

## SINGULAR COHOMOLOGY

These notes continue *Homology* and *Singular Homology*. From the singular chain complex  $C_*(X)$  of a space we manufacture, by dualizing, a cochain complex  $C^*(X; G)$ , and its cohomology  $H^*(X; G)$ . We see new structure coming from the diagonal map  $X \rightarrow X \times X$ , which endows cohomology with a ring structure<sup>1</sup>. References to the companions are given as *Theorem such-and-such of Homology*, and similarly for *Singular Homology*; all other references are internal. All maps of spaces are continuous.

### 1. SINGULAR COCHAINS AND SINGULAR COHOMOLOGY

Throughout this section  $G$  is an abelian group.

**Definition 1.1.** Let  $X$  be a space. The group of *singular  $n$ -cochains* of  $X$  with coefficients in  $G$  is

$$C^n(X; G) = \text{Hom}(C_n(X), G) \quad (n \geq 0),$$

with  $C^n(X; G) = 0$  for  $n < 0$ , and the *coboundary*  $\delta: C^n(X; G) \rightarrow C^{n+1}(X; G)$  is  $\delta = \text{Hom}(\partial, G)$ , that is

$$(\delta\varphi)(c) = \varphi(\partial c) \quad \text{for } c \in C_{n+1}(X).$$

Since  $\partial\partial = 0$  we have  $\delta\delta = 0$ , so  $(C^*(X; G), \delta)$  is a cochain complex in the sense of Remark 2.9 of *Homology*. Its cohomology

$$H^n(X; G) = \ker(\delta: C^n \rightarrow C^{n+1}) / \text{im}(\delta: C^{n-1} \rightarrow C^n)$$

is the *singular cohomology* of  $X$  with coefficients in  $G$ . Elements of the kernel are *cocycles*, elements of the image are *coboundaries*, and the class of a cocycle  $\varphi$  is written  $[\varphi]$ .

*Remark 1.2.* Because  $C_n(X)$  is free abelian on  $\text{Sing}_n(X)$ , a cochain is nothing but an arbitrary function  $\text{Sing}_n(X) \rightarrow G$ , with no continuity or coherence required. On a singular  $(n+1)$ -simplex  $\sigma$  the coboundary is

$$(1.1) \quad (\delta\varphi)(\sigma) = \sum_{i=0}^{n+1} (-1)^i \varphi(\sigma \circ \delta^i),$$

the alternating sum of the values of  $\varphi$  on the faces. Some authors insert a sign  $(-1)^{n+1}$  in the definition of  $\delta$ ; this changes nothing, since it replaces  $\delta$  by  $\pm\delta$  in each degree, and we shall not do it.

*Remark 1.3.* In general, the functor  $\text{Hom}(-, G)$  does not send short exact sequences to short exact sequences, and hence  $H^n(X; G)$  is not in general isomorphic to  $\text{Hom}(H_n(X), G)$ . Over a field  $F$  it is:  $\text{Hom}(-, F)$  is exact on  $F$ -vector spaces, so  $H^n(X; F) \cong H_n(X; F)^\vee$ . The precise relation is the universal coefficient theorem, Theorem 8.6 of *Homology*.

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<sup>1</sup>In homology the diagonal map produces a co-ring structure, which is less familiar.

### 1.1. Functoriality.

**Definition 1.4.** For a map  $f: X \rightarrow Y$  let  $f^\# = \text{Hom}(f_\#, G): C^n(Y; G) \rightarrow C^n(X; G)$ , so that  $(f^\# \varphi)(c) = \varphi(f_\# c)$ ; on a singular simplex,  $(f^\# \varphi)(\sigma) = \varphi(f \circ \sigma)$ .

**Proposition 1.5.**  $f^\#$  is a map of cochain complexes,  $\delta f^\# = f^\# \delta$ , and  $(g \circ f)^\# = f^\# g^\#$ ,  $\text{id}^\# = \text{id}$ . Hence  $X \mapsto H^n(X; G)$  and  $f \mapsto H^n(f) = f^*$  is a contravariant functor  $\mathbf{Top} \rightarrow \mathbf{Ab}$ .

*Proof.* Applying  $\text{Hom}(-, G)$  to the identity  $\partial f_\# = f_\# \partial$  of Proposition 1.4 of *Singular Homology* gives  $f^\# \delta = \delta f^\#$ , and  $\text{Hom}(-, G)$  is itself a contravariant functor, whence the two remaining identities. A cochain map carries cocycles to cocycles and coboundaries to coboundaries, so it induces maps on cohomology, functorially.  $\square$

The following lemma is the mechanism by which every homotopy-theoretic statement about  $C_*(X)$  is inherited by  $C^*(X; G)$ .

**Lemma 1.6.** Let  $u, v: C_* \rightarrow D_*$  be chain maps and let  $P$  be a chain homotopy from  $u$  to  $v$ , that is  $\partial P + P\partial = u - v$ . Put  $Q = \text{Hom}(P, G)$ , so that  $Q\varphi = \varphi \circ P$  shifts degree by  $-1$ . Then

$$\delta Q + Q\delta = \text{Hom}(u, G) - \text{Hom}(v, G).$$

Thus chain homotopic maps dualize to cochain homotopic maps, and chain homotopy equivalences dualize to cochain homotopy equivalences.

*Proof.* For  $\varphi \in \text{Hom}(D_n, G)$  we have  $Q\varphi = \varphi \circ P$  on  $D_{n-1}$ , and

$$(\delta Q\varphi + Q\delta\varphi)(c) = \varphi(P\partial c) + \varphi(\partial P c) = \varphi((u - v)c)$$

for  $c \in C_n$ , which is the assertion. The last statement follows by applying this to the two homotopies exhibiting a chain homotopy equivalence.  $\square$

**Theorem 1.7.** Homotopic maps  $f \simeq g: X \rightarrow Y$  induce the same homomorphism  $H^n(Y; G) \rightarrow H^n(X; G)$  for every  $n$ . Consequently a homotopy equivalence induces isomorphisms on cohomology, and a contractible space has  $H^0(X; G) = G$  and  $H^n(X; G) = 0$  for  $n \neq 0$ .

