrow C_p(|K|), \quad \Theta_\lambda[v_0, \dots, v_p] = \sigma_s \quad \text{when } v_0 < v_1 < \dots < v_p,$$

on the basis of increasing orientations, extended additively. Equivalently,

$$\Theta_\lambda[v_{\pi(0)}, \dots, v_{\pi(p)}] = \text{sgn}(\pi) \sigma_s$$

for every permutation π , which is consistent with the relations defining $C_p^{\text{or}}(K)$.

Lemma 5.2. Θ_λ is a chain map, and for each orientation ω of a simplex s the chain $\Theta_\lambda(\omega)$ is a chain in the closed simplex $|s|$.

Proof. The second statement holds because σ_s has image $|s|$, and the faces of s have their closed simplices inside $|s|$. For the first, let $v_0 < \dots < v_p$ span s . Deleting a vertex preserves the increasing order, so $\sigma_s \circ \delta^i = \sigma_{s_i}$, where $s_i = s \setminus \{v_i\}$, and therefore

$$\begin{aligned} \partial \Theta_\lambda[v_0, \dots, v_p] &= \partial \sigma_s = \sum_{i=0}^p (-1)^i \sigma_{s_i} \\ &= \Theta_\lambda \left(\sum_{i=0}^p (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_p] \right) = \Theta_\lambda \partial[v_0, \dots, v_p]. \quad \square \end{aligned}$$

Lemma 5.3. Let λ and λ' be total orders on $V(K)$. The chain maps

$$\Theta_\lambda, \Theta_{\lambda'}: C_*^{\text{or}}(K) \longrightarrow C_*(|K|)$$

are chain homotopic, and consequently induce the same map $H_*^{\text{or}}(K) \rightarrow H_*(|K|)$.

Proof. Choose an orientation η_s of each simplex s of K ; these form a basis of $C_*^{\text{or}}(K)$ by Lemma 9.3, so a homomorphism may be defined by prescribing it on them. We construct $D: C_p^{\text{or}}(K) \rightarrow C_{p+1}(|K|)$ by induction on p so that

$$(5.1) \quad \partial D + D \partial = \Theta_{\lambda'} - \Theta_\lambda \quad \text{on } C_p^{\text{or}}(K),$$

and so that $D(\eta_s)$ is a chain in $|s|$.

For $p = 0$ the two maps agree, both sending the orientation of a vertex v to the singular 0-simplex $\langle v \rangle$; set $D = 0$ on $C_0^{\text{or}}(K)$. Let $p \geq 1$ and suppose D has been constructed in degrees $< p$ with these properties. Let s be a p -simplex and put

$$u = \Theta_{\lambda'}(\eta_s) - \Theta_\lambda(\eta_s) - D(\partial \eta_s) \in C_p(|s|),$$

which lies in $C_p(|s|)$ because each term does: the first two by Lemma 5.2, the third because $\partial \eta_s$ is a combination of orientations of the codimension-one faces of s , whose closed simplices lie in $|s|$. Using the inductive hypothesis in degree $p - 1$,

$$\partial u = \Theta_{\lambda'} \partial \eta_s - \Theta_\lambda \partial \eta_s - \partial D \partial \eta_s = (\partial D + D \partial) \partial \eta_s - \partial D \partial \eta_s = D \partial \partial \eta_s = 0,$$

so u is a p -cycle in $|s|$. Now $|s|$ is convex, hence contractible, so $H_p(|s|) = 0$ by Corollary 4.4 of *Singular Homology*, and $u = \partial w$ for some $w \in C_{p+1}(|s|)$. Set $D(\eta_s) = w$. Then (5.1) holds on η_s , hence on all of $C_p^{\text{or}}(K)$, and $D(\eta_s)$ is a chain in $|s|$, completing the induction. A chain homotopy induces equality on homology. $\square$

Lemma 5.4. *Let K be a simplicial complex and $C \subseteq |K|$ compact. Then C meets only finitely many open simplices of K , and so is contained in $|K_0|$ for some finite subcomplex K_0 .*

Proof. Choose a point of C in each open simplex that C meets, and let S be the resulting set. Any subset of S meets each open simplex in at most one point, hence meets each closed simplex in a finite, therefore closed, set; so every subset of S is closed in $|K|$, the realization carrying the weak topology. Thus S is closed in $|K|$ and discrete, and being a closed subset of the compact set C it is compact. A compact discrete space is finite, so C meets only finitely many open simplices; the faces of those simplices form a finite subcomplex K_0 with $C \subseteq |K_0|$. $\square$

Theorem 5.5. *Let K be a simplicial complex and λ a total order on $V(K)$. Then*

$$\Theta_{\lambda*}: H_n^{\text{or}}(K) \longrightarrow H_n(|K|)$$

is an isomorphism for every n .

Proof. Suppose first that K is finite, and induct on the number of its simplices. If K has a single simplex, that simplex is a vertex v ; then $C_*^{\text{or}}(K)$ is infinite cyclic on $[v]$ in degree 0 and zero elsewhere, $|K|$ is a point, and $\Theta_{\lambda}[v] = \langle v \rangle$ carries a generator to a generator.

Let K have more than one simplex and suppose the theorem holds for complexes with fewer. Choose a simplex s of K of maximal dimension p . It is a proper face of no simplex of K , so $K' = K \setminus \{s\}$ is a subcomplex; and the open simplex $\text{int } |s|$ is open in $|K|$, with $|K| \setminus \text{int } |s| = |K'|$. Give $V(K')$ the order restricted from λ ; then Θ_{λ} restricts to the corresponding map for K' , so it carries the short exact sequence of chain complexes

$$0 \longrightarrow C_*^{\text{or}}(K') \longrightarrow C_*^{\text{or}}(K) \longrightarrow Q_* \longrightarrow 0$$

to the short exact sequence defining the relative singular chains of $(|K|, |K'|)$, and therefore maps the one long exact sequence to the other. By the inductive hypothesis the maps $H_q^{\text{or}}(K') \rightarrow H_q(|K'|)$ are isomorphisms, so by the five lemma, Exercise 4.6 of *Homology*, it is enough to prove that

$$H_q(Q_*) \longrightarrow H_q(|K|, |K'|)$$

is an isomorphism for every q .

