

CW COMPLEXES

Throughout, $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the closed unit n -ball, $S^{n-1} = \partial D^n$ is its boundary sphere, and $\text{int } D^n = D^n \setminus S^{n-1}$ is the open n -ball; by convention D^0 is a point and $S^{-1} = \emptyset$. All maps of spaces are continuous. References to the companion notes are given as *Theorem such-and-such of Singular Homology*, and similarly for *Homology*; all other references are internal.

1. THE DEFINITION

A CW complex is a space assembled from n -balls, one dimension at a time, each ball glued to what has already been built by a map of its boundary sphere. The two ingredients are therefore the attaching construction and the topology on the union of the stages.

Definition 1.1. Let A be a space and let $\{\varphi_\alpha : S^{n-1} \rightarrow A\}_{\alpha \in J}$ be a family of maps, called *attaching maps*. The space obtained from A by *attaching n -cells* along the φ_α is the quotient

$$A \cup_{\{\varphi_\alpha\}} \coprod_{\alpha \in J} D_\alpha^n = \left(A \sqcup \coprod_{\alpha \in J} D_\alpha^n \right) / (x \sim \varphi_\alpha(x) \text{ for } x \in S^{n-1} = \partial D_\alpha^n),$$

where each D_α^n is a copy of D^n and the quotient carries the quotient topology.

Definition 1.2. A *CW complex* is a space X together with a filtration by subspaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X, \quad X = \bigcup_{n \geq 0} X^n,$$

called the *skeleta*, such that

- (i) X^0 is a discrete space; we write J_0 for its set of points;
- (ii) for each $n \geq 1$ there are an index set J_n and attaching maps $\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$, $\alpha \in J_n$, such that X^n is obtained from X^{n-1} by attaching n -cells along them, as in Definition 1.1;
- (iii) X carries the *weak topology* determined by the skeleta: a subset C of X is closed if and only if $C \cap X^n$ is closed in X^n for every n .

Condition (iii) is vacuous when the filtration is finite; it has content only for complexes of infinite dimension, where it is exactly what is needed to make maps out of X constructible one skeleton at a time.

Definition 1.3. For $\alpha \in J_n$ the composite

$$\Phi_\alpha : D_\alpha^n \longrightarrow X^{n-1} \sqcup \coprod_{\beta \in J_n} D_\beta^n \longrightarrow X^n \subseteq X$$

of the inclusion with the quotient map is the *characteristic map* of the cell α , and

$$e_\alpha^n = \Phi_\alpha(\text{int } D_\alpha^n)$$

is the corresponding *open n -cell*. For $n = 0$ the characteristic maps are the inclusions of the points of X^0 , and these points are the open 0-cells. Such a filtration and characteristic maps Φ_α are called a *cell structure* on X . If $X = X^n$ for some n , the smallest such n is the *dimension* of X ; the complex is *finite* if it has finitely many cells in all.

Remark 1.4. Directly from the construction: Φ_α restricts to a homeomorphism of $\text{int } D_\alpha^n$ onto e_α^n , it carries S^{n-1} into X^{n-1} by φ_α , and X is the set-theoretic disjoint union of its open cells,

$$X = \coprod_{n \geq 0} \coprod_{\alpha \in J_n} e_\alpha^n,$$

with X^n the union of the cells of dimension at most n . A point of X therefore lies in exactly one open cell. The open cells are not in general open subsets of X .

Remark 1.5. The letters CW stand for *closure-finite* and *weak topology*.

Definition 1.6. A *subcomplex* of a CW complex X is a closed subspace $A \subseteq X$ that is a union of open cells of X with the property that the closure of each of these cells is contained in A . Filtered by $A \cap X^n$, such an A is itself a CW complex, and the pair (X, A) is called a *CW pair*. Each skeleton X^n is a subcomplex.

Definition 1.7. A *morphism of CW complexes* $f: X \rightarrow Y$, also called a *cellular* map, is a map of the underlying spaces with $f(X^n) \subseteq Y^n$ for all n .

Definition 1.8. The identity map of a CW complex is a morphism, and a composite of morphisms is a morphism, so CW complexes and morphisms of CW complexes form a category, which we write **CW**. By Definition 1.3 an object of **CW** is a space *together with* a cell structure on it, so one space with two different cell structures gives two different objects of **CW**; a morphism, on the other hand, is required only to respect the skeleta, and need carry neither cells to cells nor characteristic maps to characteristic maps. Forgetting the cell structure is a faithful functor **CW** $\rightarrow$ **Top**.

Remark 1.9. The point-set facts that will be used repeatedly are these: a map $f: X \rightarrow Z$ defined on a CW complex is continuous if and only if $f \circ \Phi_\alpha$ is continuous for every characteristic map Φ_α (Proposition 2.4); each skeleton is closed in X (Proposition 2.5); and a compact subset of X is contained in a finite subcomplex, so a CW complex is compact if and only if it is finite (Theorem 2.11). The first of these three is the usable form of the weak topology, and it is what allows maps out of a CW complex to be built cell by cell. All three are proved in §2. A CW complex is also T_1 , normal and hence Hausdorff, and locally contractible; these two further properties are stated as Theorem 2.6 and proved in Appendix A.

2. POINT-SET PROPERTIES

Throughout this section X is a CW complex with skeleta X^n , index sets J_n , attaching maps $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$ and characteristic maps $\Phi_\alpha: D_\alpha^n \rightarrow X$ for $\alpha \in J_n$.

In this section we shall establish, among other things, these basic topological facts about CW complexes.

Theorem 2.1. *Let X be a CW complex. Then:*

- (i) a map $f: X \rightarrow Z$ into a space Z is continuous if and only if $f \circ \Phi_\alpha$ is continuous for every characteristic map Φ_α of X ;
- (ii) every skeleton, and more generally every subcomplex, is a closed subspace of X , and carries as a subspace the topology it has as a CW complex;
- (iii) a compact subset of X meets only finitely many open cells and is contained in a finite subcomplex; consequently X is compact if and only if it is finite.

This theorem is proved below. Part (i) is Proposition 2.4; Part (ii) for skeleta is Proposition 2.5 and for general subcomplexes is Proposition 2.10; and Part (iii) is Proposition 2.9 together with Theorem 2.11. Two further properties, normality and local contractibility, are stated as Theorem 2.6 below and proved in the appendix. Everything rests on the elementary description of the topology of a single attaching step.

Lemma 2.2. *Let $Y = A \cup_{\{\varphi_\alpha\}} \coprod_{\alpha \in J} D_\alpha^n$ be as in Definition 1.1, with quotient map $q: A \sqcup \coprod_{\alpha} D_\alpha^n \rightarrow Y$ and $\Phi_\alpha = q|_{D_\alpha^n}$. Then:*

- (i) $q|_A$ is injective and is a homeomorphism onto a closed subspace of Y ; we identify A with that subspace.
- (ii) A subset $C \subseteq Y$ is closed in Y if and only if $C \cap A$ is closed in A and $\Phi_\alpha^{-1}(C)$ is closed in D_α^n for every α ; the same statement holds with “closed” replaced by “open”.
- (iii) $Y \setminus A = \coprod_{\alpha} e_\alpha$, where $e_\alpha = \Phi_\alpha(\text{int } D_\alpha^n)$; each e_α is an open subset of Y and Φ_α maps $\text{int } D_\alpha^n$ homeomorphically onto it.