*Proof.* By Theorem 4.1 of *Singular Homology* the prism operator is a chain homotopy between  $f_\#$  and  $g_\#$ ; dualize it by Lemma 1.6 and pass to cohomology. For the last statement, a contractible space has the cochain complex of a point up to cochain homotopy equivalence, and for a point  $P$  the complex  $C^*(P; G)$  is  $G \rightarrow G \rightarrow G \rightarrow \dots$  with differentials alternately 0 and  $\text{id}$ , as in Proposition 3.1 of *Singular Homology*.  $\square$

**Proposition 1.8.**  $H^0(X; G)$  is the group of all functions  $\pi_0(X) \rightarrow G$ , that is, of functions  $X \rightarrow G$  that are constant on path components.

*Proof.* A 0-cochain is a function  $\varphi: X \rightarrow G$ , and there are no coboundaries in degree 0. For a singular 1-simplex  $\tau$ ,  $\partial\tau = \tau(e_1) - \tau(e_0)$ , so  $(\delta\varphi)(\tau) = \varphi(\tau(e_1)) - \varphi(\tau(e_0))$ . Thus  $\varphi$  is a cocycle exactly when it takes the same value at the two ends of every path, that is, when it is constant on path components.  $\square$

*Remark 1.9.* Compare Proposition 3.2 of *Singular Homology*:  $H_0(X)$  is the free abelian group on  $\pi_0(X)$ , a direct *sum*, whereas  $H^0(X; G)$  is a direct *product* of copies of  $G$ . Dualizing turns sums into products. We see this also at the chain and cochain level:  $C_n(X)$  is a direct sum of copies of  $\mathbb{Z}$ , one for each singular simplex, whereas  $C^n(X; G)$  is the corresponding direct product of copies of  $G$ . When  $\text{Sing}_n(X)$  is infinite the product  $C^n(X; \mathbb{Z}) = \prod_{\text{Sing}_n(X)} \mathbb{Z}$  is not free abelian, by a theorem of Baer and Specker.

## 1.2. Relative cohomology and the long exact sequence.

**Definition 1.10.** For a pair  $(X, A)$  put  $C^n(X, A; G) = \text{Hom}(C_n(X, A), G)$  with the coboundary induced by  $\partial$ , and  $H^n(X, A; G) = H^n(C^*(X, A; G))$ . Since  $C_n(X, A) = C_n(X)/C_n(A)$ , the inclusion  $C^n(X, A; G) \subseteq C^n(X; G)$  identifies the relative cochains with the cochains of  $X$  that vanish on every singular simplex with image in  $A$ .

**Theorem 1.11.** For every pair  $(X, A)$  there is a long exact sequence

$$\cdots \longrightarrow H^n(X, A; G) \longrightarrow H^n(X; G) \longrightarrow H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \longrightarrow \cdots,$$

natural for maps of pairs, in which the unlabelled maps are induced by the inclusions.

*Proof.* The short exact sequence  $0 \rightarrow C_*(A) \rightarrow C_*(X) \rightarrow C_*(X, A) \rightarrow 0$  of Proposition 6.2 of *Singular Homology* is split in each degree. A degreewise split short exact sequence stays exact under  $\text{Hom}(-, G)$ , so

$$0 \longrightarrow C^*(X, A; G) \longrightarrow C^*(X; G) \longrightarrow C^*(A; G) \longrightarrow 0$$

is a short exact sequence of cochain complexes; now apply Theorem 4.3 of *Homology*, read through the relabelling  $C^n = C_{-n}$  of Remark 2.9 there, and Proposition 4.4 for naturality.  $\square$

*Remark 1.12.* The splitting is essential:  $\text{Hom}(-, G)$  does not preserve exactness in general, and for a non-split surjection the dual map need not be injective. Here  $C_n(X)$  is free on  $\text{Sing}_n(X)$  and  $\text{Sing}_n(A) \subseteq \text{Sing}_n(X)$ , which is what produces the splitting.

**Exercise 1.13.** Show that a relative cocycle in  $C^n(X, A; G)$  is the same thing as a cochain  $\varphi \in C^n(X; G)$  with  $\varphi|_{C_n(A)} = 0$  and  $\delta\varphi = 0$ , and that  $\delta$  in Theorem 1.11 is computed by extending a cocycle of  $A$  arbitrarily to  $X$  and taking its coboundary.

**Exercise 1.14.** Show that maps of pairs homotopic through maps of pairs induce the same homomorphisms on relative cohomology, by dualizing the proof of Theorem 6.6 of *Singular Homology*.

## 2. THE ALEXANDER–WHITNEY MAP AND THE CUP PRODUCT

From now on  $R$  is a commutative ring with unit, and cochains have coefficients in  $R$ . Then  $C^n(X; R)$  is an  $R$ -module and  $H^*(X; R) = \bigoplus_{n \geq 0} H^n(X; R)$  is a graded  $R$ -module.

The extra structure on cohomology comes from the diagonal  $\Delta: X \rightarrow X \times X$ ,  $x \mapsto (x, x)$ . To use it one needs a chain-level model for the diagonal, and this is what the Alexander–Whitney map provides. Recall from §6 of *Homology* that the tensor product complex has  $(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$  with  $\partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y$  for  $x \in C_p$ .

**Definition 2.1.** Let  $\sigma$  be a singular  $n$ -simplex of  $X$  and let  $p + q = n$ . The *front  $p$ -face* and the *back  $q$ -face* of  $\sigma$  are the singular simplices

$$\sigma \circ [e_0, \dots, e_p] \in \text{Sing}_p(X), \quad \sigma \circ [e_p, \dots, e_n] \in \text{Sing}_q(X),$$

where  $[e_{i_0}, \dots, e_{i_k}]: \Delta^k \rightarrow \Delta^n$  is the affine linear map of §4 of *Singular Homology*. The two faces meet exactly in the vertex  $e_p$ .