The quotient Q_* is infinite cyclic in degree p and zero elsewhere, a generator being the orientation η_s of s given by the increasing ordering of its vertices; so $H_q(Q_*)$ is infinite cyclic on η_s for $q = p$ and zero otherwise, and $\Theta_{\lambda}(\eta_s) = \sigma_s$. It remains to identify the target.

Suppose first that $p = 0$. Then every simplex of K is a vertex, s is a vertex v and $|K| = |K'| \sqcup \{v\}$, so $H_q(|K|) = H_q(|K'|) \oplus H_q(\{v\})$ by Proposition 3.3 of *Singular Homology* and the long exact sequence of the pair gives $H_q(|K|, |K'|) \cong H_q(\{v\})$, which is infinite cyclic on the class of $\langle v \rangle = \sigma_s$ for $q = 0$ and zero otherwise, as required.

Now let $p \geq 1$ and let b be the barycenter of s . The radial formula of Lemma 1.3, applied in the plane of $|s|$, retracts $|s| \setminus \{b\}$ onto $|\partial s|$, and together with the identity of $|K'|$ it exhibits $|K'|$ as a deformation retract of $|K| \setminus \{b\}$, the two prescriptions agreeing on the closed set $|\partial s|$. Comparing the long exact sequences of the pairs $(|K|, |K'|)$ and $(|K|, |K| \setminus \{b\})$, the five lemma gives

$$H_q(|K|, |K'|) \cong H_q(|K|, |K| \setminus \{b\}).$$

Excise $Z = |K'| \setminus |\partial s|$: its closure lies in $|K'|$, which misses b and so lies in the open set $|K| \setminus \{b\}$, while $|K| \setminus Z = |s|$ and $(|K| \setminus \{b\}) \setminus Z = |s| \setminus \{b\}$. So Corollary 8.2 of *Singular Homology* gives

$$H_q(|s|, |s| \setminus \{b\}) \cong H_q(|K|, |K| \setminus \{b\}),$$

and the radial retraction again replaces $|s| \setminus \{b\}$ by $|\partial s|$. Finally σ_s is a homeomorphism of pairs $(\Delta^p, \partial\Delta^p) \rightarrow (|s|, |\partial s|)$ carrying ι_p to σ_s , so by Proposition 1.4 the group $H_p(|s|, |\partial s|)$ is infinite cyclic on $[\sigma_s]$. For $q \neq p$ it vanishes: Δ^p is contractible, so the reduced sequence of the pair gives $H_q(\Delta^p, \partial\Delta^p) \cong \tilde{H}_{q-1}(\partial\Delta^p) \cong \tilde{H}_{q-1}(S^{p-1})$, which is zero by Theorem 2.5.

Composing these isomorphisms, $H_q(|K|, |K'|)$ is infinite cyclic on the image of $[\sigma_s]$ for $q = p$ and zero otherwise; and the map from $H_q(Q_*)$ carries the generator η_s to σ_s . It is therefore an isomorphism, and the induction is complete.

Now let K be arbitrary and let K_0 range over its finite subcomplexes, a family directed by inclusion whose union is K . Then $C_*^{\text{or}}(K) = \varinjlim C_*^{\text{or}}(K_0)$, an oriented chain being a finite sum; and $C_*(|K|) = \varinjlim C_*(|K_0|)$, since a singular simplex has compact image and so lands in some $|K_0|$ by Lemma 5.4, and a singular chain is a finite sum. Giving each $V(K_0)$ the order restricted from λ makes Θ_λ the direct limit of the maps for the K_0 . By Proposition 2.12 of *Homology* homology commutes with direct limits, so $\Theta_{\lambda*}$ is the direct limit of the isomorphisms already proved, hence an isomorphism. $\square$

Proposition 5.6. *Let $f: K \rightarrow L$ be a simplicial map of simplicial complexes and let λ and μ be total orders on $V(K)$ and $V(L)$. Then*

$$f_\# \circ \Theta_\lambda, \quad \Theta_\mu \circ f_\#: C_*^{\text{or}}(K) \longrightarrow C_*(|L|)$$

are chain homotopic. Consequently the isomorphisms of Theorem 5.5 constitute a natural isomorphism $H_^{\text{or}} \cong H_*(|\cdot|)$ on simplicial complexes and simplicial maps.*

Proof. For a simplex s of K both maps carry an orientation of s into $C_*(|f(s)|)$: the first because $f \circ \sigma_s$ has image $|f(s)|$, the second because $f_\#$ sends the orientation either to 0 or to an orientation of $f(s)$, by Proposition 9.5 of *Homology*. Since $|f(s)|$ is convex, hence contractible, the construction of the chain homotopy in the proof of Lemma 5.3 applies verbatim, with $|f(s)|$ in place of $|s|$: one builds D by induction on p , taking $D = 0$ in degree 0, where both maps send the orientation of a vertex v to $\langle f(v) \rangle$. Naturality on homology follows, and Theorem 5.5 makes the resulting maps isomorphisms. $\square$

Exercise 5.7. Show that the chain homotopy D of Lemma 5.3 can be chosen so that it is zero on every simplex on which the two orders agree, and explain why no such D can be canonical.

6. THE KÜNNETH THEOREM FOR SINGULAR HOMOLOGY

This section supplies the geometric half of the computation of $H_*(X \times Y)$ and combines it with the algebra of §7 of *Homology*. We construct the Eilenberg–Zilber map EZ , a chain-level inverse of the Alexander–Whitney map AW , together with chain homotopies from $\text{EZ} \circ \text{AW}$ and $\text{AW} \circ \text{EZ}$ to the identities; the Künneth theorem for spaces follows at the end. Everything rests on one picture: the standard triangulation of a product of two simplices by staircase paths in a rectangular lattice. The constructions of the chain homotopies are examples of the theorem of acyclic models.