Proof. The equivalence relation defining Y has as its classes the sets $\{a\} \cup \bigcup_{\alpha} \varphi_\alpha^{-1}(a)$ for $a \in A$, together with the singletons $\{z\}$ for $z \in \text{int } D_\alpha^n$. In particular distinct points of A are not identified, which gives injectivity of $q|_A$ and the decomposition in (iii).

For (i), let $B \subseteq A$ be closed. Then $q^{-1}(q(B)) = B \sqcup \coprod_{\alpha} \varphi_\alpha^{-1}(B)$, and each $\varphi_\alpha^{-1}(B)$ is closed in S^{n-1} , hence closed in D_α^n ; so $q(B)$ is closed in Y . Thus $q|_A$ is a closed continuous injection, that is, a homeomorphism onto its image, and taking $B = A$ shows the image is closed.

For (ii), $q^{-1}(C) = (C \cap A) \sqcup \coprod_{\alpha} \Phi_\alpha^{-1}(C)$, and C is closed in Y exactly when this set is closed, that is, when each summand is closed in its own piece. The open case follows by taking complements.

For (iii), let $O \subseteq \text{int } D_\alpha^n$ be open. The preimage under q of $Y \setminus \Phi_\alpha(O)$ is $A \sqcup (D_\alpha^n \setminus O) \sqcup \coprod_{\beta \neq \alpha} D_\beta^n$, which is closed; hence $\Phi_\alpha(O)$ is open in Y . Taking $O = \text{int } D_\alpha^n$ shows e_α is open, and since Φ_α is injective on $\text{int } D_\alpha^n$ by the description of the equivalence classes, it is a continuous open bijection onto e_α , that is, a homeomorphism. $\square$

Proposition 2.3. *A subset $C \subseteq X$ is closed if and only if $\Phi_\alpha^{-1}(C)$ is closed in D_α^n for every $n \geq 1$ and every $\alpha \in J_n$. The same holds with “closed” replaced by “open”.*

Proof. Necessity is the continuity of the Φ_α . Conversely, assume the condition and argue by induction that $C \cap X^n$ is closed in X^n . For $n = 0$ this is automatic, X^0 being discrete. If $C \cap X^{n-1}$ is closed in X^{n-1} , then Lemma 2.2(ii) applied to $X^n = X^{n-1} \cup_{\{\varphi_\alpha\}} \coprod_{\alpha \in J_n} D_\alpha^n$ shows that $C \cap X^n$ is closed in X^n . Now C is closed by the weak topology, Definition 1.2(iii). $\square$

Proposition 2.4. *A map of sets $f: X \rightarrow Z$ into a space Z is continuous if and only if $f \circ \Phi_\alpha$ is continuous for every characteristic map Φ_α of X .*

Proof. If f is continuous so is each composite. Conversely, let $C \subseteq Z$ be closed. Then

$$\Phi_\alpha^{-1}(f^{-1}(C)) = (f \circ \Phi_\alpha)^{-1}(C)$$

is closed for every α , so $f^{-1}(C)$ is closed by Proposition 2.3. $\square$

This is part (i) of Theorem 2.1.

Proposition 2.5. *For every n the skeleton X^n is a closed subspace of X , and its topology as a subspace of X is the one it carries as a CW complex.*

Proof. By Lemma 2.2(i), X^{m-1} is closed in X^m for every $m \geq 1$, so by induction $X^n \cap X^m = X^{\min(n,m)}$ is closed in X^m for every m ; by the weak topology X^n is closed in X . A subset $C \subseteq X^n$ closed in the topology of X^n is then closed in X : indeed $C \cap X^m$ is closed in X^m for $m \leq n$ because X^m is a subspace of X^n , and for $m > n$ one applies Lemma 2.2(ii) repeatedly, noting $\Phi_\alpha^{-1}(C) = \varphi_\alpha^{-1}(C)$ is closed for cells of dimension $> n$. Hence the two topologies agree. $\square$

This is the case of a skeleton in part (ii) of Theorem 2.1; the general case is Proposition 2.10 below.

Theorem 2.6. *A CW complex X is T_1 and normal, and hence Hausdorff; and X is locally contractible.*

The proofs of these results use local topological arguments, somewhat intricate but not deep, and are postponed to the appendix. The only use made of them in what follows is that of the Hausdorff property, to see that the compact set $\Phi_\alpha(D_\alpha^n)$ is closed, in Corollary 2.7 immediately below.

Corollary 2.7. $\overline{e_\alpha^n} = \Phi_\alpha(D_\alpha^n)$; it is compact and contained in $e_\alpha^n \cup X^{n-1}$.

Proof. $\Phi_\alpha(D_\alpha^n)$ is compact, hence closed since X is Hausdorff by Theorem 2.6, and it contains e_α^n ; so $\overline{e_\alpha^n} \subseteq \Phi_\alpha(D_\alpha^n)$. In the other direction D_α^n is the closure of $\text{int } D_\alpha^n$, so $\Phi_\alpha(D_\alpha^n) \subseteq \overline{\Phi_\alpha(\text{int } D_\alpha^n)} = \overline{e_\alpha^n}$ by continuity. Finally $\Phi_\alpha(D_\alpha^n) = e_\alpha^n \cup \varphi_\alpha(S^{n-1})$ and $\varphi_\alpha(S^{n-1}) \subseteq X^{n-1}$. $\square$

Lemma 2.8. *Let $S \subseteq X$ meet each open cell of X in at most one point. Then S is a closed subspace of X with the discrete topology.*

Proof. We first show that S is closed, by induction on n : $S \cap X^n$ is closed in X^n . For $n = 0$ this holds because X^0 is discrete. Assume $S \cap X^{n-1}$ is closed in X^{n-1} and let $\alpha \in J_n$. Then

$$\Phi_\alpha^{-1}(S) = \varphi_\alpha^{-1}(S \cap X^{n-1}) \cup \Phi_\alpha^{-1}(S \cap e_\alpha^n),$$

because $\Phi_\alpha(D_\alpha^n) \subseteq e_\alpha^n \cup X^{n-1}$. The first set is closed in S^{n-1} , hence in D_α^n ; the second is at most one point. So $\Phi_\alpha^{-1}(S)$ is closed, and $S \cap X^n$ is closed in X^n by Lemma 2.2(ii). The weak topology now gives that S is closed in X . Every subset of S again meets each open cell in at most one point, so is closed in X by what we have just proved, and hence closed in S . Applying this to the complement in S of one of its points shows that each point of S is open in S ; so S is discrete. $\square$

Proposition 2.9. *A compact subset K of a CW complex meets only finitely many open cells. In particular the closure of a cell meets only finitely many cells, so the complex is closure-finite.*

Proof. Suppose K met infinitely many open cells and choose a point of K in each of infinitely many of them, obtaining an infinite set $S \subseteq K$ meeting each cell in at most one point. By Lemma 2.8 S is a closed subspace of X with the discrete topology, so S is a closed subset of the compact set K and therefore compact; an infinite discrete space is not compact, a contradiction. For the second statement apply this to the compact set $\overline{e}_\alpha^n$ of Corollary 2.7. $\square$

This is the first assertion of part (iii) of Theorem 2.1.