**Definition 2.2.** The *Alexander–Whitney map* is the homomorphism

$$\text{AW}: C_n(X) \longrightarrow (C_*(X) \otimes C_*(X))_n, \quad \text{AW}(\sigma) = \sum_{0 \leq p \leq n} (\sigma \circ [e_0, \dots, e_p]) \otimes (\sigma \circ [e_p, \dots, e_n]),$$

defined on generators and extended additively.

*Remark 2.3.* AW is a natural chain map, and on a 0-simplex it is  $\sigma \mapsto \sigma \otimes \sigma$ , which is the chain-level record of  $\Delta(x) = (x, x)$ . So AW is a *diagonal approximation*: a chain-level substitute for the diagonal map, which cannot be used directly because  $C_*(X \times X)$  is not  $C_*(X) \otimes C_*(X)$ . Its being a chain map is the content of Lemma 2.6 below, read on chains rather than on cochains; note that AW is not cocommutative — interchanging the two factors gives a different chain map — and §3 is devoted to measuring the difference.

**Definition 2.4.** For  $\varphi \in C^p(X; R)$  and  $\psi \in C^q(X; R)$  the *cup product*  $\varphi \smile \psi \in C^{p+q}(X; R)$  is defined on a singular  $(p+q)$ -simplex  $\sigma$  by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma \circ [e_0, \dots, e_p]) \cdot \psi(\sigma \circ [e_p, \dots, e_{p+q}]),$$

the product being taken in  $R$ . Equivalently  $\varphi \smile \psi = (\varphi \otimes \psi) \circ \text{AW}$ , where  $\varphi \otimes \psi$  is the homomorphism  $(C_* \otimes C_*)_{p+q} \rightarrow R$  that is  $\varphi \cdot \psi$  on the summand  $C_p \otimes C_q$  and 0 on the others.

*Remark 2.5.* The evaluation  $(\varphi \otimes \psi)(x \otimes y) = \varphi(x)\psi(y)$  used here carries no sign. That is the correct partner for the unsigned coboundary  $(\delta\varphi)(c) = \varphi(\partial c)$  of Definition 1.1: with the graded dual  $(\delta\varphi)(c) = (-1)^{|\varphi|+1}\varphi(\partial c)$  one would have to use  $(\varphi \otimes \psi)(x \otimes y) = (-1)^{|\psi||x|}\varphi(x)\psi(y)$  instead, the two conventions differing by the rescaling  $\varphi \mapsto (-1)^{p(p+1)/2}\varphi$  on  $C^p$ . See Remark 6.7 of *Homology*. It is the unsigned convention that gives the Leibniz rule in the form (2.1) below.

**Lemma 2.6.** For  $\varphi \in C^p(X; R)$  and  $\psi \in C^q(X; R)$ ,

$$(2.1) \quad \delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^p \varphi \smile \delta\psi.$$

*Proof.* Let  $\sigma$  be a singular  $(p+q+1)$ -simplex and abbreviate  $\sigma_{[\cdot]} = \sigma \circ [\cdot]$ . By the definition of the cup product and (1.1),

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{i=0}^{p+1} (-1)^i \varphi(\sigma_{[e_0, \dots, \widehat{e_i}, \dots, e_{p+1}]}) \psi(\sigma_{[e_{p+1}, \dots, e_{p+q+1}]})$$

and

$$(-1)^p (\varphi \smile \delta\psi)(\sigma) = \sum_{i=p}^{p+q+1} (-1)^i \varphi(\sigma_{[e_0, \dots, e_p]}) \psi(\sigma_{[e_p, \dots, \widehat{e_i}, \dots, e_{p+q+1}]})$$

where in the second line the sign  $(-1)^p$  has been absorbed by re-indexing the sum defining  $\delta\psi$ , whose  $j$ -th term omits the vertex  $e_{p+j}$ . In the first sum the term  $i = p+1$  is

$$\varphi(\sigma_{[e_0, \dots, e_p]}) \psi(\sigma_{[e_{p+1}, \dots, e_{p+q+1}]}) \cdot (-1)^{p+1},$$

and in the second sum the term  $i = p$  is the same expression with the opposite sign; they cancel. What remains are the terms with  $i \leq p$  in the first sum and those with  $i \geq p+1$  in the second, one for each vertex of  $\sigma$ : the term indexed by  $i$  omits  $e_i$  from the front face if  $i \leq p$  and from the back face if  $i \geq p+1$ , and in either case it equals  $(-1)^i$  times the value of  $\varphi \smile \psi$  on  $\sigma \circ \delta^i$ . Summing, what is left is  $\sum_i (-1)^i (\varphi \smile \psi)(\sigma \circ \delta^i) = (\varphi \smile \psi)(\partial\sigma) = \delta(\varphi \smile \psi)(\sigma)$ .  $\square$

**Theorem 2.7.** The cup product of cochains induces a well-defined product

$$H^p(X; R) \otimes_R H^q(X; R) \longrightarrow H^{p+q}(X; R), \quad [\varphi] \smile [\psi] = [\varphi \smile \psi].$$

It is  $R$ -bilinear and associative, the class  $1 \in H^0(X; R)$  of the cochain taking the value 1 on every singular 0-simplex is a two-sided unit, and for every map  $f: X \rightarrow Y$  the homomorphism  $f^*$  satisfies  $f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta$  and  $f^*(1) = 1$ . Thus  $H^*(X; R)$  is a graded  $R$ -algebra and  $H^*(-; R)$  is a contravariant functor to graded  $R$ -algebras.

*Proof.* If  $\delta\varphi = 0$  and  $\delta\psi = 0$  then  $\delta(\varphi \smile \psi) = 0$  by (2.1), so the cup product of cocycles is a cocycle. If moreover  $\varphi = \delta\eta$  then, again by (2.1),  $\delta(\eta \smile \psi) = \delta\eta \smile \psi = \varphi \smile \psi$ , so  $\varphi \smile \psi$  is a coboundary; and if  $\psi = \delta\theta$  then  $\delta((-1)^p\varphi \smile \theta) = \varphi \smile \psi$  likewise. Hence the class of  $\varphi \smile \psi$  depends only on  $[\varphi]$  and  $[\psi]$ , and the product is well defined; bilinearity is clear from the formula.