6.1. Staircase paths and the triangulation of $\Delta^p \times \Delta^q$. Fix integers $p, q \geq 0$ and consider the lattice box

$$B_{p,q} = \{0, 1, \dots, p\} \times \{0, 1, \dots, q\}.$$

Its points are matched with the vertices of $\Delta^p \times \Delta^q$ by

$$(i, j) \mapsto (e_i, e_j).$$

Definition 6.1. A *staircase path* in $B_{p,q}$ is a sequence

$$\gamma = ((i_0, j_0), (i_1, j_1), \dots, (i_{p+q}, j_{p+q}))$$

of lattice points with $(i_0, j_0) = (0, 0)$ and $(i_{p+q}, j_{p+q}) = (p, q)$ in which each point is obtained from its predecessor by one step to the right or one step up:

$$(i_k, j_k) - (i_{k-1}, j_{k-1}) \in \{(1, 0), (0, 1)\} \quad (1 \leq k \leq p+q).$$

A staircase path thus has $p+q$ steps, of which p are horizontal and q vertical, and there are $\binom{p+q}{p}$ possibilities.

Definition 6.2. For a staircase path γ let

$$\theta_\gamma: \Delta^{p+q} \longrightarrow \Delta^p \times \Delta^q$$

be the affine linear map carrying the vertex e_k of Δ^{p+q} to the vertex (e_{i_k}, e_{j_k}) of the product, $0 \leq k \leq p+q$. It is a singular $(p+q)$ -simplex of $\Delta^p \times \Delta^q$, and it is injective: the differences of its consecutive vertices are $(e_i - e_{i-1}, 0)$ at the horizontal step from column $i-1$ to column i and $(0, e_j - e_{j-1})$ at the vertical step from row $j-1$ to row j , each of these $p+q$ vectors occurring exactly once, and they are linearly independent, so the vertices of θ_γ are affinely independent.

Lemma 6.3. *The images of the θ_γ , as γ runs over the staircase paths in $B_{p,q}$, are the top-dimensional simplices of a triangulation of $\Delta^p \times \Delta^q$: they cover it, they have disjoint interiors, and any two of them meet in a common face.*

The lemma is the standard triangulation of a product of simplices. It is recorded to give the reader a picture that explains why the combinatorics given below works; nothing below uses it, the construction and its proofs being combinatorial. Here is the argument in outline: a point of $\Delta^p \times \Delta^q$ with barycentric coordinates (u, v) lies in the image of θ_γ for the path determined by merging the partial sums of the u_i and of the v_j in increasing order, and that path is unique when no partial sum of the one equals a partial sum of the other.

6.2. Signs. Each staircase path cuts the box in two. Write $A(\gamma)$ for the area of the part lying below and to the right of γ , that is, of the region bounded by the bottom edge $[0, p] \times \{0\}$, the right edge $\{p\} \times [0, q]$ and the path; it is the number of unit squares of the box on that side, so

$$A(\gamma) = \sum_{\text{horizontal steps of } \gamma} j,$$

the sum of the heights at which the horizontal steps are taken.

Definition 6.4. The *sign* of a staircase path is $\text{sgn}(\gamma) = (-1)^{A(\gamma)}$.

Thus the path that runs along the bottom edge and then up the right-hand edge has $A = 0$ and sign $+1$, and each elementary move of the path across a single square reverses the sign.

Remark 6.5. The sign is a comparison of orientations. Order the vertices of θ_γ as in Definition 6.2; this orients the simplex. Orient $\Delta^p \times \Delta^q$ by the product of the standard orientations of the two factors. Then $\text{sgn}(\gamma) = +1$ exactly when the two agree, and $\text{sgn}(\gamma) = -1$ when they are opposite. Indeed the path of sign $+1$ described above is orientation compatible with the product, and when the path moves across one square the two simplices meet in the common face obtained by deleting the vertex that moved and lie on opposite sides of it, so exactly one of them is orientation compatible with the product; this is the cancellation of interior faces in the proof of Proposition 6.14.

Remark 6.6. Staircase paths are the same thing as (p, q) -shuffles: numbering the steps $0, 1, \dots, p + q - 1$ and recording at which of them the vertical moves occur gives a q -element subset ν of $\{0, 1, \dots, p + q - 1\}$, whose complement μ has p elements; and $A(\gamma)$ is the number of inversions of the corresponding shuffle permutation, that is, the number of pairs with $\mu_b > \nu_a$. In the simplicial formulation EZ of §6.4 is therefore written

$$\text{EZ}(x \otimes y) = \sum_{(\mu, \nu)} (-1)^{\text{inv}(\mu, \nu)} (s_\nu x, s_\mu y),$$

the sum over the (p, q) -shuffles: the pairs (μ, ν) of complementary subsets of $\{0, 1, \dots, p + q - 1\}$ with $\mu = (\mu_1 < \dots < \mu_p)$ and $\nu = (\nu_1 < \dots < \nu_q)$, with $\text{inv}(\mu, \nu)$ the number of pairs (μ_b, ν_a) with $\mu_b > \nu_a$ and $s_\nu = s_{\nu_q} \dots s_{\nu_1}$, $s_\mu = s_{\mu_p} \dots s_{\mu_1}$, so that $s_\nu x$ and $s_\mu y$ both have degree $p + q$. Here s_i is the degeneracy operator $s_i z = z \circ s^i$, where $s^i: \Delta^n \rightarrow \Delta^{n-1}$, $0 \leq i \leq n - 1$, is the affine surjection repeating the i -th vertex, $s^i(e_j) = e_j$ for $j \leq i$ and $s^i(e_j) = e_{j-1}$ for $j > i$. For $p = q = 1$ the formula reads $(s_1 x, s_0 y) - (s_0 x, s_1 y)$, which is Example 6.7.

Example 6.7. For $p = q = 1$ the box is a single square and there are two paths:

$$\gamma_+ = ((0, 0), (1, 0), (1, 1)), \quad \gamma_- = ((0, 0), (0, 1), (1, 1)),$$

with $A(\gamma_+) = 0$ and $A(\gamma_-) = 1$. The two triangles $\theta_{\gamma_\pm}$ are the halves of the square $\Delta^1 \times \Delta^1$ cut by the diagonal from (e_0, e_0) to (e_1, e_1) , and the signed sum is $\theta_{\gamma_+} - \theta_{\gamma_-}$.

