

CATEGORIES AND FUNCTORS

Category Theory is the setting in which (most of) modern mathematics is formulated. It emphasizes exactly those central structures and notions that appear over and over again in mathematics, and it provides the language to formulate these concepts. Its abstraction is both its power and an impediment to coming to grips with it on first contact.

1. CATEGORIES

Definition 1.1. A *category* $\mathcal{C}$ consists of

- a collection $\text{Ob}(\mathcal{C})$ of *objects*;
- for each ordered pair (X, Y) of objects a set $\text{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* (or *arrows*) from X to Y , written $f: X \rightarrow Y$; and
- for each triple (X, Y, Z) a *composition* law

$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(X, Z), \quad (g, f) \longmapsto g \circ f,$$

subject to two axioms:

- (C1) *Associativity.* $h \circ (g \circ f) = (h \circ g) \circ f$ whenever the composites are defined.
- (C2) *Identities.* For each object X there is a morphism $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$ with $f \circ \text{id}_X = f$ and $\text{id}_Y \circ g = g$ for all $f: X \rightarrow Y$ and $g: W \rightarrow X$.

The sets $\text{Hom}_{\mathcal{C}}(X, Y)$ are disjoint for distinct pairs (X, Y) , so that a morphism determines its source and target.

Here is our first result in category theory:

Lemma 1.2. *The identity id_X is unique.*

Proof. If id'_X is another, then $\text{id}_X = \text{id}_X \circ \text{id}'_X = \text{id}'_X$. □

Remark 1.3 (a word on size). In most of the examples below the objects form a *proper class*, not a set: there is no set of all sets, nor of all groups. There is a well-known contradiction showing that there cannot be a set of all sets that are not members of themselves. Were there a set of all sets, that collection would be one of its subsets, hence a set; so there is no set of all sets; the collection of sets is too big to be a set. It is a class. But we are not doing logic here, so this point will not arise again.

Definition 1.4. A morphism $f: X \rightarrow Y$ is an *isomorphism* if there is $g: Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. Such a g is unique and is written f^{-1} . Objects X, Y are *isomorphic*, $X \cong Y$, if such an f exists.

Definition 1.5. A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ consists of a subcollection $\text{Ob}(\mathcal{C}') \subseteq \text{Ob}(\mathcal{C})$ together with, for each pair X, Y of objects of $\mathcal{C}'$, a subset $\text{Hom}_{\mathcal{C}'}(X, Y) \subseteq \text{Hom}_{\mathcal{C}}(X, Y)$, such that

- (S1) $\text{id}_X \in \text{Hom}_{\mathcal{C}'}(X, X)$ for every object X of $\mathcal{C}'$; and
- (S2) $g \circ f \in \text{Hom}_{\mathcal{C}'}(X, Z)$ whenever $f \in \text{Hom}_{\mathcal{C}'}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}'}(Y, Z)$.

With the composition inherited from $\mathcal{C}$, a subcategory is itself a category. It is called a *full* subcategory if

$$\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \quad \text{for all objects } X, Y \text{ of } \mathcal{C}',$$

that is, if no morphisms between the retained objects have been discarded. A full subcategory is therefore specified by its objects alone, and (S1), (S2) are then automatic: *any* subcollection of objects determines a full subcategory.

2. EXAMPLES OF CATEGORIES

Example 2.1 (Sets). **Set** has as objects all sets, and $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is the set of all set functions $X \rightarrow Y$, with the usual composition. Isomorphisms are exactly the bijections.

Example 2.2 (Topological spaces). **Top** has as objects all topological spaces, with $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ the set of continuous maps $X \rightarrow Y$. Composition of continuous maps is continuous and id_X is continuous, so this is a category. An isomorphism in **Top** is a continuous bijection with continuous inverse, that is, a *homeomorphism*.

Based Spaces and Maps: **Top**_{*}, whose objects are pairs (X, x_0) with a base point $x_0 \in X$ and whose morphisms are continuous maps carrying basepoint to basepoint; and the *homotopy category* **hTop**, with the same objects as **Top** but with $\mathrm{Hom}_{\mathbf{hTop}}(X, Y) = [X, Y]$ the set of homotopy classes of continuous maps. That **hTop** is a category is exactly the statement that homotopy is an equivalence relation compatible with composition. Isomorphism in **hTop** is homotopy equivalence.