Proposition 2.10. *Let $A \subseteq X$ be a union of open cells of X such that $\overline{e} \subseteq A$ for every cell e contained in A . Then A is closed, so A is a subcomplex in the sense of Definition 1.6, and A , filtered by the skeleta $A \cap X^n$, is a CW complex whose cells are the cells of X contained in A .*

Proof. We show $A \cap X^n$ is closed in X^n by induction, the case $n = 0$ being clear. Let $\alpha \in J_n$. If $e_\alpha^n \subseteq A$ then $\Phi_\alpha(D_\alpha^n) = \overline{e}_\alpha^n \subseteq A$ and $\Phi_\alpha^{-1}(A) = D_\alpha^n$. Otherwise $e_\alpha^n \cap A = \emptyset$, since A is a union of open cells and the open cells are disjoint, and then $\Phi_\alpha^{-1}(A) = \varphi_\alpha^{-1}(A \cap X^{n-1})$ is closed in S^{n-1} by the inductive hypothesis. Either way $\Phi_\alpha^{-1}(A)$ is closed, so $A \cap X^n$ is closed in X^n by Lemma 2.2(ii), and A is closed by the weak topology.

For the cell structure: if $e_\alpha^n \subseteq A$ then $\varphi_\alpha(S^{n-1}) \subseteq \overline{e}_\alpha^n \subseteq A$, so φ_α takes values in $A \cap X^{n-1}$, and $A \cap X^n$ is the union of $A \cap X^{n-1}$ with the open n -cells of X lying in A . That $A \cap X^n$ carries the quotient topology of $(A \cap X^{n-1}) \sqcup \coprod D_\alpha^n$, the coproduct over those cells, is Lemma 2.2(ii) applied inside the closed subspace $A \cap X^n$ of X^n . Finally A has the weak topology: if $C \subseteq A$ meets each $A \cap X^n$ in a closed set, then $C \cap X^n$ is closed in $A \cap X^n$, which is closed in X^n , so $C \cap X^n$ is closed in X^n for all n and C is closed in X . $\square$

This completes part (ii) of Theorem 2.1.

Theorem 2.11. *Every compact subset of a CW complex X is contained in a finite subcomplex. Consequently X is compact if and only if it is finite.*

Proof. First, every $\overline{e}_\alpha^n$ lies in a finite subcomplex; we induct on n . For $n = 0$ the cell is a point and $\{e_\alpha^0\}$ is itself a finite subcomplex by Proposition 2.10. For $n \geq 1$ the set $\varphi_\alpha(S^{n-1})$ is compact and contained in X^{n-1} , so by Proposition 2.9 it meets only finitely many cells $e_1, \dots, e_r$, all of dimension $< n$, and it is contained in their union. By induction each $\overline{e}_i$ lies in a finite subcomplex A_i , and

$$A = A_1 \cup \dots \cup A_r \cup e_\alpha^n$$

is a union of finitely many open cells containing the closure of each of them, since $\overline{e}_\alpha^n \subseteq e_\alpha^n \cup \varphi_\alpha(S^{n-1}) \subseteq A$. By Proposition 2.10 A is a finite subcomplex containing $\overline{e}_\alpha^n$.

Now let $K \subseteq X$ be compact. By Proposition 2.9 K meets only finitely many open cells, say $e_1, \dots, e_s$, and $K \subseteq e_1 \cup \dots \cup e_s$. Choosing for each i a finite subcomplex $B_i \supseteq \overline{e}_i$ as above, the union $B_1 \cup \dots \cup B_s$ is a finite subcomplex containing K .

If X is compact, then X is contained in a finite subcomplex, hence has finitely many cells. Conversely a finite complex is the union of the finitely many compact sets $\overline{e}_\alpha^n$, hence compact. $\square$

This completes part (iii) of Theorem 2.1.

Two constructions are recorded here for later use. The first is attaching one CW complex to another along a subcomplex. Attaching a single cell, Definition 1.1, is the case $Z = D^n$, $A = S^{n-1}$.

Proposition 2.12 (attaching a complex along a subcomplex). *Let Y and Z be CW complexes, let $A \subseteq Z$ be a subcomplex, and let $g: A \rightarrow Y$ be a cellular map. Form*

$$W = Y \cup_g Z = (Y \sqcup Z) / (a \sim g(a) \text{ for } a \in A),$$

with quotient map q . Then q carries Y homeomorphically onto a closed subspace of W , and W is a CW complex with skeleta $W^n = q(Y^n \cup Z^n)$ whose cells are the images of the cells of Y together with the images of the cells of Z not contained in A , the characteristic maps being the $q \circ \Phi$. In particular $q(Y)$ is a subcomplex of W , and so is the image of any subcomplex of Z containing A .

Proof. The equivalence classes are the sets $\{y\} \cup g^{-1}(y)$ for $y \in Y$ together with the singletons $\{z\}$ for $z \in Z \setminus A$. So q is injective on Y and on $Z \setminus A$; and a cell of Z , being either contained in A or disjoint from it, is either collapsed into Y or carried injectively into W . For $B \subseteq Y$ closed, $q^{-1}(q(B)) = B \sqcup g^{-1}(B)$, and $g^{-1}(B)$ is closed in A , hence in Z because A is closed; so $q(B)$ is closed. Thus $q|_Y$ is a closed embedding, and we identify Y with its image.

The weak topology. Let $C \subseteq W$ be such that $(q\Phi)^{-1}(C)$ is closed for every characteristic map Φ of a cell of Y and of a cell of $Z \setminus A$. Then $q^{-1}(C) \cap Y$ is closed, by Proposition 2.3 applied to Y . If Φ is the characteristic map of a cell of A then $q \circ \Phi$ is $g \circ \Phi$ followed by the inclusion of Y , so

$$\Phi^{-1}(q^{-1}(C)) = (g\Phi)^{-1}(q^{-1}(C) \cap Y)$$

is closed too; hence $q^{-1}(C) \cap Z$ is closed by Proposition 2.3 applied to Z , and C is closed in W . The converse is the continuity of the $q\Phi$.

The skeleta. A cell of A of dimension at most n is carried by g into Y^n , g being cellular, so $W^n = q(Y^n \cup Z^n)$ is the union of Y^n with the images of the cells of $Z \setminus A$ of dimension at most n ; in particular the W^n increase with union W , and W^0 is discrete. The set $q^{-1}(W^n)$ is $Y^n \sqcup (Z^n \cup g^{-1}(Y^n))$, which is closed and saturated, so W^n is closed and is a quotient of it; and it is equally a quotient of the smaller set $Y^n \sqcup Z^n$, since a subset of W^n whose preimage there is closed has closed preimage in $g^{-1}(Y^n)$ as well, that preimage being g^{-1} of its intersection with Y^n .