Example 6.8. For $p = 2$, $q = 1$ there are three paths, with heights of horizontal steps $0, 0$; $0, 1$; and $1, 1$, so with signs $+1$, -1 and $+1$. The three triangles triangulate the prism $\Delta^2 \times \Delta^1$ in the familiar way, and the alternating signs are those of the prism operator used in proving homotopy invariance, Theorem 4.1.

6.3. The Alexander–Whitney map. Let $p_X: X \times Y \rightarrow X$ and $p_Y: X \times Y \rightarrow Y$ be the projections.

Definition 6.9. The *Alexander–Whitney map* is the natural homomorphism

$$\begin{aligned} \text{AW}: C_n(X \times Y) &\longrightarrow (C_*(X) \otimes C_*(Y))_n, \\ \text{AW}(\sigma) &= \sum_{p+q=n} ((p_X \sigma) \circ [e_0, \dots, e_p]) \otimes ((p_Y \sigma) \circ [e_p, \dots, e_n]), \end{aligned}$$

the front face of the first coordinate tensor the back face of the second.

Lemma 6.10. *AW is a chain map: $\partial \circ \text{AW} = \text{AW} \circ \partial$.*

Proof. Let σ be a singular n -simplex of $X \times Y$ and abbreviate $\alpha = p_X \sigma$, $\beta = p_Y \sigma$. In $\partial \text{AW}(\sigma)$ the summand indexed by $p+q = n$ contributes the terms obtained by omitting a vertex e_i with $i \leq p$ from the front face, with sign $(-1)^i$, and those obtained by omitting a vertex e_j with $j \geq p$ from the back face, with sign $(-1)^p(-1)^{j-p} = (-1)^j$, by the sign rule of (6.1). For fixed $i < p$ or $j > p$ these are exactly the terms of $\text{AW}(\sigma \circ \delta^i)$, respectively $\text{AW}(\sigma \circ \delta^j)$, with the same signs, so they sum to $\text{AW}(\partial \sigma)$. The remaining terms are those omitting the last vertex of the front face or the first vertex of the back face; from the summand indexed by p they are

$$\begin{aligned} & (-1)^p (\alpha \circ [e_0, \dots, e_{p-1}]) \otimes (\beta \circ [e_p, \dots, e_n]), \\ & (-1)^p (\alpha \circ [e_0, \dots, e_p]) \otimes (\beta \circ [e_{p+1}, \dots, e_n]). \end{aligned}$$

The first occurs only for $p \geq 1$ and the second only for $p \leq n-1$, a 0-simplex having zero boundary in the unaugmented complex. The second is cancelled by the first term of the summand indexed by $p+1$, which is the same expression with the sign $(-1)^{p+1}$; as p runs from 0 to $n-1$ this pairs off both families completely, so they telescope to zero. $\square$

6.4. The Eilenberg–Zilber map.

Definition 6.11. The *model chain* of the pair (p, q) is

$$E_{p,q} = \sum_{\gamma} \text{sgn}(\gamma) \theta_{\gamma} \in C_{p+q}(\Delta^p \times \Delta^q),$$

the sum over the staircase paths in $B_{p,q}$.

Definition 6.12. Let X and Y be spaces. For singular simplices $\sigma: \Delta^p \rightarrow X$ and $\tau: \Delta^q \rightarrow Y$ put

$$\text{EZ}(\sigma \otimes \tau) = (\sigma \times \tau)_{\#}(E_{p,q}) = \sum_{\gamma} \text{sgn}(\gamma) (\sigma \times \tau) \circ \theta_{\gamma} \in C_{p+q}(X \times Y),$$

where $\sigma \times \tau: \Delta^p \times \Delta^q \rightarrow X \times Y$, and extend additively. This defines

$$\text{EZ}: (C_*(X) \otimes C_*(Y))_n \longrightarrow C_n(X \times Y).$$

Proposition 6.13. *EZ is natural: for maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$,*

$$(f \times g)_{\#} \circ \text{EZ} = \text{EZ} \circ (f_{\#} \otimes g_{\#}).$$

Proof. Both sides applied to $\sigma \otimes \tau$ give $\sum_{\gamma} \text{sgn}(\gamma) ((f\sigma) \times (g\tau)) \circ \theta_{\gamma}$, since $(f \times g) \circ (\sigma \times \tau) = (f\sigma) \times (g\tau)$. $\square$

Proposition 6.14. *EZ is a chain map.*

Proof. By naturality it is enough to treat the model chain, that is, to prove

$$\partial E_{p,q} = \text{EZ}(\partial(\iota_p \otimes \iota_q)) = \text{EZ}(\partial \iota_p \otimes \iota_q) + (-1)^p \text{EZ}(\iota_p \otimes \partial \iota_q),$$

the differential of the tensor product being that of (6.1).

A face of a θ_{γ} is obtained by deleting one lattice point of γ , the face omitting (i_k, j_k) occurring in $\partial \theta_{\gamma}$ with the sign $(-1)^k$, where $k = i_k + j_k$ is the number of steps taken to reach that point. Two signs are thus in play: this one, and the sign $\text{sgn}(\gamma) = (-1)^{A(\gamma)}$ of

the path. In the case when removing one point from γ removes the only occurrence in the path of a column index i , or of a row index j , the resulting sequence of pairs of vertices can be re-read as a staircase path γ' in the smaller box obtained by deleting that vertex from the corresponding factor; and in this case $A(\gamma')$ is not necessarily equal to $A(\gamma)$.

Consider first an interior point (i_k, j_k) , $0 < k < p + q$, at which the path *turns*, one of the two steps at it being horizontal and the other vertical. In this case every column index and every row index still occurs in the remaining sequence. Deleting the point produces a simplex which is the corresponding face of exactly one other staircase path, namely of the path obtained from γ by replacing that point by the opposite corner of the square that the two steps cut; those two paths differ in area by one, so their signs are opposite, while the two faces occur with the same sign $(-1)^k$ in the two boundaries, the two points being reached in the same number of steps. Hence all these terms cancel in pairs.