Pairs of Topological spaces: **TopPair**, whose objects are pairs, (A, B) , consisting of a topological space and a subspace and whose morphisms $\mathrm{Hom}_{\mathbf{TopPair}}((A, B), (C, D))$ are continuous maps $f: A \rightarrow C$ with $f(B) \subset D$.

Example 2.3 (Vector spaces). Fix a field k . The category **Vect** _{k} has as objects all k -vector spaces and as morphisms all k -linear maps. Isomorphisms are the linear bijections. We write **Vect**^{fd} _{k} for the full subcategory of finite-dimensional spaces: same morphisms, fewer objects.

Example 2.4 (Modules). Let R be a ring with 1. The category **Mod** _{R} has as objects all left R -modules and as morphisms all R -linear maps. Taking $R = k$ a field recovers **Vect** _{k} ; taking $R = \mathbb{Z}$ recovers **Ab**, the category of abelian groups. Here $\mathrm{Hom}_{\mathbf{Mod}_R}(M, N)$ is itself an abelian group, and a k -vector space when $R = k$. This makes the category *additive*: each Hom is an abelian group, composition is biadditive, there is a zero object — the zero module, which is both initial and terminal — and any two objects have a direct sum, which serves as both their product and their coproduct. This structure is lacking in **Set** and **Top**.

Example 2.5 (Rings). **Ring** has as objects all rings with 1, and as morphisms the ring homomorphisms sending 1 to 1. We write **CRing** for the full subcategory of commutative rings. Isomorphisms are the bijective ring homomorphisms.

Example 2.6 (Groups). **Grp** has groups as objects and group homomorphisms as morphisms; **Ab** $\subset$ **Grp** is the full subcategory of abelian groups.

The next two examples are of a different flavor: unlike all of the examples so far, a general category need not be a category “of sets with extra structure”.

Example 2.7 (A monoid is a category). Let M be a monoid, i.e., a set with an associative multiplication and a unit. Let $\mathcal{C}_M$ have a *single* object $*$, with $\text{Hom}(*, *) = M$, composition is the multiplication of M and id_* its unit. The axioms (C1), (C2) say exactly that M is a monoid. When M is a group, every morphism of $\mathcal{C}_M$ is an isomorphism.

Example 2.8 (A partially ordered set is a category). Let $(P, \leq)$ be a poset. Let $\mathcal{C}_P$ have the elements of P as objects, with $\text{Hom}(x, y)$ a single point if $x \leq y$ and empty otherwise. Transitivity gives composition, reflexivity gives identities, and antisymmetry says that the only isomorphisms are the identities.

Example 2.9 (Groupoids, and a non-full subcategory). A *groupoid* is a category in which every morphism is an isomorphism. Given any category $\mathcal{C}$, keep all its objects but only its isomorphisms: the result is a subcategory $\mathcal{C}^{\cong}$, the *core* of $\mathcal{C}$, and it is a groupoid. It is a subcategory in the sense of Definition 1.5 — identities are isomorphisms and isomorphisms compose — but it is emphatically *not* full, since a non-invertible morphism of $\mathcal{C}$ between two of its objects has been thrown away. For instance $\mathbf{Set}^{\cong}$ has all sets as objects and only the bijections as morphisms.

Example 2.10 (The opposite category). Given $\mathcal{C}$, the *opposite* category $\mathcal{C}^{\text{op}}$ has the same objects and $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$, with composition reversed. Formally one just turns every arrow around. This trivial-looking construction is what makes “contravariant” a special case of “covariant” in §4.

Example 2.11 (Initial and terminal objects). An object I is *initial* if $\text{Hom}(I, X)$ has exactly one element for every X , and an object T is *terminal* if $\text{Hom}(X, T)$ has a single element for every X . In $\mathbf{Set}$ the empty set is initial and any one-point set is terminal; in $\mathbf{Grp}$ and in $\mathbf{Mod}_R$ the zero object is both; in $\mathbf{Ring}$ the ring $\mathbb{Z}$ is initial (a homomorphism $\mathbb{Z} \rightarrow S$ is determined by $1 \mapsto 1$). Any two initial objects are isomorphic by a unique isomorphism — a first taste of how categorical language formulates “canonical”.