Associativity holds already on cochains: for  $\varphi, \psi, \eta$  of degrees  $p, q, r$  and  $\sigma$  a singular  $(p+q+r)$ -simplex, both  $((\varphi \smile \psi) \smile \eta)(\sigma)$  and  $(\varphi \smile (\psi \smile \eta))(\sigma)$  equal

$$\varphi(\sigma \circ [e_0, \dots, e_p]) \psi(\sigma \circ [e_p, \dots, e_{p+q}]) \eta(\sigma \circ [e_{p+q}, \dots, e_{p+q+r}]).$$

For the unit, let 1 denote the 0-cochain with  $1(\sigma) = 1$  for every 0-simplex; it is a cocycle by Proposition 1.8, and  $(1 \smile \varphi)(\sigma) = \varphi(\sigma)$  and  $(\varphi \smile 1)(\sigma) = \varphi(\sigma)$  directly from the definition, the front 0-face and the back 0-face being vertices of  $\sigma$ . Finally  $f_\#$  commutes with passage to front and back faces, since  $(f \circ \sigma) \circ [e_0, \dots, e_p] = f \circ (\sigma \circ [e_0, \dots, e_p])$ , whence  $f^\#(\varphi \smile \psi) = f^\#\varphi \smile f^\#\psi$  on cochains; passing to cohomology gives the last assertion.  $\square$

*Remark 2.8.* The same formula defines cup products  $H^p(X, A; R) \otimes H^q(X; R) \rightarrow H^{p+q}(X, A; R)$  and  $H^p(X, A; R) \otimes H^q(X, B; R) \rightarrow H^{p+q}(X, A \cup B; R)$  for suitable pairs, since a cochain vanishing on chains of  $A$  cups to a cochain vanishing on chains of  $A$ . We shall use the absolute case only.

**Exercise 2.9.** Let  $X$  be path connected and  $\varphi$  a 0-cocycle. Show that  $[\varphi]$  is a multiple of 1, say  $[\varphi] = r \cdot 1$  with  $r \in R$ , and deduce that the product  $H^0 \otimes H^n \rightarrow H^n$  given by cupping with  $[\varphi]$  is multiplication by  $r$ .

**Exercise 2.10.** Write  $AW'$  for the Alexander–Whitney map  $C_*(X \times X) \rightarrow C_*(X) \otimes C_*(X)$  of Definition 6.9 of *Further Properties of Singular Homology*. Show that  $AW = AW' \circ \Delta_\#$ , where  $\Delta_\#$  is induced by the diagonal  $\Delta: X \rightarrow X \times X$ , and deduce that the cup product of cochains is given by  $\varphi \smile \psi = (\varphi \otimes \psi) \circ AW' \circ \Delta_\#$ .

### 3. GRADED COMMUTATIVITY

The cup product is not commutative on cochains: the front and back faces of a simplex play different roles, and  $\varphi \smile \psi$  and  $\psi \smile \varphi$  are genuinely different functions. On cohomology the asymmetry disappears up to a sign. The reason is that  $AW$  and the map obtained from it by interchanging the two tensor factors are chain homotopic, and the homotopy can be written down explicitly on the faces of  $\Delta^n$ .

Throughout this section we abbreviate, for a singular  $n$ -simplex  $\sigma$  of  $X$  and  $0 \leq k_0 < k_1 < \dots < k_r \leq n$ ,

$$\sigma\langle k_0, k_1, \dots, k_r \rangle = \sigma \circ [e_{k_0}, e_{k_1}, \dots, e_{k_r}] \in \text{Sing}_r(X),$$

so that  $\sigma\langle 0, 1, \dots, n \rangle = \sigma$  and, writing  $AW_k(\sigma) = \sigma\langle 0, \dots, k \rangle \otimes \sigma\langle k, \dots, n \rangle$  for the  $k$ -th summand of Definition 2.2,

$$AW(\sigma) = \sum_{k=0}^n AW_k(\sigma).$$

**Definition 3.1.** The *switch of factors* is the homomorphism

$$\tau: (C_*(X) \otimes C_*(X))_n \longrightarrow (C_*(X) \otimes C_*(X))_n, \quad \tau(x \otimes y) = (-1)^{pq} y \otimes x \quad (x \in C_p, y \in C_q).$$

The Koszul sign is what makes  $\tau$  a chain map: for  $x \in C_p$  and  $y \in C_q$  both  $\partial\tau(x \otimes y)$  and  $\tau\partial(x \otimes y)$  equal  $(-1)^{pq}\partial y \otimes x + (-1)^{pq+q}y \otimes \partial x$ .

**Definition 3.2.** Let  $\sigma$  be a singular  $n$ -simplex of  $X$ . For  $0 \leq i < j \leq n$  put

$$C_\sigma(i, j) = \sigma\langle 0, \dots, i, j, \dots, n \rangle \otimes \sigma\langle i, \dots, j \rangle,$$

the tensor product of the face spanned by the vertices outside the open interval  $(i, j)$  and the face spanned by the vertices in the closed interval  $[i, j]$ , and define

$$P: C_n(X) \longrightarrow (C_*(X) \otimes C_*(X))_{n+1}, \quad P(\sigma) = \sum_{0 \leq i < j \leq n} \varepsilon_{ij} C_\sigma(i, j),$$

where

$$\varepsilon_{ij} = (-1)^{j(n-j)+i(n-j+1)},$$

on generators and extend additively.

The two factors of  $C_\sigma(i, j)$  have dimensions  $i + n - j + 1$  and  $j - i$ , which add up to  $n + 1$ ; so  $P$  does raise degree by one, and the two faces meet exactly in the two vertices  $e_i$  and  $e_j$ .