What survive are exactly the deletions that make a vertex of one of the two factors disappear from the remaining sequence: those at a point that is the only point of γ in its column, and those at a point that is the only point of γ in its row. These are the deletions at the first point, at the last point, and at the interior points at which the path runs straight on, and they fall into two families.

- (1) *The deleted point is the only point of γ in the column i , $0 \leq i \leq p$, and it lies at the height j ; this is the case at an interior point at which the path runs horizontally straight through that column, at the first point when the first step is horizontal, where $i = j = 0$, and at the last point when the last step is horizontal, where $i = p$ and $j = q$. In each case the remaining sequence omits the vertex e_i of the first factor and loses exactly one horizontal step, taken at the height j ; so $k = i + j$ and $A(\gamma) = A(\gamma') + j$, and the two signs combine to $(-1)^i \operatorname{sgn}(\gamma')$. For each i the deletion is a bijection onto the staircase paths of $B_{p-1,q}$, so these terms sum to $(-1)^i (\delta^i \times \operatorname{id})_{\#} E_{p-1,q}$.*
- (2) *The deleted point is the only point of γ in the row j , $0 \leq j \leq q$, and it lies in the column i ; this is the case at an interior point at which the path runs vertically straight through that row, at the first point when the first step is vertical, where $i = j = 0$, and at the last point when the last step is vertical, where $i = p$ and $j = q$. In each case the remaining sequence omits the vertex e_j of the second factor, and of the p horizontal steps the $p - i$ taken above that row are lowered by one row; so $k = i + j$ and $A(\gamma) = A(\gamma') + p - i$, and the two signs combine to $(-1)^{p+j} \operatorname{sgn}(\gamma')$. For each j the deletion is a bijection onto the staircase paths of $B_{p,q-1}$, so these terms sum to $(-1)^p (-1)^j (\operatorname{id} \times \delta^j)_{\#} E_{p,q-1}$.*

Summing the first family over i and the second over j gives

$$\begin{aligned} \partial E_{p,q} &= \sum_{i=0}^p (-1)^i (\delta^i \times \operatorname{id})_{\#} E_{p-1,q} + (-1)^p \sum_{j=0}^q (-1)^j (\operatorname{id} \times \delta^j)_{\#} E_{p,q-1} \\ &= \operatorname{EZ}(\partial \iota_p \otimes \iota_q) + (-1)^p \operatorname{EZ}(\iota_p \otimes \partial \iota_q), \end{aligned}$$

which is the asserted identity. $\square$

6.5. The Eilenberg–Zilber theorem. Theorem 6.15 allows the Künneth theorem for a tensor product of chain complexes, Theorem 7.6, to be imported into singular homology

for a product of spaces: it identifies $C_*(X \times Y)$, up to natural chain homotopy, with $C_*(X) \otimes C_*(Y)$.

Theorem 6.15 (Eilenberg–Zilber). *AW and EZ are mutually inverse natural chain homotopy equivalences: there are natural chain homotopies J on $C_*(X) \otimes C_*(Y)$ and H on $C_*(X \times Y)$, raising degree by one, with*

$$\partial J + J\partial = \text{AW} \circ \text{EZ} - \text{id}, \quad \partial H + H\partial = \text{EZ} \circ \text{AW} - \text{id}.$$

In particular

$$H_n(X \times Y) \cong H_n(C_*(X) \otimes C_*(Y)) \quad \text{for every } n.$$

The homotopy J is constructed in §6.6 and H in §6.7, where the proof is completed.

All four maps are defined on generators, so they extend R -linearly for any commutative ring R , and the identities persist: the theorem holds verbatim with coefficients in R , which is the form in which it is used in §6.8.

6.6. An explicit homotopy $\text{AW} \circ \text{EZ} \simeq \text{id}$. We construct J by induction over the models (Δ^p, Δ^q) .

A basis of $(C_*(X) \otimes C_*(Y))_n$ consists of the elements $\sigma \otimes \tau$ with σ a singular p -simplex of X , τ a singular q -simplex of Y and $p + q = n$, and each of these is $(\sigma_{\#} \otimes \tau_{\#})(\iota_p \otimes \iota_q)$. A natural transformation is therefore determined by its values on the universal elements $\iota_p \otimes \iota_q$. All the simplices involved are affine — EZ produces staircase simplices and AW front and back faces — so

$$(\text{AW EZ} - \text{id})(\iota_p \otimes \iota_q) \in (LC_*(\Delta^p) \otimes LC_*(\Delta^q))_{p+q},$$

where $LC_*(K)$ is the augmented affine complex of Definition 7.1. What replaces the choice of a bounding chain at each stage is the following contraction of a tensor product of two such complexes.

Lemma 6.16. *Let K and L be convex sets and let $b \in K$, $c \in L$. Write ε for the augmentation, $\varepsilon(\sum_i n_i [p_i]) = \sum_i n_i$ in degree 0 and $\varepsilon = 0$ in positive degrees, and define*

$$\Phi(\lambda \otimes \mu) = (b \cdot \lambda) \otimes \mu + \varepsilon(\lambda) [b] \otimes (c \cdot \mu)$$

on elementary tensors in $LC_(K) \otimes LC_*(L)$, extended additively, where $[b]$ is the 0-simplex at b ; it raises degree by one. Then*

$$\partial\Phi + \Phi\partial = \text{id} - \pi, \quad \pi(\lambda \otimes \mu) = \varepsilon(\lambda)\varepsilon(\mu) [b] \otimes [c].$$

In particular $\partial\Phi + \Phi\partial = \text{id}$ in positive total degree.