Now Y^n is a quotient of $Y^{n-1} \sqcup \coprod_{\beta} D_{\beta}^n$ and Z^n of $Z^{n-1} \sqcup \coprod_{\gamma} D_{\gamma}^n$, the γ running over all n -cells of Z ; composing, W^n is a quotient of

$$Y^{n-1} \sqcup Z^{n-1} \sqcup \coprod_{\beta} D_{\beta}^n \sqcup \coprod_{\gamma} D_{\gamma}^n.$$

The summands D_{γ}^n belonging to cells of A may be deleted without changing the quotient: each maps into Y^n , through g , so the map out of it is determined by the map out of $Y^{n-1} \sqcup \coprod_{\beta} D_{\beta}^n$, and its preimage of a set is closed as soon as that set's preimage in Y^n is. What is left is $Y^{n-1} \sqcup Z^{n-1}$, which maps onto W^{n-1} by a quotient map by the previous paragraph in degree $n-1$, together with one D^n for each n -cell of Y and each n -cell of Z not in A . So W^n is obtained from W^{n-1} by attaching those cells along the composites of their attaching maps with q , which is Definition 1.2(ii); and (iii) holds because a subset

of W meeting each W^n in a closed set pulls back closed under every $q\Phi$, so is closed by the previous paragraph.

Finally Y is a union of cells of W containing the closure of each, so it is a subcomplex by Proposition 2.10, and the same holds for the image of a subcomplex of Z containing A . $\square$

The second is the homotopy extension property, in the form in which it will be used: a homotopy prescribed on a subcomplex, together with a map on the whole complex, extends to a homotopy of that map.

Proposition 2.13 (the homotopy extension property). *Let (X, A) be a CW pair. Then $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. Consequently, given a map $f: X \rightarrow Z$ and a homotopy $g: A \times I \rightarrow Z$ with $g(a, 0) = f(a)$ for $a \in A$, there is a homotopy $F: X \times I \rightarrow Z$ with $F(x, 0) = f(x)$ for all $x \in X$ and $F(a, t) = g(a, t)$ for all $(a, t) \in A \times I$.*

Proof. Write $r: D^n \times I \rightarrow D^n \times \{0\} \cup S^{n-1} \times I$ for the radial projection from a point above the cylinder, which is Lemma 6.3 of *The homotopy category*, and the homotopy groups applied to the compact convex body D^n .

Construct retractions $R_n: (A \cup X^n) \times I \rightarrow (A \cup X^n) \times \{0\} \cup A \times I$ by induction, R_{-1} being the identity of $A \times I$. Suppose R_{n-1} is given and let Φ be the characteristic map of an n -cell of X not lying in A . For $z \in D^n$ and $t \in I$ set

$$R_n(\Phi(z), t) = \begin{cases} (\Phi(z'), 0), & r(z, t) = (z', 0), \\ R_{n-1}(\Phi(z'), t'), & r(z, t) = (z', t') \text{ with } z' \in S^{n-1}, \end{cases}$$

and let $R_n = R_{n-1}$ on $(A \cup X^{n-1}) \times I$. The two clauses agree where they overlap, that is when $z' \in S^{n-1}$ and $t' = 0$, because R_{n-1} is the identity on $(A \cup X^{n-1}) \times \{0\}$; and the prescription is independent of the choice of z with $\Phi(z)$ a given point of the open cell, Φ being injective there; on the attaching sphere it reduces to R_{n-1} . It is continuous because $(A \cup X^n) \times I$ carries the weak topology determined by $(A \cup X^{n-1}) \times I$ and the sets $D^n \times I$, the product of a quotient map with the identity of the locally compact space I being again a quotient map. It is the identity on $(A \cup X^n) \times \{0\} \cup A \times I$.

Let $R: X \times I \rightarrow X \times \{0\} \cup A \times I$ agree with R_n on $(A \cup X^n) \times I$; these agree on overlaps, and R is continuous because $X \times I$ carries the weak topology determined by the $X^n \times I$, again because I is locally compact. It is a retraction.

For the last statement, f and g agree on $X \times \{0\} \cap A \times I = A \times \{0\}$ and the two sets are closed in $X \times \{0\} \cup A \times I$, so they define a continuous map there; and F is that map composed with R . Since R fixes $X \times \{0\}$ and $A \times I$ pointwise, F has the required restrictions. $\square$

3. EXAMPLES

In each example below the verification consists of exhibiting the skeleta and the characteristic maps and checking the conditions of Definition 1.2; condition (iii) is automatic for the finite complexes, and for the infinite ones (S^∞ , $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$) the weak topology is imposed as part of the definition of the union.

Example 3.1 (Spheres with two cells). Give S^n , $n \geq 1$, the structure with one 0-cell and one n -cell: take $X^0 = \{*\}$, let $X^k = X^0$ for $0 \leq k < n$, and attach a single n -cell along the

constant map $\varphi: S^{n-1} \rightarrow \{*\}$. The resulting space is $D^n/\partial D^n$, which is homeomorphic to S^n , and the characteristic map $\Phi: D^n \rightarrow S^n$ is the quotient map collapsing the boundary. Thus $S^n = e^0 \cup e^n$, and this is the smallest cell structure on S^n : a complex with a single cell is a point, so at least two are needed.

Example 3.2 (Spheres with two cells in each dimension). Alternatively give S^n two cells in each dimension $0, 1, \dots, n$. Take the k -skeleton to be the subsphere

$$S^k = \{x \in S^n : x_{k+2} = \dots = x_{n+1} = 0\} \subseteq S^n,$$

so that S^0 is a pair of points, and let the two k -cells be the open hemispheres $e_{\pm}^k = \{x \in S^k : \pm x_{k+1} > 0\}$. The characteristic maps

$$\Phi_{\pm}^k: D^k \longrightarrow S^k, \quad \Phi_{\pm}^k(x) = (x, \pm \sqrt{1 - |x|^2}),$$

are homeomorphisms of D^k onto the closed upper and lower hemispheres, and both restrict on S^{k-1} to the identity map of S^{k-1} : the two hemispheres of S^k are attached to S^{k-1} along the identity. Taking the union over all k gives a CW structure on $S^\infty = \bigcup_n S^n$ with two cells in every dimension. This structure is the one compatible with the antipodal map, and it is what makes Example 3.3 work.

Example 3.3 (Real projective spaces). Let $\mathbb{R}P^k = S^k/(x \sim -x)$ and let $\pi: S^k \rightarrow \mathbb{R}P^k$ be the quotient map. The inclusions $\mathbb{R}P^0 \subseteq \mathbb{R}P^1 \subseteq \dots \subseteq \mathbb{R}P^k$ are a cell structure with one cell in each dimension $0, 1, \dots, k$: the closed upper hemisphere of S^n is a copy of D^n meeting each antipodal pair exactly once except along the equator, where it contains both points of the pair, so

$$\Phi_n: D^n \longrightarrow \mathbb{R}P^n, \quad \Phi_n(x) = \pi(x, \sqrt{1 - |x|^2}),$$

is a characteristic map: it is injective on $\text{int } D^n$ with image $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$, and on S^{n-1} it is the double covering $\varphi_n = \pi: S^{n-1} \rightarrow \mathbb{R}P^{n-1}$. Equivalently

$$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_{\varphi_n} D^n,$$

real projective n -space is obtained from $\mathbb{R}P^{n-1}$ by attaching a single n -cell along the double cover of $\mathbb{R}P^{n-1}$ by S^{n-1} . Letting $k \rightarrow \infty$ gives a CW structure on $\mathbb{R}P^\infty$ with one cell in each dimension. For $n = 1$ the attaching map $\varphi_1: S^0 \rightarrow \mathbb{R}P^0$ is the constant map and $\mathbb{R}P^1 = S^1$; for $n = 2$ the attaching map is the double cover $S^1 \rightarrow \mathbb{R}P^1 = S^1$, that is, the map $z \mapsto z^2$.