*Remark 3.3.* The two summands in the exponent of  $\varepsilon_{ij}$  have a meaning:  $j(n - j)$  is the Koszul sign by which  $\tau$  acts on  $\text{AW}_j(\sigma)$ , and  $i(n - j + 1)$  is the sign of the shuffle moving the front block  $e_0, \dots, e_i$  of the first factor past the back block  $e_j, \dots, e_n$ . For the sign manipulations of the next proof it is convenient to have the equivalent form

$$(3.1) \quad j(n - j) + i(n - j + 1) \equiv ij + (n + 1)(i + j) \pmod{2},$$

which follows from  $-j^2 \equiv j$  and  $-ij \equiv ij$ .

**Theorem 3.4.** *As homomorphisms  $C_n(X) \rightarrow (C_*(X) \otimes C_*(X))_n$ ,*

$$\partial \circ P + P \circ \partial = \tau \circ \text{AW} - \text{AW}.$$

*That is,  $P$  is a natural chain homotopy from  $\text{AW}$  to  $\tau \circ \text{AW}$ .*

*Proof.* Naturality is clear from the formula, since  $(f \circ \sigma)(\cdot) = f \circ \sigma(\cdot)$ , so it suffices to evaluate the two sides on a singular  $n$ -simplex  $\sigma$ . All sign identities below are congruences of exponents modulo 2, and we use  $\varepsilon_{ij}$  in the form (3.1), writing  $\varepsilon_{ij}^{(n)}$  when the dimension has to be recorded. Abbreviate

$$A_{ij} = \sigma\langle 0, \dots, i, j, \dots, n \rangle, \quad B_{ij} = \sigma\langle i, \dots, j \rangle, \quad a_{ij} = \deg A_{ij} = i + n - j + 1.$$

By the definition of the differential of the tensor product complex,

$$\partial P(\sigma) = \sum_{0 \leq i < j \leq n} \varepsilon_{ij} \left( \partial A_{ij} \otimes B_{ij} + (-1)^{a_{ij}} A_{ij} \otimes \partial B_{ij} \right).$$

The vertex  $e_r$  with  $r \leq i$  stands in place  $r$  of  $A_{ij}$ , the vertex  $e_s$  with  $s \geq j$  stands in place  $i + 1 + s - j$ , and the vertex  $e_t$  with  $i \leq t \leq j$  stands in place  $t - i$  of  $B_{ij}$ . Hence

$$\partial A_{ij} = \sum_{r=0}^i (-1)^r \sigma\langle 0, \dots, \widehat{r}, \dots, i, j, \dots, n \rangle + \sum_{s=j}^n (-1)^{i+1+s-j} \sigma\langle 0, \dots, i, j, \dots, \widehat{s}, \dots, n \rangle,$$

$$\partial B_{ij} = \sum_{t=i}^j (-1)^{t-i} \sigma\langle i, \dots, \widehat{t}, \dots, j \rangle.$$

We split the resulting terms into two kinds. A term is *outer* if the deleted vertex is  $e_r$  with  $r < i$ , or  $e_s$  with  $s > j$ , or  $e_t$  with  $i < t < j$ : such a vertex belongs to only one of the two faces  $A_{ij}$ ,  $B_{ij}$ , so it disappears from the boundary term altogether. The remaining terms are the four families coming from  $r = i$ ,  $s = j$ ,  $t = i$ ,  $t = j$ ; there the deleted vertex is one of the two vertices shared by  $A_{ij}$  and  $B_{ij}$ , so it survives in the boundary. All  $n + 1$  vertices appear in these boundary terms, and one of them appears twice, whereas an outer term involves only  $n$  of the vertices; so no cancellation can occur between the two kinds.

*Step 1: the outer terms of  $\partial P(\sigma)$  cancel  $P(\partial\sigma)$ .* Fix  $m$  and consider  $P(\sigma \circ \delta^m)$ , where  $\sigma \circ \delta^m = \sigma\langle 0, \dots, \widehat{m}, \dots, n \rangle$ . In  $\sigma \circ \delta^m$  the vertex  $e_k$  with  $k < m$  stands in place  $k$  and the vertex  $e_k$  with  $k > m$  stands in place  $k - 1$ . Thus the terms of  $(-1)^m P(\sigma \circ \delta^m)$  are indexed by the pairs  $i < j$  of vertices other than  $e_m$ , and for such a pair the corresponding element is obtained from  $C_\sigma(i, j)$  by deleting  $e_m$  from the first factor if  $m < i$  or  $m > j$ , and from the second factor if  $i < m < j$ . This is exactly the indexing of the outer terms, and the correspondence is bijective. It remains to compare signs. If  $m < i$  the outer term carries  $\varepsilon_{ij}^{(n)}(-1)^m$  and the term of  $(-1)^m P(\sigma \circ \delta^m)$  carries  $(-1)^m \varepsilon_{i-1, j-1}^{(n-1)}$ , and

$$(ij + (n+1)(i+j)) - ((i-1)(j-1) + n(i+j-2)) \equiv (i+j) + (i+j) + 1 \equiv 1.$$

If  $m > j$  the two signs are  $\varepsilon_{ij}^{(n)}(-1)^{i+1+m-j}$  and  $(-1)^m \varepsilon_{ij}^{(n-1)}$ , and

$$(ij + (n+1)(i+j) + i + 1 + m - j) - (m + ij + n(i+j)) \equiv (i+j) + i + j + 1 \equiv 1.$$

If  $i < m < j$  the two signs are  $\varepsilon_{ij}^{(n)}(-1)^{a_{ij}+m-i}$  and  $(-1)^m \varepsilon_{i, j-1}^{(n-1)}$ , and

$$(ij + (n+1)(i+j) + i + n - j + 1 + m - i) - (m + i(j-1) + n(i+j-1)) \equiv 1.$$

In each case the two signs are opposite, so the outer part of  $\partial P(\sigma)$  equals  $-P(\partial\sigma)$ .