Proof. In the unaugmented complex the cone identity of Lemma 7.3 reads

$$\partial(b \cdot \lambda) + b \cdot \partial\lambda = \lambda - \varepsilon(\lambda) [b]$$

in every degree ≥ 0 . Write $\Phi = T_1 + T_2$ for the two terms of the definition. If λ has degree m , the two contributions to $\partial T_1(\lambda \otimes \mu) + T_1\partial(\lambda \otimes \mu)$ that involve $\partial\mu$ carry the signs $(-1)^{m+1}$ and $(-1)^m$ and cancel, so

$$(\partial T_1 + T_1\partial)(\lambda \otimes \mu) = [\partial(b \cdot \lambda) + b \cdot \partial\lambda] \otimes \mu = \lambda \otimes \mu - \varepsilon(\lambda) [b] \otimes \mu.$$

For T_2 note that $\partial[b] = 0$, that $\varepsilon(\partial\lambda) = 0$, and that $\varepsilon(\lambda)$ vanishes unless $m = 0$, in which case the sign $(-1)^m$ is $+1$; hence

$$(\partial T_2 + T_2 \partial)(\lambda \otimes \mu) = \varepsilon(\lambda) [b] \otimes [\partial(c \cdot \mu) + c \cdot \partial\mu] = \varepsilon(\lambda) [b] \otimes \mu - \varepsilon(\lambda) \varepsilon(\mu) [b] \otimes [c].$$

Adding the two lines gives $\partial\Phi + \Phi\partial = \text{id} - \pi$, and π vanishes except in total degree 0. $\square$

Definition 6.17. Let Φ be the operator of Lemma 6.16 on $LC_*(\Delta^p) \otimes LC_*(\Delta^q)$ for the vertices $b = e_0$ of Δ^p and $c = e_0$ of Δ^q . Define $J(\iota_0 \otimes \iota_0) = 0$ and, for $p + q \geq 1$, assuming that $J(\iota_r \otimes \iota_s)$ has been defined for all $r + s < p + q$, set

$$(6.1) \quad u = (\text{AW EZ} - \text{id})(\iota_p \otimes \iota_q) - J(\partial(\iota_p \otimes \iota_q)), \quad J(\iota_p \otimes \iota_q) = \Phi(u),$$

where

$$J(\partial(\iota_p \otimes \iota_q)) = \sum_{i=0}^p (-1)^i ((\delta^i)_\# \otimes \text{id}) J(\iota_{p-1} \otimes \iota_q) + (-1)^p \sum_{j=0}^q (-1)^j (\text{id} \otimes (\delta^j)_\#) J(\iota_p \otimes \iota_{q-1}),$$

the first sum being absent when $p = 0$ and the second when $q = 0$. For a basis element $\sigma \otimes \tau$ of $C_*(X) \otimes C_*(Y)$, with σ of degree p and τ of degree q , set $J(\sigma \otimes \tau) = (\sigma_\# \otimes \tau_\#) J(\iota_p \otimes \iota_q)$ and extend additively. This gives an inductive definition of J , which raises degree by one.

The displayed sum is J of $\partial(\iota_p \otimes \iota_q) = \partial\iota_p \otimes \iota_q + (-1)^p \iota_p \otimes \partial\iota_q$, computed from the definition: the i -th face of ι_p is $(\delta^i)_\# \iota_{p-1}$, so J of the corresponding basis element is $((\delta^i)_\# \otimes \text{id}) J(\iota_{p-1} \otimes \iota_q)$, and similarly in the second variable.

Proposition 6.18. J is natural and satisfies

$$\partial J + J\partial = \text{AW} \circ \text{EZ} - \text{id} \quad \text{on } C_*(X) \otimes C_*(Y).$$

Proof. Naturality is built into the definition: for $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$,

$$(f_\# \otimes g_\#) J(\sigma \otimes \tau) = ((f\sigma)_\# \otimes (g\tau)_\#) J(\iota_p \otimes \iota_q) = J(f\sigma \otimes g\tau).$$

Both sides of the identity are natural, so it is enough to prove it on the universal elements $\iota_p \otimes \iota_q$, and there we argue by induction on $p + q$.

For $p = q = 0$ both sides vanish: the box $B_{0,0}$ has one staircase path, so $\text{EZ}(\iota_0 \otimes \iota_0)$ is the 0-simplex (e_0, e_0) of $\Delta^0 \times \Delta^0$, of which AW returns the two coordinates, giving $\text{AW EZ}(\iota_0 \otimes \iota_0) = \iota_0 \otimes \iota_0$; and $J(\iota_0 \otimes \iota_0) = 0$ while $\partial(\iota_0 \otimes \iota_0) = 0$.

Let $p + q \geq 1$ and suppose the result is known for all $r + s < p + q$, and let u be the chain of (6.1), so that $J(\iota_p \otimes \iota_q) = \Phi(u)$. Then u is a cycle: since, by Proposition 6.14 and Lemma 6.10, EZ and AW are chain maps, we have

$$\begin{aligned} \partial u &= (\text{AW EZ} - \text{id})(\partial(\iota_p \otimes \iota_q)) - \partial J(\partial(\iota_p \otimes \iota_q)) \\ &= (\text{AW EZ} - \text{id})(\partial(\iota_p \otimes \iota_q)) - \left[(\text{AW EZ} - \text{id})(\partial(\iota_p \otimes \iota_q)) - J(\partial\partial(\iota_p \otimes \iota_q)) \right] = 0, \end{aligned}$$

the middle step being the inductive hypothesis applied in degree $p + q - 1$ to the chain $\partial(\iota_p \otimes \iota_q)$. As u has degree $p + q \geq 1$, Lemma 6.16 gives $\partial\Phi(u) = u - \Phi(\partial u) = u$, that is

$$\partial J(\iota_p \otimes \iota_q) = (\text{AW EZ} - \text{id})(\iota_p \otimes \iota_q) - J(\partial(\iota_p \otimes \iota_q)),$$

which is the assertion for $\iota_p \otimes \iota_q$. $\square$

Remark 6.19. Here $C_*(X) \otimes C_*(Y)$ and $C_*(X \times Y)$ are regarded as functors of the pair (X, Y) , with values chain complexes of abelian groups, and AW and EZ as natural transformations between them; $\text{AW} \circ \text{EZ}$ and the identity are then two natural transformations from the first functor to itself.

This is the proof mechanism of the Acyclic Models Theorem, applied with $C_*(X) \otimes C_*(Y)$ in both the source and the target role. That functor is free on the models (Δ^p, Δ^q) , which is the statement that a natural transformation may be prescribed on the $\iota_p \otimes \iota_q$ alone, and it is acyclic on them, which is Lemma 6.16; so any two natural chain maps from it to itself agreeing in degree 0 — here $\text{AW} \circ \text{EZ}$ and the identity — are naturally chain homotopic. The contraction Φ makes any cycle bound, in general by invoking the acyclicity of the models, but in this case by using the cone to the first vertex. §6.7 is the same argument for the functor $C_*(X \times Y)$.