Example 3.4 (Complex projective spaces). Let $\mathbb{C}P^k$ be the space of complex lines through the origin in $\mathbb{C}^{k+1}$, with homogeneous coordinates $[z_0 : \dots : z_k]$. Regard D^{2n} as the unit ball $\{z \in \mathbb{C}^n : |z| \leq 1\}$ and define

$$\Phi_n: D^{2n} \longrightarrow \mathbb{C}P^n, \quad \Phi_n(z_0, \dots, z_{n-1}) = [z_0 : \dots : z_{n-1} : \sqrt{1 - |z|^2}].$$

Every line whose last coordinate is non-zero has a unique representative of norm 1 with last coordinate real and positive, so Φ_n is injective on $\text{int } D^{2n}$ with image $\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}$; on the boundary S^{2n-1} it is $\varphi_n(z) = [z_0 : \dots : z_{n-1} : 0]$, the quotient map $S^{2n-1} \rightarrow \mathbb{C}P^{n-1}$ by the action of the unit complex numbers. Hence

$$\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_{\varphi_n} D^{2n},$$

and $\mathbb{C}P^k$ is a CW complex with one cell in each even dimension $0, 2, \dots, 2k$ and no cells in odd dimensions; $\mathbb{C}P^\infty$ has one cell in each even dimension. For $n = 1$ the attaching map $\varphi_1: S^1 \rightarrow \mathbb{C}P^0 = \{*\}$ is constant, which recovers $\mathbb{C}P^1 = S^2$ as in Example 3.1; for $n = 2$ the attaching map $\varphi_2: S^3 \rightarrow \mathbb{C}P^1 = S^2$ is the Hopf map.

Example 3.5 (Lens spaces). Fix $m \geq 2$, let $\zeta = e^{2\pi i/m}$, and let the cyclic group $\mathbb{Z}/m$ act on the unit sphere $S^{2n-1} = \{z \in \mathbb{C}^n : |z| = 1\}$ diagonally by

$$\zeta \cdot (z_1, \dots, z_n) = (\zeta^{\ell_1} z_1, \dots, \zeta^{\ell_n} z_n),$$

where $\ell_1, \dots, \ell_n$ are integers, which we assume are relatively prime to m . The action is then free, no non-trivial ζ^r fixing a point with some $z_j \neq 0$. The quotient

$$L = L^{2n-1}(m; \ell_1, \dots, \ell_n) = S^{2n-1}/(\mathbb{Z}/m)$$

is the associated *lens space*, a closed $(2n-1)$ -manifold; the quotient map $\pi: S^{2n-1} \rightarrow L$ is an m -fold covering. For $m = 2$ (necessarily ℓ_j odd) the action is the antipodal map and $L = \mathbb{R}P^{2n-1}$, and for $n = 1$ it is S^1 covering S^1 .

There is a CW structure on L with exactly one cell in each dimension $0, 1, \dots, 2n-1$. It comes from an equivariant cell structure upstairs. Write a point of S^{2n-1} as $(z_1, \dots, z_k, w, 0, \dots, 0)$ when its last $n-k-1$ coordinates vanish, so that $S^{2k+1} = \{z \in S^{2n-1} : z_{k+2} = \dots = z_n = 0\}$ is an invariant subsphere and $S^{-1} \subseteq S^1 \subseteq S^3 \subseteq \dots \subseteq S^{2n-1}$. For $0 \leq k \leq n-1$ and $0 \leq j < m$ put

$$E_j^{2k} = \{(z_1, \dots, z_k, w, 0, \dots, 0) \in S^{2k+1} : \arg w = 2\pi j/m\},$$

$$E_j^{2k+1} = \{(z_1, \dots, z_k, w, 0, \dots, 0) \in S^{2k+1} : 2\pi j/m < \arg w < 2\pi(j+1)/m\},$$

in both cases with $w \neq 0$. The set E_j^{2k} is an open $2k$ -cell, parametrized homeomorphically by $z = (z_1, \dots, z_k)$ in the open unit ball of $\mathbb{C}^k$ with $|w| = \sqrt{1 - |z|^2}$; the set E_j^{2k+1} is an open $(2k+1)$ -cell, the open sector cut out of $S^{2k+1} \setminus S^{2k-1}$ by two consecutive rays. Since $S^{2k+1} \setminus S^{2k-1} = \{w \neq 0\}$ is the disjoint union of the m rays $\arg w = 2\pi j/m$ and the m open sectors between them, these sets, for all k and j , partition S^{2n-1} into m cells in each dimension $0, 1, \dots, 2n-1$, with S^{2k+1} as $(2k+1)$ -skeleton. The characteristic maps are the evident ones: for E_j^{2k} it is

$$\tilde{\Phi}_j: D^{2k} \longrightarrow S^{2n-1}, \quad \tilde{\Phi}_j(z) = (z, e^{2\pi i j/m} \sqrt{1 - |z|^2}, 0, \dots, 0),$$

a homeomorphism onto $\overline{E}_j^{2k}$ which is the identity of S^{2k-1} on the boundary; and for E_j^{2k+1} it is the closed sector, which is homeomorphic to D^{2k+1} with boundary sphere the union of the two closed faces $\arg w = 2\pi j/m$ and $\arg w = 2\pi(j+1)/m$, glued along S^{2k-1} .

The group permutes these cells: multiplication by ζ carries E_j^{2k} to $E_{j+\ell_{k+1}}^{2k}$ and E_j^{2k+1} to $E_{j+\ell_{k+1}}^{2k+1}$, indices mod m , and since ℓ_{k+1} is prime to m this permutation of the m cells of a given dimension is a cyclic permutation, in particular free and transitive. The action is therefore cellular and permutes the cells of each dimension freely, so the images $\pi(E_0^k)$ are open k -cells of L , one for each $k = 0, 1, \dots, 2n-1$; we write L^k for the k -skeleton, as for any CW complex. The odd skeleta are again lens spaces, $L^{2k+1} = S^{2k+1}/(\mathbb{Z}/m)$. The even ones are not: there L^{2k} is no more than a name for the $2k$ -skeleton, which is the lens space