3. PRODUCTS AND CO-PRODUCTS

Definition 3.1. A category $\mathcal{C}$ has *binary products* if for every pair of objects A, B of $\mathcal{C}$ there is an object P together with structure morphisms $\varepsilon_A: P \rightarrow A$ and $\varepsilon_B: P \rightarrow B$, universal among objects equipped with a morphism to A and a morphism to B . This means given any object Q and morphisms $f: Q \rightarrow A$ and $g: Q \rightarrow B$ there is a unique morphism $\eta: Q \rightarrow P$ such that $f = \varepsilon_A \circ \eta$ and $g = \varepsilon_B \circ \eta$.

By the usual argument, if P exists then it is unique up to canonical isomorphism commuting with its structure morphisms. We often denote P by $A \times B$. The construction is natural in the sense that $f: A \rightarrow C$ and $g: B \rightarrow D$ lead to a morphism $(f \times g): A \times B \rightarrow C \times D$.

More generally, we can consider all finite products, countably infinite products and products indexed by any set. The question is always do they exist or not.

For sets, groups, commutative groups, rings, modules over a ring, vector spaces, topological spaces all products indexed by sets exist. For sets it is the usual Cartesian product. The category of smooth manifolds has finite products but not infinite products. The category of topological spaces has all products indexed by sets but in the case of infinite products, the definition of the topology is more delicate.

Definition 3.2. The category $\mathcal{C}$ has *binary co-products* if for any pair of objects A, B there is an object C together with structure morphisms $\varepsilon_A: A \rightarrow C$ and $\varepsilon_B: B \rightarrow C$, universal among objects equipped with a morphism from A and a morphism from B . This means given any object X of $\mathcal{C}$ and morphisms $f: A \rightarrow X$ and $g: B \rightarrow X$, there is a unique morphism $\eta: C \rightarrow X$ such that $f = \eta \circ \varepsilon_A$ and $g = \eta \circ \varepsilon_B$.

If the co-product exists, by the universal property it is unique up to canonical isomorphism.

As with products one can consider finite co-products, and co-products indexed by sets. The question is always whether or not they exist. For sets, co-products are disjoint union. For abelian groups, modules over a ring, and vector spaces they are direct sum. For groups the co-product is free product.

4. FUNCTORS

Definition 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (*covariant*) *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ assigns to each object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, and to each morphism $f: X \rightarrow Y$ a morphism $F(f): F(X) \rightarrow F(Y)$, such that

$$F(g \circ f) = F(g) \circ F(f) \quad \text{whenever } g \circ f \text{ is defined,} \quad F(\text{id}_X) = \text{id}_{F(X)} \text{ for every } X.$$

A *contravariant functor* $\mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$; concretely, it sends any morphism $f: X \rightarrow Y$ of $\mathcal{C}$ to $F(f): F(Y) \rightarrow F(X)$ and satisfies $F(g \circ f) = F(f) \circ F(g)$.

Lemma 4.2. *A functor carries isomorphisms to isomorphisms: if f is an isomorphism then so is $F(f)$, with $F(f)^{-1} = F(f^{-1})$.*

Proof. $F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(\text{id}) = \text{id}$, and symmetrically. $\square$

Lemma 4.2 is the formal reason that functors produce invariants: if $F(X) \not\cong F(Y)$ then $X \not\cong Y$. All of algebraic topology is an elaboration of this one line.

Example 4.3 (Forgetful functors). There are functors

$$\begin{aligned} \mathbf{Top} &\rightarrow \mathbf{Set}, & \mathbf{Grp} &\rightarrow \mathbf{Set}, & \mathbf{Ring} &\rightarrow \mathbf{Set}, \\ \mathbf{Mod}_R &\rightarrow \mathbf{Ab}, & \mathbf{CRing} &\hookrightarrow \mathbf{Ring}, \end{aligned}$$

each forgetting part of the structure and keeping the underlying map.