6.7. An explicit homotopy $\text{EZ} \circ \text{AW} \simeq \text{id}$. The construction runs parallel to that of §6.6, with the cone operator of Definition 7.2 in place of Φ : the models are convex, so the cone contracts them, and the recursion again has no choices left in it.

A singular n -simplex of $X \times Y$ is a pair $\sigma = (\alpha, \beta)$ with $\alpha: \Delta^n \rightarrow X$ and $\beta: \Delta^n \rightarrow Y$, so $C_n(X \times Y)$ is free on such pairs and a natural transformation is determined by its value on the universal one. Write

$$\text{diag}_n = (\iota_n, \iota_n): \Delta^n \longrightarrow \Delta^n \times \Delta^n,$$

the diagonal, a singular n -simplex of $\Delta^n \times \Delta^n$; then $\sigma = (\alpha \times \beta) \circ \text{diag}_n$, so H is determined by its values $H(\text{diag}_n)$ on the diagonals.

Both diag_n and $\text{EZ AW}(\text{diag}_n)$ are affine chains in $\Delta^n \times \Delta^n$: AW takes front and back faces, EZ forms the staircase simplices, and all the maps involved are affine.

Our method of constructing the chain homotopy is to produce, for each n , an element $H(\text{diag}_n) \in LC_{n+1}(\Delta^n \times \Delta^n)$ by $H(\text{diag}_0) = 0$ and, for $n \geq 1$, assuming that $H(\text{diag}_{n-1})$ has been defined,

$$(6.2) \quad \begin{aligned} u &= (\text{EZ AW} - \text{id})(\text{diag}_n) - \sum_{i=0}^n (-1)^i (\delta^i \times \delta^i)_{\#} H(\text{diag}_{n-1}), \\ H(\text{diag}_n) &= (e_0, e_0) \cdot u, \end{aligned}$$

the cone being taken on the vertex (e_0, e_0) of $\Delta^n \times \Delta^n$. In the sum, $H(\text{diag}_{n-1})$ is a chain in $\Delta^{n-1} \times \Delta^{n-1}$, carried into $\Delta^n \times \Delta^n$ by $\delta^i \times \delta^i$: the face map δ^i identifies Δ^{n-1} with the i -th face of Δ^n , and one pushes forward by that identification in each of the two factors.

Definition 6.20. For a singular n -simplex $\sigma = (\alpha, \beta)$ of $X \times Y$ set $H(\sigma) = (\alpha \times \beta)_{\#} H(\text{diag}_n)$ and extend additively to $C_*(X \times Y)$. This gives an inductive definition of H .

The sum in (6.2) is $H(\partial \text{diag}_n)$: the i -th face of diag_n is $((\delta^i)_{\#} \iota_{n-1}, (\delta^i)_{\#} \iota_{n-1}) = (\delta^i \times \delta^i) \circ \text{diag}_{n-1}$, so Definition 6.20 gives $(\delta^i \times \delta^i)_{\#} H(\text{diag}_{n-1})$ for it.

Proposition 6.21. H is natural and satisfies

$$\partial H + H \partial = \text{EZ} \circ \text{AW} - \text{id} \quad \text{on } C_*(X \times Y).$$

Proof. Naturality holds by construction: if $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ then $(f \times g)_\# H(\alpha, \beta) = (f\alpha \times g\beta)_\# H(\text{diag}_n) = H(f\alpha, g\beta)$. By naturality again it is enough to prove the identity on each diag_n , and there we argue by induction on n .

For $n = 0$ both sides vanish: $\text{AW}(\text{diag}_0) = \iota_0 \otimes \iota_0$ and $\text{EZ}(\iota_0 \otimes \iota_0) = \text{diag}_0$, the box $B_{0,0}$ having one staircase path, so $\text{EZ AW}(\text{diag}_0) = \text{diag}_0$; and $H(\text{diag}_0) = 0$ while $\partial \text{diag}_0 = 0$.

Let $n \geq 1$ and suppose the result is known in degrees less than n , and let u be the chain of (6.2), so that $H(\text{diag}_n) = (e_0, e_0) \cdot u$. Then u is a cycle: since, by Proposition 6.14 and Lemma 6.10, EZ and AW are chain maps, we have

$$\begin{aligned} \partial u &= (\text{EZ AW} - \text{id})(\partial \text{diag}_n) - \partial H(\partial \text{diag}_n) \\ &= (\text{EZ AW} - \text{id})(\partial \text{diag}_n) - \left[(\text{EZ AW} - \text{id})(\partial \text{diag}_n) - H(\partial \partial \text{diag}_n) \right] = 0, \end{aligned}$$

the middle step being the inductive hypothesis applied in degree $n-1$ to the chain ∂diag_n . Since u has degree $n \geq 1$, the cone satisfies $\partial((e_0, e_0) \cdot u) = u - (e_0, e_0) \cdot \partial u$, so

$$\partial H(\text{diag}_n) = u = (\text{EZ AW} - \text{id})(\text{diag}_n) - H(\partial \text{diag}_n),$$

which is the assertion for diag_n . $\square$

Proof of Theorem 6.15. Proposition 6.18 provides the natural homotopy J with $\partial J + J\partial = \text{AW} \circ \text{EZ} - \text{id}$, and Proposition 6.21 the natural homotopy H with $\partial H + H\partial = \text{EZ} \circ \text{AW} - \text{id}$. So AW and EZ are mutually inverse natural chain homotopy equivalences, and they induce mutually inverse isomorphisms $H_n(X \times Y) \cong H_n(C_*(X) \otimes C_*(Y))$ for every n . $\square$