L^{2k-1} with a single $2k$ -cell attached along the m -fold cyclic covering $\pi: S^{2k-1} \rightarrow L^{2k-1}$. That covering projection is the attaching map,

$$\varphi_{2k} = \pi: S^{2k-1} \longrightarrow L^{2k-1},$$

exactly as for $\mathbb{R}P^n$ in Example 3.3, and the characteristic map of the $2k$ -cell of L , defined on D^{2k} , is $\Phi_{2k} = \pi \circ \tilde{\Phi}_0$. In dimension $2k+1$ the attaching map $\varphi_{2k+1}: S^{2k} \rightarrow L^{2k}$ is π restricted to the boundary sphere $S^{2k} = \overline{E}_0^{2k} \cup \overline{E}_1^{2k}$ of the closed sector; in terms of the parametrizations $\tilde{\Phi}_0, \tilde{\Phi}_1$ of the two hemispheres this says

$$\varphi_{2k+1} \circ \tilde{\Phi}_0 = \Phi_{2k}, \quad \varphi_{2k+1} \circ \tilde{\Phi}_1 = \Phi_{2k} \circ T,$$

where

$$T(z_1, \dots, z_k) = (\zeta^{r\ell_1} z_1, \dots, \zeta^{r\ell_k} z_k), \quad r\ell_{k+1} \equiv -1 \pmod{m},$$

is the action on the first k coordinates of the deck transformation ζ^r , the one carrying $\overline{E}_1^{2k}$ onto $\overline{E}_0^{2k}$. The two prescriptions agree on S^{2k-1} : there $\tilde{\Phi}_0$ and $\tilde{\Phi}_1$ are both the identity, and Φ_{2k} restricts to π , which is invariant under the group. Notice that while L^{2k-1} has one cell in each dimension $\leq 2k-1$, for $m > 2$ the even-dimensional skeleta of positive dimension are not manifolds. For $m = 2$ one has $T = -\text{id}$, the two hemispheres are antipodal, φ_{2k+1} is the double cover $S^{2k} \rightarrow \mathbb{R}P^{2k}$, and these prescriptions reduce to the structure on $\mathbb{R}P^{2n-1}$ with one cell in each dimension.

Example 3.6 (Products, and $S^p \times S^q$). If X and Y are CW complexes, the products $e \times e'$ of an open m -cell of X with an open n -cell of Y partition $X \times Y$ into open cells of dimension $m+n$, with characteristic maps

$$\Phi \times \Phi': D^m \times D^n \longrightarrow X \times Y,$$

where $D^m \times D^n$ is identified with D^{m+n} by a homeomorphism of pairs carrying $\partial(D^m \times D^n) = (S^{m-1} \times D^n) \cup (D^m \times S^{n-1})$ to S^{m+n-1} . The skeleta are $(X \times Y)^k = \bigcup_{m+n=k} X^m \times Y^n$. This is a cell structure in the sense of Definition 1.2 provided the product topology is the weak topology determined by these skeleta, which holds whenever X or Y is locally compact — in particular whenever one of them is a finite complex. (In general one must retopologize $X \times Y$ with the compactly generated topology; the two agree in every example we use.)

Taking $X = S^p$ and $Y = S^q$ with the two-cell structures of Example 3.1, $p, q \geq 1$, gives a CW structure on $S^p \times S^q$ with four cells,

$$\{*\} \times \{*\}, \quad e^p \times \{*\}, \quad \{*\} \times e^q, \quad e^p \times e^q,$$

of dimensions 0, p , q and $p+q$, so that the $(p+q-1)$ -skeleton is the wedge $S^p \vee S^q$. The characteristic map of the top cell is $\Phi_p \times \Phi_q$, and its attaching map $\varphi: S^{p+q-1} = \partial(D^p \times D^q) \rightarrow S^p \vee S^q$ is

$$\varphi(x, y) = (*, \Phi_q(y)) \text{ for } x \in S^{p-1}, \quad \varphi(x, y) = (\Phi_p(x), *) \text{ for } y \in S^{q-1},$$

the two prescriptions agreeing on $S^{p-1} \times S^{q-1}$, where both give the wedge point. For $p = q = 1$ this is the torus, presented as a square with its boundary attached along the word $aba^{-1}b^{-1}$, the case $g = 1$ of Example 3.7.

Example 3.7 (Closed surfaces). Let Σ_g be the closed orientable surface of genus $g \geq 1$, realized as the quotient of a $4g$ -gon P by the boundary identifications given by the word

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

All $4g$ vertices of P are identified to a single point in the quotient, so the identifications produce a CW structure with

$$X^0 = \{*\}, \quad X^1 = \bigvee_{2g} S^1, \quad X^2 = \Sigma_g,$$

having one 0-cell, $2g$ one-cells $a_1, b_1, \dots, a_g, b_g$, and one 2-cell. The characteristic map of the 2-cell is the quotient map $P \rightarrow \Sigma_g$, after a homeomorphism $D^2 \cong P$, and its attaching map $S^1 \rightarrow \bigvee_{2g} S^1$ is the loop that traverses the $2g$ circles in the order and with the orientations prescribed by the word above. For $g = 0$ one takes instead the two-cell structure on S^2 of Example 3.1.

The closed non-orientable surface N_h , $h \geq 1$, is the quotient of a $2h$ -gon by the word $a_1^2 a_2^2 \cdots a_h^2$; again all vertices are identified, and the resulting CW structure has one 0-cell, h one-cells and one 2-cell. Here $N_1 = \mathbb{R}P^2$, recovering the structure of Example 3.3 with $k = 2$, and N_2 is the Klein bottle.

Example 3.8 (Simplicial complexes). Let K be a simplicial complex with geometric realization $|K|$, topologized as usual by the weak topology with respect to the closed simplices, and let $|K^n|$ denote the realization of the n -skeleton of K , that is, the union of the simplices of dimension at most n . Then $|K|$ is a CW complex with skeleta $|K^n|$ and with one n -cell for each n -simplex of K . Indeed, fix once and for all a homeomorphism of pairs $(D^n, S^{n-1}) \cong (\Delta^n, \partial\Delta^n)$ with the standard n -simplex. For an n -simplex σ of K the characteristic map Φ_σ is the composite

$$D^n \xrightarrow{\cong} \Delta^n \rightarrow |K|,$$

the second map being the canonical affine embedding of σ ; it carries $\text{int } D^n$ homeomorphically onto the open simplex $\text{int } \sigma$, and it carries S^{n-1} onto the union of the proper faces of σ , which lies in $|K^{n-1}|$. Since $|K|$ is the disjoint union of its open simplices, $|K^n|$ is obtained from $|K^{n-1}|$ by attaching one n -cell for each n -simplex along these attaching maps, $|K^0|$ is the discrete set of vertices, and the weak topology condition holds by the definition of the topology on $|K|$. So every simplicial complex carries a natural CW structure, and every simplicial map is cellular in the sense of Definition 1.7.

This structure is special in two ways. First, each characteristic map Φ_σ is injective on all of D^n , not merely on the interior, and each closed cell is the union of cells; a CW structure with these properties is called *regular*. Second, it is very inefficient: the smallest simplicial complex with realization S^n is the boundary of Δ^{n+1} , with $2^{n+2} - 2$ simplices in all, whereas the CW structure of Example 3.1 uses two cells. The gain in flexibility is exactly that a CW structure allows the attaching maps to be arbitrary maps of spheres rather than face identifications; this is why $\mathbb{R}P^n$ and $\mathbb{C}P^n$ have such small cell structures above.