Example 4.4 (Free functors). The free R -module functor $F: \mathbf{Set} \rightarrow \mathbf{Mod}_R$ sends a set S to $F(S) = \bigoplus_{s \in S} R$ and a function $\varphi: S \rightarrow T$ to the unique R -linear map with $e_s \mapsto e_{\varphi(s)}$. Similarly one has the free group functor $\mathbf{Set} \rightarrow \mathbf{Grp}$ and the polynomial ring functor $S \mapsto \mathbb{Z}[x_s : s \in S]$, $\mathbf{Set} \rightarrow \mathbf{CRing}$.

Example 4.5 (Linear algebra). Fix k . Dualization $V \mapsto V^* = \text{Hom}_k(V, k)$ is a *contravariant* functor $\mathbf{Vect}_k \rightarrow \mathbf{Vect}_k$: a linear map $f: V \rightarrow W$ gives $f^*: W^* \rightarrow V^*$, $\lambda \mapsto \lambda \circ f$, and $(g \circ f)^* = f^* \circ g^*$. More generally, for a fixed R -module A ,

$$\text{Hom}_R(A, -): \mathbf{Mod}_R \rightarrow \mathbf{Ab} \quad \text{is covariant,}$$

$$\text{Hom}_R(-, A): \mathbf{Mod}_R \rightarrow \mathbf{Ab} \quad \text{is contravariant,}$$

and for a fixed right R -module B , $B \otimes_R (-): \mathbf{Mod}_R \rightarrow \mathbf{Ab}$ is covariant.

Example 4.6 (Between algebraic categories). Abelianization $G \mapsto G/[G, G]$ is a functor $\mathbf{Grp} \rightarrow \mathbf{Ab}$. For each n , $S \mapsto GL_n(S)$ and $S \mapsto S^\times$ are functors $\mathbf{CRing} \rightarrow \mathbf{Grp}$. Restriction of scalars along a ring homomorphism $\varphi: R \rightarrow S$ is a functor $\mathbf{Mod}_S \rightarrow \mathbf{Mod}_R$, and extension of scalars $M \mapsto S \otimes_R M$ is a functor $\mathbf{Mod}_R \rightarrow \mathbf{Mod}_S$.

Example 4.7 (Topology to algebra). $X \mapsto C(X; \mathbb{R})$, the ring of continuous real-valued functions, is a contravariant functor $\mathbf{Top} \rightarrow \mathbf{Ring}$: a continuous $f: X \rightarrow Y$ induces $f^*: C(Y; \mathbb{R}) \rightarrow C(X; \mathbb{R})$ by $\lambda \mapsto \lambda \circ f$.

Definition 4.8. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is

- (i) *faithful* if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is injective for all X, Y ;
- (ii) *full* if that map is surjective for all X, Y ;
- (iii) *fully faithful* if it is both; and
- (iv) *essentially surjective* if every object of $\mathcal{D}$ is isomorphic to some $F(X)$.

A subcategory $\mathcal{C}' \subseteq \mathcal{C}$ comes with an *inclusion functor* $\mathcal{C}' \rightarrow \mathcal{C}$, which is always faithful; it is full, in the sense of Definition 4.8(ii), exactly when $\mathcal{C}'$ is a full subcategory in the sense of Definition 1.5. Thus the inclusions $\mathbf{Ab} \subset \mathbf{Grp}$ and $\mathbf{Vect}_k^{\mathrm{fd}} \subset \mathbf{Vect}_k$ are fully faithful, while the inclusion $\mathbf{Set}^{\cong} \subset \mathbf{Set}$ of Example 2.9 is faithful and not full. The forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ is likewise faithful but not full.

5. NATURAL TRANSFORMATIONS

With appropriate smallness assumptions, functors $\mathcal{C} \rightarrow \mathcal{D}$ are themselves the objects of a category; the morphisms between them are natural transformations. In general, we have:

Definition 5.1. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. A *natural transformation* $\eta: F \Rightarrow G$ assigns to each object X of $\mathcal{C}$ a morphism $\eta_X: F(X) \rightarrow G(X)$ of $\mathcal{D}$, called the *component* at X , such that for every morphism $f: X \rightarrow Y$ of $\mathcal{C}$ the square

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

commutes, that is $G(f) \circ \eta_X = \eta_Y \circ F(f)$. If every η_X is an isomorphism, η is a *natural isomorphism*, written $F \cong G$; in that case the inverses η_X^{-1} are themselves the components of a natural transformation $G \Rightarrow F$.