Remark 6.22. $\text{AW} \circ \text{EZ}$ is the identity modulo degenerate simplices. Indeed, fix $\sigma: \Delta^p \rightarrow X$ and $\tau: \Delta^q \rightarrow Y$ and a staircase path γ , and consider the term of $\text{AW}((\sigma \times \tau) \circ \theta_\gamma)$ belonging to a splitting $p' + q' = p + q$: the front p' -face of the first coordinate tensor the back q' -face of the second. The first coordinate runs through the lattice indices $i_0, \dots, i_{p+q}$, with values in $\{0, \dots, p\}$, and the second through $j_0, \dots, j_{p+q}$, with values in $\{0, \dots, q\}$. If $p' > p$ the front face repeats a vertex in the first coordinate, and if $p' < p$ the back face repeats one in the second; in either case that face factors through an s^i and the term is degenerate in one of the two factors. For $p' = p$ neither repeats exactly when $i_k = k$ for $k \leq p$ and $j_{p+l} = l$ for $l \leq q$, that is, exactly for the path along the bottom edge and then up the right-hand edge, whose sign is $+1$ and whose term is $\sigma \otimes \tau$. The degenerate terms are genuinely there: for $X = Y$ a point and $p = q = 1$ the two staircase simplices coincide, so $\text{EZ}(\iota_1 \otimes \iota_1) = 0$ while $\iota_1 \otimes \iota_1 \neq 0$.

Remark 6.23. The recursions (6.1) and (6.2) are explicit in the sense that each computes its value at one model from the values at the models below, with no choices, starting from $J(\iota_0 \otimes \iota_0) = 0$ and $H(\text{diag}_0) = 0$; unwinding the second expresses $H(\text{diag}_n)$ as an iterated cone on chains built from staircase simplices. Neither is the classical closed formula: Eilenberg and Mac Lane wrote down a natural homotopy given by a signed sum indexed by shuffle-like data, which can moreover be arranged to satisfy side conditions such as

$$H \circ H = 0, \quad \text{AW} \circ H = 0, \quad H \circ \text{EZ} = 0,$$

making $(\text{AW}, \text{EZ}, H)$ a contraction in the sense of homological perturbation theory. Neither the homotopy of Definition 6.20 nor that of Definition 6.17 satisfies such conditions in general, and for a first pass it is the existence, not the shape, that is wanted.

Exercise 6.24. Write out $E_{2,1}$ explicitly, as a signed sum of three singular 3-simplices of $\Delta^2 \times \Delta^1$, and verify the identity of Proposition 6.14 in this case by computing both sides.

Exercise 6.25. Compute $\text{AW} \circ \text{EZ}(\iota_1 \otimes \iota_1)$ in $C_*(\Delta^1) \otimes C_*(\Delta^1)$ and check that it is not $\iota_1 \otimes \iota_1$; then compute $J(\iota_1 \otimes \iota_1)$ from (6.1) and verify $\partial J + J\partial = \text{AW} \circ \text{EZ} - \text{id}$ on $\iota_1 \otimes \iota_1$. Verify also that $\text{EZ} \circ \text{AW}$ is not the identity on $C_2(\Delta^1 \times \Delta^1)$.

6.8. The Künneth theorem for spaces.

Theorem 6.26. *Let X and Y be spaces and let R be a principal ideal domain. For every n there is a short exact sequence*

$$\begin{aligned} 0 \longrightarrow \bigoplus_{p+q=n} H_p(X; R) \otimes H_q(Y; R) &\longrightarrow H_n(X \times Y; R) \\ &\longrightarrow \bigoplus_{p+q=n-1} \text{Tor}(H_p(X; R), H_q(Y; R)) \longrightarrow 0, \end{aligned}$$

natural in X and Y , and it splits, although not naturally. The first map sends $[x] \otimes [y]$ to $[\text{EZ}(x \otimes y)]$, the homology class of the product of the two cycles.

Proof. The complexes $C_*(X; R)$ and $C_*(Y; R)$ are free, so Theorem 7.6 applies to them, and Theorem 6.15 identifies $H_n(C_*(X; R) \otimes C_*(Y; R))$ with $H_n(X \times Y; R)$ naturally. $\square$

Corollary 6.27. *If F is a field then for every n*

$$H_n(X \times Y; F) \cong \bigoplus_{p+q=n} H_p(X; F) \otimes_F H_q(Y; F).$$

Proof. Every module over a field is free, so all Tor terms vanish by Proposition 7.3(ii). $\square$

Corollary 6.28. *If $H_*(Y; R)$ is free in every degree — for instance if Y is a sphere, or a point — then $H_n(X \times Y; R) \cong \bigoplus_{p+q=n} H_p(X; R) \otimes H_q(Y; R)$ for every n .*

Proof. The Tor terms are $\text{Tor}(H_p(X; R), H_q(Y; R))$, whose second variable is free, so they vanish by Proposition 7.3(v). $\square$

Example 6.29. Taking $R = \mathbb{Z}$ and using Theorem 2.5, for $m, n \geq 1$

$$H_i(S^m \times S^n) \cong \begin{cases} \mathbb{Z}, & i = 0, i = m + n, \\ \mathbb{Z}^2, & i = m = n, \\ \mathbb{Z}, & i = m \text{ or } i = n, m \neq n, \\ 0, & \text{otherwise.} \end{cases}$$

In particular the torus $T^2 = S^1 \times S^1$ has $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}^2$, $H_2 = \mathbb{Z}$, and $H_i = 0$ for $i > 2$. Iterating, $H_*(T^k)$ is free with $H_i(T^k) \cong \mathbb{Z}^{\binom{k}{i}}$.

Example 6.30. The Tor term is not decoration. Take $X = Y = \mathbb{R}P^2$, the real projective plane, for which $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}/2$ and $H_i = 0$ for $i \geq 2$. Then all the tensor terms in degree 3 vanish, while

$$H_3(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \text{Tor}(H_1(\mathbb{R}P^2), H_1(\mathbb{R}P^2)) = \text{Tor}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$$

by Proposition 7.3(iv), so the product has homology in a degree where the naive guess predicts none.

Exercise 6.31. Verify Theorem 6.26 directly for Y a point, and check that the sequence then reduces to an identity.

Exercise 6.32. Let X and Y be spaces whose homology is finitely generated and vanishes above some degree. Show that the Euler characteristics satisfy $\chi(X \times Y) = \chi(X)\chi(Y)$, and explain why the Tor terms do not affect the count.