4. CELLULAR HOMOLOGY

All homology in this section is singular homology with coefficients $\mathbb{Z}$, and we use freely the long exact sequence of a pair and its naturality (Theorem 6.4 and Proposition 6.5 of *Singular Homology*), homotopy invariance (Corollary 4.2 of *Singular Homology*), excision (Corollary 8.2 of *Singular Homology*), the homology of spheres (Theorem 2.5 of *Further Properties of Singular Homology*) and the behaviour of homology on disjoint unions (Proposition 3.3 of *Singular Homology*). We write $X^{-1} = \emptyset$, so that $H_k(X^0, X^{-1}) = H_k(X^0)$.

The point of the section is that the skeletal filtration of a CW complex has homology concentrated in a single degree at each stage, so that the whole of $H_*(X)$ is squeezed into a chain complex whose n -th group is free abelian on the n -cells.

4.1. The homology of the skeletal pairs.

Lemma 4.1. *Let $n \geq 1$. The characteristic maps induce an isomorphism*

$$\bigoplus_{\alpha \in J_n} H_k(D_\alpha^n, S^{n-1}) \xrightarrow{\cong} H_k(X^n, X^{n-1})$$

for every k . Consequently $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$, while $H_n(X^n, X^{n-1})$ is free abelian with an infinite cyclic summand for each n -cell. For $n = 0$, $H_k(X^0)$ is free abelian on the 0-cells for $k = 0$ and vanishes otherwise.

Proof. The statement for $n = 0$ is Proposition 3.3 of *Singular Homology*, together with the homology of a point (Proposition 3.1 of *Singular Homology*), X^0 being discrete.

Let $n \geq 1$. In the long exact sequence of (D^n, S^{n-1}) the groups $\tilde{H}_k(D^n)$ vanish, D^n being convex and so contractible, whence an isomorphism

$$\partial: H_k(D^n, S^{n-1}) \xrightarrow{\cong} \tilde{H}_{k-1}(S^{n-1}),$$

which by Theorem 2.5 of *Further Properties of Singular Homology* is $\mathbb{Z}$ for $k = n$ and 0 otherwise.

Put $p_\alpha = \Phi_\alpha(0) \in e_\alpha^n$ and $S = \{p_\alpha : \alpha \in J_n\}$. The set S meets each open cell of the CW complex X^n in at most one point, so by Lemma 2.8 it is a closed subspace of X^n with the discrete topology, and $A = X^n \setminus S$ is open in X^n .

X^{n-1} is a deformation retract of A . Define $R: A \times I \rightarrow A$ by $R(v, t) = v$ for $v \in X^{n-1}$ and

$$R(\Phi_\alpha(z), t) = \Phi_\alpha\left((1-t)z + t\frac{z}{|z|}\right), \quad z \in D_\alpha^n, \quad z \neq 0.$$

This is well defined, since for $|z| = 1$ both prescriptions give $\varphi_\alpha(z)$, and it is continuous: it is continuous on $X^{n-1} \times I$ and on each $(D_\alpha^n \setminus \{0\}) \times I$, and $A \times I$ carries the weak topology determined by those subspaces, the product of a quotient map with the identity map of the locally compact space I being again a quotient map. It is the identity at $t = 0$, fixes X^{n-1} throughout, and retracts A onto X^{n-1} at $t = 1$.

Compare the long exact sequences of the pairs (X^n, X^{n-1}) and (X^n, A) . In the resulting ladder the maps $H_k(X^{n-1}) \rightarrow H_k(A)$ are isomorphisms by Corollary 4.2 of *Singular Homology*, and the maps $H_k(X^n) \rightarrow H_k(X^n)$ are identities, so the five lemma (Exercise 4.6 of *Homology*) gives $H_k(X^n, X^{n-1}) \cong H_k(X^n, A)$.

Next excise $Z = X^{n-1}$ from (X^n, A) : this is legitimate because X^{n-1} is closed (Proposition 2.5) and $X^{n-1} \subseteq A = \text{int } A$. By Corollary 8.2 of *Singular Homology*,

$$H_k\left(\coprod_{\alpha} e_{\alpha}^n, \coprod_{\alpha} (e_{\alpha}^n \setminus \{p_{\alpha}\})\right) \xrightarrow{\cong} H_k(X^n, A),$$

the open cells being disjoint open subsets of X^n by Lemma 2.2(iii). By Proposition 3.3 of *Singular Homology* the singular chain complex of a disjoint union is the direct sum of the singular chain complexes, hence so is the relative chain complex, and homology commutes with direct sums (Remark 2.13 of *Homology*); so the left-hand group is $\bigoplus_{\alpha} H_k(e_{\alpha}^n, e_{\alpha}^n \setminus \{p_{\alpha}\})$.

Finally, for each α the characteristic map carries the diagram of pairs

$$(D^n, S^{n-1}) \longrightarrow (D^n, D^n \setminus \{0\}) \longleftarrow (\text{int } D^n, \text{int } D^n \setminus \{0\})$$

to the corresponding diagram $(X^n, X^{n-1}) \rightarrow (X^n, A) \leftarrow (e_{\alpha}^n, e_{\alpha}^n \setminus \{p_{\alpha}\})$. In the upper row the left-hand map induces an isomorphism because S^{n-1} is a deformation retract of $D^n \setminus \{0\}$, and the right-hand map is an excision as before. Since Φ_{α} maps $(\text{int } D^n, \text{int } D^n \setminus \{0\})$ homeomorphically onto $(e_{\alpha}^n, e_{\alpha}^n \setminus \{p_{\alpha}\})$, the isomorphisms above assemble to the asserted isomorphism $\bigoplus_{\alpha} H_k(D_{\alpha}^n, S^{n-1}) \cong H_k(X^n, X^{n-1})$ induced by the Φ_{α} . $\square$

Definition 4.2. Fix once and for all the generator ι_1 of $H_1(D^1, S^0) \cong \mathbb{Z}$ with $\partial \iota_1 = [1] - [-1] \in \tilde{H}_0(S^0)$, and a homeomorphism $h_n: D^n/S^{n-1} \rightarrow S^n$ for each $n \geq 1$, and define generators ι_n of $H_n(D^n, S^{n-1})$ inductively by requiring that

$$\partial \iota_n \in \tilde{H}_{n-1}(S^{n-1})$$

be the image of ι_{n-1} under the isomorphism

$$H_{n-1}(D^{n-1}, S^{n-2}) \cong \tilde{H}_{n-1}(D^{n-1}/S^{n-2}) \cong \tilde{H}_{n-1}(S^{n-1})$$

given by collapsing the boundary and by h_{n-1} ; that isomorphism exists by Proposition 1.2 of *Further Properties of Singular Homology*, the pair (D^{n-1}, S^{n-2}) being a good pair by Lemma 1.3 of *Further Properties of Singular Homology*. Write $s_{n-1} = \partial \iota_n$. For a cell e_{α}^n put

$$[e_{\alpha}^n] = (\Phi_{\alpha})_*(\iota_n) \in H_n(X^n, X^{n-1}),$$

so that by Lemma 4.1 the classes $[e_{\alpha}^n]$, $\alpha \in J_n$, form a basis. They depend on the chosen characteristic maps; a change of Φ_{α} can change $[e_{\alpha}^n]$ by a sign, and choosing them is what is meant by orienting the cells.