Naturality is the precise form of the informal “defined without choices, uniformly in X ”. The following examples are the ones to remember.

Example 5.2 (Double dual: natural). For a k -vector space V define $\eta_V: V \rightarrow V^{**}$ by $\eta_V(v)(\lambda) = \lambda(v)$. This is linear, and for any linear $f: V \rightarrow W$ the square of Definition 5.1 commutes: both ways round send v to the functional $\lambda \mapsto \lambda(f(v))$ on W^* . So η is a natural transformation from the identity functor of $\mathbf{Vect}_k$ to the double-dual functor $V \mapsto V^{**}$ (a composite of two contravariant functors, hence covariant). Restricted to $\mathbf{Vect}_k^{\mathrm{fd}}$ each η_V is an isomorphism, so there η is a natural isomorphism $\mathrm{id} \cong (-)^{**}$.

Example 5.3 (The determinant). For each n , $\det: GL_n(-) \Rightarrow (-)^\times$ is a natural transformation between the functors $\mathbf{CRing} \rightarrow \mathbf{Grp}$ of Example 4.6: its component at a commutative ring S is $\det_S: GL_n(S) \rightarrow S^\times$, and naturality says that for a ring homomorphism $\varphi: S \rightarrow S'$ one has $\det(\varphi(A)) = \varphi(\det A)$, where φ is applied to A entrywise — which is clear, the determinant being a polynomial in the entries with integer coefficients. Naturality here is exactly the statement that “the determinant is given by a universal formula”.

Example 5.4. Let $F: \mathbf{Set} \rightarrow \mathbf{Mod}_R$ be the free module functor and $U: \mathbf{Mod}_R \rightarrow \mathbf{Set}$ the forgetful one. The maps $\eta_S: S \rightarrow UF(S)$, $s \mapsto e_s$, are the components of a natural transformation $\text{id}_{\mathbf{Set}} \Rightarrow UF$, and the maps $\varepsilon_M: FU(M) \rightarrow M$, $e_m \mapsto m$, of a natural transformation $FU \Rightarrow \text{id}_{\mathbf{Mod}_R}$. Together they express the universal property of the free module as a bijection

$$\text{Hom}_{\mathbf{Mod}_R}(F(S), M) \cong \text{Hom}_{\mathbf{Set}}(S, U(M))$$

natural in S and in M : this is the statement that F is *left adjoint* to U . Adjunctions are beyond these notes, but they are one of the main reasons the language pays off.

Remark 5.5. Natural transformations compose: if $\eta: F \Rightarrow G$ and $\theta: G \Rightarrow H$, then $(\theta\eta)_X = \theta_X \circ \eta_X$ is natural. Also, id_F is given by $(\text{id}_F)_X = \text{id}_{F(X)}$. So, under the smallness proviso above, functors $\mathcal{C} \rightarrow \mathcal{D}$ and natural transformations form the *functor category* $\mathcal{D}^{\mathcal{C}}$, whose isomorphisms are exactly the natural isomorphisms.

6. EQUIVALENCE OF CATEGORIES

One might guess that the right notion of sameness for categories is *isomorphism*: functors F, G with $GF = \text{id}$ and $FG = \text{id}$ on the nose. This is almost never the right notion, because it demands equalities of objects, and there are too many objects. Here is the useful notion:

Definition 6.1. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence of categories* if there exist a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms

$$\eta: \text{id}_{\mathcal{C}} \xrightarrow{\cong} GF, \quad \varepsilon: FG \xrightarrow{\cong} \text{id}_{\mathcal{D}}.$$

We then say $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, $\mathcal{C} \simeq \mathcal{D}$. If $\mathcal{C}^{\text{op}} \simeq \mathcal{D}$, then these categories are *anti-equivalent*, and one says that there is a *duality* between $\mathcal{C}$ and $\mathcal{D}$.

The following criterion for equivalence of categories is the one used in practice; it lets one verify an equivalence without ever constructing the inverse functor.

Theorem 6.2. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.