*Step 2: the four remaining families.* Write

$$\begin{aligned} T_1(i, j) &= \sigma\langle 0, \dots, i-1, j, \dots, n \rangle \otimes \sigma\langle i, \dots, j \rangle, & \text{from } r = i, \\ T_2(i, j) &= \sigma\langle 0, \dots, i, j+1, \dots, n \rangle \otimes \sigma\langle i, \dots, j \rangle, & \text{from } s = j, \\ T_3(i, j) &= A_{ij} \otimes \sigma\langle i+1, \dots, j \rangle, & \text{from } t = i, \\ T_4(i, j) &= A_{ij} \otimes \sigma\langle i, \dots, j-1 \rangle, & \text{from } t = j, \end{aligned}$$

with respective coefficients

$$\varepsilon_{ij}(-1)^i, \quad \varepsilon_{ij}(-1)^{i+1}, \quad \varepsilon_{ij}(-1)^{a_{ij}}, \quad \varepsilon_{ij}(-1)^{a_{ij}+j-i}.$$

As elements of  $C_*(X) \otimes C_*(X)$ ,

$$T_1(i, j) = T_3(i-1, j) \quad (i \geq 1), \quad T_2(i, j) = T_4(i, j+1) \quad (j \leq n-1),$$

and in both cases the coefficients are opposite: for the first,

$$(ij + (n+1)(i+j) + i) - ((i-1)j + (n+1)(i+j-1) + i + n - j) \equiv 2j + 1 \equiv 1,$$

and for the second,

$$(ij + (n+1)(i+j) + i + 1) - (i(j+1) + (n+1)(i+j+1) + (i+n-j) + (j+1-i)) \equiv 1.$$

So all these terms cancel in pairs except for  $T_1(0, j)$  with  $1 \leq j \leq n$ , for  $T_2(i, n)$  with  $0 \leq i \leq n-1$ , and for those  $T_3(i, j)$  and  $T_4(i, j)$  with  $j = i+1$ .

*Step 3: the survivors.* For  $j = i+1$  we have  $A_{i, i+1} = \sigma$  and  $a_{i, i+1} = n$ , so

$$T_3(i, i+1) = \sigma \otimes \sigma\langle i+1 \rangle, \quad T_4(i, i+1) = \sigma \otimes \sigma\langle i \rangle,$$

with coefficients  $\varepsilon_{i, i+1}(-1)^n$  and  $\varepsilon_{i, i+1}(-1)^{n+1}$ . Here  $\varepsilon_{i, i+1} = (-1)^{(i+1)(n-i-1)+i(n-i)} = (-1)^{n+1}$ , so  $\varepsilon_{i, i+1}(-1)^n = -1$  and these terms contribute

$$\sum_{i=0}^{n-1} \left( -\sigma \otimes \sigma\langle i+1 \rangle + \sigma \otimes \sigma\langle i \rangle \right) = \sigma \otimes \sigma\langle 0 \rangle - \sigma \otimes \sigma\langle n \rangle,$$

the sum telescoping. Next,  $T_1(0, j) = \sigma\langle j, \dots, n \rangle \otimes \sigma\langle 0, \dots, j \rangle$  has coefficient  $\varepsilon_{0j} = (-1)^{j(n-j)}$ , which is precisely the sign in  $\tau(\text{AW}_j(\sigma)) = (-1)^{j(n-j)} \sigma\langle j, \dots, n \rangle \otimes \sigma\langle 0, \dots, j \rangle$ ; so these terms contribute  $\sum_{j=1}^n \tau(\text{AW}_j(\sigma))$ . Finally  $T_2(i, n) = \sigma\langle 0, \dots, i \rangle \otimes \sigma\langle i, \dots, n \rangle = \text{AW}_i(\sigma)$ , and  $\varepsilon_{in} =$

$(-1)^i$ , so its coefficient is  $\varepsilon_{in}(-1)^{i+1} = -1$  and these terms contribute  $-\sum_{i=0}^{n-1} \text{AW}_i(\sigma)$ . Since  $\tau(\text{AW}_0(\sigma)) = \sigma \otimes \sigma\langle 0 \rangle$  and  $\text{AW}_n(\sigma) = \sigma \otimes \sigma\langle n \rangle$ , adding the three contributions gives

$$\sum_{j=1}^n \tau(\text{AW}_j(\sigma)) - \sum_{i=0}^{n-1} \text{AW}_i(\sigma) + \tau(\text{AW}_0(\sigma)) - \text{AW}_n(\sigma) = \tau(\text{AW}(\sigma)) - \text{AW}(\sigma).$$

Together with Step 1 this proves the theorem.  $\square$

*Remark 3.5.* Two comments. First,  $\tau$  must carry the Koszul sign: without it  $\tau$  is not a chain map,  $\tau \circ \text{AW}$  is not a chain map, and no  $P$  whatever could satisfy the identity. Second, the signs  $\varepsilon_{ij}$  are not a matter of taste. Granting the formula in dimensions  $< n$ , the terms of  $C_\sigma(i, j)$  occurring in  $\partial P + P\partial$  determine the  $\varepsilon_{ij}$  in dimension  $n$  uniquely: the pattern of Definition 3.2 is the only one for which the identity holds.

**Definition 3.6.** For  $\varphi \in C^p(X; R)$  and  $\psi \in C^q(X; R)$  with  $p + q \geq 1$  the *cup-one product*  $\varphi \smile_1 \psi \in C^{p+q-1}(X; R)$  is

$$\varphi \smile_1 \psi = (\varphi \otimes \psi) \circ P,$$

with  $\varphi \otimes \psi$  as in Definition 2.4: it is  $\varphi \cdot \psi$  on the summand  $C_p \otimes C_q$  and 0 on the others.

**Lemma 3.7.** For  $\varphi \in C^p(X; R)$  and  $\psi \in C^q(X; R)$ ,

$$(3.2) \quad \delta(\varphi \smile_1 \psi) = (-1)^{pq} \psi \smile \varphi - \varphi \smile \psi - \delta\varphi \smile_1 \psi - (-1)^p \varphi \smile_1 \delta\psi.$$

*Proof.* First,  $(\varphi \otimes \psi) \circ \tau \circ \text{AW} = (-1)^{pq} \psi \smile \varphi$ . Indeed, on a singular  $(p+q)$ -simplex  $\sigma$  the only summand of  $\tau(\text{AW}(\sigma))$  lying in  $C_p \otimes C_q$  is  $\tau(\text{AW}_q(\sigma)) = (-1)^{q(p+q-q)} \sigma\langle q, \dots, p+q \rangle \otimes \sigma\langle 0, \dots, q \rangle$ , on which  $\varphi \otimes \psi$  takes the value  $(-1)^{pq} \varphi(\sigma\langle q, \dots, p+q \rangle) \psi(\sigma\langle 0, \dots, q \rangle)$ ; as  $R$  is commutative this is  $(-1)^{pq} (\psi \smile \varphi)(\sigma)$ .

Second,  $(\varphi \otimes \psi) \circ \partial = \delta\varphi \otimes \psi + (-1)^p \varphi \otimes \delta\psi$  on  $(C_* \otimes C_*)_{p+q+1}$ , because for  $x \in C_{p+1}$ ,  $y \in C_q$  one has  $(\varphi \otimes \psi)(\partial(x \otimes y)) = \varphi(\partial x) \psi(y) = (\delta\varphi \otimes \psi)(x \otimes y)$ , for  $x \in C_p$ ,  $y \in C_{q+1}$  one has  $(\varphi \otimes \psi)(\partial(x \otimes y)) = (-1)^p \varphi(x) \psi(\partial y)$ , and the other summands are killed on both sides. Composing with  $P$  gives

$$(\varphi \otimes \psi) \circ \partial \circ P = \delta\varphi \smile_1 \psi + (-1)^p \varphi \smile_1 \delta\psi.$$

Now apply  $\varphi \otimes \psi$  to the identity of Theorem 3.4 in the form  $P \circ \partial = \tau \circ \text{AW} - \text{AW} - \partial \circ P$  and use  $\delta(\varphi \smile_1 \psi) = (\varphi \smile_1 \psi) \circ \partial = (\varphi \otimes \psi) \circ P \circ \partial$ .  $\square$

**Theorem 3.8.** For  $\alpha \in H^p(X; R)$  and  $\beta \in H^q(X; R)$ ,

$$\alpha \smile \beta = (-1)^{pq} \beta \smile \alpha.$$

Thus  $H^*(X; R)$  is a graded commutative  $R$ -algebra.

*Proof.* Represent  $\alpha$  and  $\beta$  by cocycles  $\varphi$  and  $\psi$ . Both  $\varphi \smile \psi$  and  $\psi \smile \varphi$  are cocycles by Lemma 2.6, and (3.2) with  $\delta\varphi = 0$  and  $\delta\psi = 0$  reads

$$\delta(\varphi \smile_1 \psi) = (-1)^{pq} \psi \smile \varphi - \varphi \smile \psi.$$

So the two cocycles  $\varphi \smile \psi$  and  $(-1)^{pq} \psi \smile \varphi$  differ by a coboundary, and by Theorem 2.7 their classes are  $\alpha \smile \beta$  and  $(-1)^{pq} \beta \smile \alpha$ .  $\square$

**Corollary 3.9.** If  $\alpha \in H^p(X; R)$  with  $p$  odd then  $2\alpha \smile \alpha = 0$ . In particular  $\alpha \smile \alpha = 0$  whenever  $H^{2p}(X; R)$  has no 2-torsion. Without that hypothesis the square of an odd-degree class can be non-zero: we shall see later that in  $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$  the square of the generator of  $H^1$  is the generator of  $H^2$ .

*Proof.* Theorem 3.8 gives  $\alpha \smile \alpha = (-1)^{p^2} \alpha \smile \alpha = -\alpha \smile \alpha$  for  $p$  odd.  $\square$

**Exercise 3.10.** Carry out in detail the identification, asserted in Step 1 of the proof of Theorem 3.4, of the outer terms of  $\partial P(\sigma)$  with the terms of  $-P(\partial\sigma)$ .

**Exercise 3.11.** Give an example of a space  $X$  and cochains  $\varphi, \psi \in C^1(X; \mathbb{Z})$  with  $\varphi \smile \psi \neq -\psi \smile \varphi$ , so that graded commutativity really does fail before passing to cohomology.

**Exercise 3.12.** Compute  $\varphi \smile_1 \psi$  for  $\varphi, \psi \in C^1(X; R)$  directly from Definition 3.2, and check (3.2) by hand in that case.

**Exercise 3.13.** Let  $\varphi \in C^p(X; R)$  be a cocycle with  $p$  odd. Deduce from (3.2) that  $2(\varphi \smile \varphi) = -\delta(\varphi \smile_1 \varphi)$ , which is Corollary 3.9 at the level of cochains. Note that for  $R = \mathbb{Z}/2$  the identity says nothing about the square, as it must, since squaring is not zero in  $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$ ; it degenerates to  $\delta(\varphi \smile_1 \varphi) = 0$ .

#### 4. THE CUP PRODUCT AND THE EILENBERG–ZILBER MAP

The definition of the cup product used the Alexander–Whitney map, which is one particular chain-level diagonal approximation. This section says what the product is independently of that choice: it is the map induced by the diagonal  $\Delta: X \rightarrow X \times X$ , read through the identification of the cohomology of  $X \times X$  with the cohomology of the dual of  $C_*(X) \otimes C_*(X)$ . The Alexander–Whitney map then appears not as the definition but as the cochain-level formula that computes it.