Sketch. If F is an equivalence with inverse G and units η, ε , then essential surjectivity is ε , and faithfulness follows from the naturality square for η : $f = \eta_Y^{-1} \circ GF(f) \circ \eta_X$, so $F(f) = F(f')$ forces $f = f'$. The same computation applied to ε shows that G is faithful, and this gives fullness of F : given $g: F(X) \rightarrow F(Y)$, put $f = \eta_Y^{-1} \circ G(g) \circ \eta_X$; naturality of η at f gives $GF(f) \circ \eta_X = \eta_Y \circ f = G(g) \circ \eta_X$, so $GF(f) = G(g)$ and hence $F(f) = g$.

Conversely, suppose F is fully faithful and essentially surjective. For each object D of $\mathcal{D}$ choose an object $G(D)$ of $\mathcal{C}$ and an isomorphism $\varepsilon_D: FG(D) \rightarrow D$; this uses the

axiom of choice. For $g: D \rightarrow D'$, the morphism $\varepsilon_{D'}^{-1} \circ g \circ \varepsilon_D: FG(D) \rightarrow FG(D')$ is F of a unique morphism $G(g): G(D) \rightarrow G(D')$, because F is fully faithful. Uniqueness makes G a functor and ε natural; the isomorphism η is produced the same way. $\square$

Example 6.3 (Matrices). Let $\mathbf{Mat}_k$ be the category whose objects are the natural numbers $0, 1, 2, \dots$, with $\text{Hom}_{\mathbf{Mat}_k}(m, n)$ the set of $n \times m$ matrices over k , composition being matrix multiplication and id_n the identity matrix. The functor $\mathbf{Mat}_k \rightarrow \mathbf{Vect}_k^{\text{fd}}$ sending $n \mapsto k^n$ and a matrix to the corresponding linear map is fully faithful (a linear map $k^m \rightarrow k^n$ is exactly a matrix) and essentially surjective (every finite-dimensional space has a basis, hence is isomorphic to some k^n). By Theorem 6.2 it is an equivalence. It is emphatically not an isomorphism of categories: $\mathbf{Mat}_k$ has a set of objects, $\mathbf{Vect}_k^{\text{fd}}$ a proper class of them. This is the precise sense in which “linear algebra is the theory of matrices”, and it also shows why equivalence, not isomorphism, is the right notion.

Example 6.4 (Duality for finite-dimensional spaces). $V \mapsto V^*$ is a contravariant functor $\mathbf{Vect}_k^{\text{fd}} \rightarrow \mathbf{Vect}_k^{\text{fd}}$, i.e. a functor $(\mathbf{Vect}_k^{\text{fd}})^{\text{op}} \rightarrow \mathbf{Vect}_k^{\text{fd}}$, and Example 5.2 provides the natural isomorphism $\text{id} \cong (-)^{**}$ making it an equivalence. So $\mathbf{Vect}_k^{\text{fd}}$ is anti-equivalent to itself. Note that the corresponding statement on all of $\mathbf{Vect}_k$ is false: η_V fails to be an isomorphism for infinite-dimensional V .

7. EXERCISES

Exercise 7.1. Show that a functor F need not reflect isomorphisms: find F and f with $F(f)$ an isomorphism but f not. Show that a fully faithful functor *does* reflect isomorphisms.

Exercise 7.2. Let $\mathcal{C}_M$ and $\mathcal{C}_N$ be the one-object categories attached to monoids M, N . Show that functors $\mathcal{C}_M \rightarrow \mathcal{C}_N$ are exactly monoid homomorphisms, and describe the natural transformations between two such functors.

Exercise 7.3. Show that $\text{Hom}_{\mathcal{C}}(X, -): \mathcal{C} \rightarrow \mathbf{Set}$ is a functor for each object X , and that a morphism $u: X \rightarrow Y$ induces a natural transformation $\text{Hom}_{\mathcal{C}}(Y, -) \Rightarrow \text{Hom}_{\mathcal{C}}(X, -)$. (The Yoneda lemma asserts that *every* natural transformation between such functors arises this way from a unique u .)

Exercise 7.4. Assuming products exist, show that they are unique up to canonical isomorphism and are functorial up to canonical isomorphism. Same question for co-products.