For a chain complex  $D_*$  of abelian groups write  $D^*$  for the cochain complex with  $D^n = \text{Hom}(D_n, R)$  and  $\delta f = f \circ \partial$ , the convention of Definition 1.1; so  $C^*(X; R) = C_*(X)^*$ , and  $(C_*(X) \otimes C_*(X))^*$  is the cochain complex whose elements in degree  $n$  are the homomorphisms  $(C_*(X) \otimes C_*(X))_n \rightarrow R$ . For  $\varphi \in C^p(X; R)$  and  $\psi \in C^q(X; R)$  the element  $\varphi \otimes \psi$  of that complex is the one of Definition 2.4: it is  $\varphi \cdot \psi$  on the summand  $C_p \otimes C_q$  and 0 on the others. Finally, write

$$\text{EZ}: C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$$

for the Eilenberg–Zilber map, Definition 6.12 of *Further Properties of Singular Homology*, and  $\text{AW}': C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y)$  for the Alexander–Whitney map of pairs of spaces, Definition 6.9 of those notes — not to be confused with the diagonal approximation AW of Definition 2.2. For a chain map  $u$  we write  $u^\vee = \text{Hom}(u, R)$  for the dual cochain map  $f \mapsto f \circ u$ .

**Theorem 4.1.** *Let  $X$  be a space and  $R$  a commutative ring.*

- (i) *If  $\varphi$  and  $\psi$  are cocycles then so is  $\varphi \otimes \psi$ , and  $[\varphi] \otimes [\psi] \mapsto [\varphi \otimes \psi]$  is a well-defined homomorphism*

$$\mu: H^p(X; R) \otimes H^q(X; R) \longrightarrow H^{p+q}\left((C_*(X) \otimes C_*(X))^*\right).$$

- (ii) *The dual  $\text{EZ}^\vee$  of the Eilenberg–Zilber map is a cochain homotopy equivalence with homotopy inverse  $(\text{AW}')^\vee$ ; in particular it induces an isomorphism*

$$\text{EZ}^*: H^n(X \times X; R) \xrightarrow{\cong} H^n\left((C_*(X) \otimes C_*(X))^*\right) \quad \text{for every } n.$$

- (iii) *The cup product is the composite*

$$H^p(X; R) \otimes H^q(X; R) \xrightarrow{\mu} H^{p+q}\left((C_*(X) \otimes C_*(X))^*\right) \xrightarrow{(\text{EZ}^*)^{-1}} H^{p+q}(X \times X; R) \xrightarrow{\Delta^*} H^{p+q}(X; R),$$

where  $\Delta: X \rightarrow X \times X$  is the diagonal.

*Proof.* (i) For  $x \in C_i(X)$  and  $y \in C_j(X)$  with  $i + j = p + q + 1$ ,

$$\delta(\varphi \otimes \psi)(x \otimes y) = (\varphi \otimes \psi)(\partial x \otimes y + (-1)^i x \otimes \partial y) = \varphi(\partial x)\psi(y) + (-1)^i \varphi(x)\psi(\partial y),$$

in which the first term survives only for  $(i, j) = (p+1, q)$  and the second only for  $(i, j) = (p, q+1)$ . Hence

$$\delta(\varphi \otimes \psi) = \delta\varphi \otimes \psi + (-1)^p \varphi \otimes \delta\psi,$$

the Leibniz rule (2.1) again. So a tensor of cocycles is a cocycle; and if  $\varphi = \delta\alpha$  with  $\psi$  a cocycle then  $\varphi \otimes \psi = \delta(\alpha \otimes \psi)$ , while if  $\psi = \delta\beta$  with  $\varphi$  a cocycle then  $\varphi \otimes \psi = (-1)^p \delta(\varphi \otimes \beta)$ . The class of  $\varphi \otimes \psi$  therefore depends only on  $[\varphi]$  and  $[\psi]$ , and it is additive in each, so  $\mu$  is defined.

(ii) The Eilenberg–Zilber theorem, Theorem 6.15 of *Further Properties of Singular Homology*, makes EZ and  $AW'$  mutually inverse chain homotopy equivalences between  $C_*(X) \otimes C_*(Y)$  and  $C_*(X \times Y)$ . By Lemma 1.6 their duals are then mutually inverse cochain homotopy equivalences, and in particular induce mutually inverse isomorphisms on cohomology. Note what is dualized: the two chain homotopies, not a homology isomorphism. No universal-coefficient argument enters, and the conclusion holds over every  $R$ .

(iii) Everything rests on the identity

$$AW = AW' \circ \Delta_{\#},$$

which is the first assertion of Exercise 2.10 and is immediate: for a singular  $n$ -simplex  $\sigma$  of  $X$  the simplex  $\Delta_{\#}\sigma = \Delta \circ \sigma$  of  $X \times X$  has both coordinate projections equal to  $\sigma$ , so Definition 6.9 gives

$$AW'(\Delta \circ \sigma) = \sum_{p+q=n} (\sigma \circ [e_0, \dots, e_p]) \otimes (\sigma \circ [e_p, \dots, e_n]) = AW(\sigma).$$

Now let  $\varphi$  and  $\psi$  be cocycles. At the level of cochains,

$$\Delta^{\#}((AW')^{\vee}(\varphi \otimes \psi)) = (\varphi \otimes \psi) \circ AW' \circ \Delta_{\#} = (\varphi \otimes \psi) \circ AW = \varphi \smile \psi,$$

the last step by Definition 2.4. Passing to cohomology and using  $(EZ^*)^{-1} = ((AW')^{\vee})^*$  from (ii), the composite of the theorem carries  $[\varphi] \otimes [\psi]$  to  $[\varphi \smile \psi]$ .  $\square$

*Remark 4.2.* The three maps have three different characters. The first is pure algebra, the pairing of two dual complexes; the second is the Eilenberg–Zilber theorem, which says that the two chain-level models of  $X \times X$  agree up to homotopy and is where all the work sits; and only the third mentions  $X$  itself, through the diagonal. So the cup product is the diagonal map, transported through an identification that has nothing to do with  $X$  — and the asymmetry of the Alexander–Whitney formula, which forced the sign of Theorem 3.8, is an artifact of choosing  $AW'$  rather than some other homotopy inverse of EZ. Any other choice gives a cochain-level product homotopic to  $\smile$ , hence the same product on cohomology.