4.2. The homology of the skeleta.

Lemma 4.3. *Let X be a CW complex. Every class in $H_k(X)$ is in the image of $H_k(X^N) \rightarrow H_k(X)$ for some N , and if a class in $H_k(X^n)$ maps to 0 in $H_k(X)$ then it maps to 0 in $H_k(X^N)$ for some $N \geq n$.*

Proof. A singular chain is a finite sum of singular simplices, so its image is a finite union of compact sets and hence compact; by Theorem 2.11 that image lies in a finite subcomplex, and a finite subcomplex lies in some X^N . Apply this to a cycle representing a given class, and to a chain whose boundary is a given cycle. $\square$

Proposition 4.4. *Let X be a CW complex. Then*

- (i) $H_k(X^n) = 0$ for $k > n$;
- (ii) the inclusion $X^n \hookrightarrow X^m$ induces an isomorphism $H_k(X^n) \rightarrow H_k(X^m)$ whenever $k < n \leq m$;
- (iii) the inclusion $X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n) \rightarrow H_k(X)$ for $k < n$.

In particular $H_k(X) = 0$ for $k > \dim X$, and $H_k(X)$ is generated by at most $\#J_k$ elements.

Proof. (i) Induct on n . For $n = 0$ this is Lemma 4.1. For $n \geq 1$ the long exact sequence of (X^n, X^{n-1}) contains

$$H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1}),$$

whose outer terms vanish for $k > n$, the first by induction and the second by Lemma 4.1.

(ii) It suffices to treat $m = n + 1$ and iterate. The long exact sequence of (X^{n+1}, X^n) contains

$$H_{k+1}(X^{n+1}, X^n) \longrightarrow H_k(X^n) \longrightarrow H_k(X^{n+1}) \longrightarrow H_k(X^{n+1}, X^n),$$

and for $k < n$ both outer terms vanish by Lemma 4.1, since $k + 1 \leq n < n + 1$ and $k \neq n + 1$.

(iii) Surjectivity: a class in $H_k(X)$ comes from some $H_k(X^N)$ by Lemma 4.3, and we may take $N \geq n$, whence it comes from $H_k(X^n)$ by (ii). Injectivity: a class in $H_k(X^n)$ dying in $H_k(X)$ dies in some $H_k(X^N)$ with $N \geq n$, and the map to that group is injective by (ii).

For the last statements, if $X = X^d$ then $H_k(X) = H_k(X^d) = 0$ for $k > d$ by (i); and $H_k(X) \cong H_k(X^{k+1})$ is a quotient of a subgroup of $H_k(X^k, X^{k-1})$ by the sequences just used, a group free of rank $\#J_k$. $\square$

4.3. The cellular chain complex.

Definition 4.5. The *cellular chain complex* of a CW complex X is

$$C_n^{\text{cw}}(X) = H_n(X^n, X^{n-1}), \quad d_n = j_{n-1} \circ \partial_n: C_n^{\text{cw}}(X) \rightarrow C_{n-1}^{\text{cw}}(X).$$

Here ∂_n is the connecting homomorphism of the pair (X^n, X^{n-1}) , and j_{n-1} is the map

$$j_{n-1}: H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

induced by the inclusion of pairs. By Lemma 4.1, $C_n^{\text{cw}}(X)$ is free abelian with basis the n -cells with a choice of orientation for each. Its homology is denoted $H_n^{\text{cw}}(X)$.

Lemma 4.6. $d_n \circ d_{n+1} = 0$, so $C_*^{\text{cw}}(X)$ is a chain complex.

Proof. $d_n d_{n+1} = j_{n-1}(\partial_n j_n) \partial_{n+1}$, and $\partial_n j_n = 0$ because j_n and ∂_n are consecutive maps in the long exact sequence of (X^n, X^{n-1}) . $\square$

Theorem 4.7. For every CW complex X and every $n \geq 0$ there is an isomorphism

$$H_n^{\text{cw}}(X) \xrightarrow{\cong} H_n(X).$$

Proof. Write $C_n = C_n^{\text{cw}}(X)$. By Proposition 4.4(i), $H_n(X^{n-1}) = 0$ and $H_{n-1}(X^{n-2}) = 0$, so the exact sequences

$$0 = H_n(X^{n-1}) \longrightarrow H_n(X^n) \xrightarrow{j_n} C_n, \quad 0 = H_{n-1}(X^{n-2}) \longrightarrow H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} C_{n-1}$$

show that j_n and j_{n-1} are injective. Since j_{n-1} is injective,

$$\ker d_n = \ker(j_{n-1}\partial_n) = \ker \partial_n = \operatorname{im} j_n = j_n(H_n(X^n)),$$

the third equality by exactness of $H_n(X^n) \xrightarrow{j_n} C_n \xrightarrow{\partial_n} H_{n-1}(X^{n-1})$. Also

$$\operatorname{im} d_{n+1} = j_n(\operatorname{im} \partial_{n+1}).$$

Therefore, j_n being injective,

$$H_n^{\text{cw}}(X) = \ker d_n / \operatorname{im} d_{n+1} = j_n(H_n(X^n)) / j_n(\operatorname{im} \partial_{n+1}) \cong H_n(X^n) / \operatorname{im} \partial_{n+1}.$$

On the other hand the long exact sequence of (X^{n+1}, X^n) contains

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) = 0,$$

the last group vanishing by Lemma 4.1; so $H_n(X^n) / \operatorname{im} \partial_{n+1} \cong H_n(X^{n+1})$. Finally $H_n(X^{n+1}) \cong H_n(X)$ by Proposition 4.4(iii). Composing the three isomorphisms gives the theorem. $\square$

Addendum 4.8. A cellular map $f: X \rightarrow Y$ is a map of pairs $(X^n, X^{n-1}) \rightarrow (Y^n, Y^{n-1})$ for every n and so induces $f_*: C_n^{\text{cw}}(X) \rightarrow C_n^{\text{cw}}(Y)$; these maps commute with the differentials, so C_*^{cw} is a functor $\mathbf{CW} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ from the category of Definition 1.8 to the category of non-negative chain complexes of abelian groups (Remark 2.11 of Homology). The isomorphism of Theorem 4.7 is then a natural isomorphism between the two functors $\mathbf{CW} \rightarrow \mathbf{Ab}$ that it relates: the square

$$\begin{array}{ccc} H_n^{\text{cw}}(X) & \xrightarrow{\cong} & H_n(X) \\ H_n^{\text{cw}}(f) \downarrow & & \downarrow H_n(f) \\ H_n^{\text{cw}}(Y) & \xrightarrow{\cong} & H_n(Y) \end{array}$$

commutes for every cellular $f: X \rightarrow Y$ and every n , the right-hand vertical map being taken along the forgetful functor $\mathbf{CW} \rightarrow \mathbf{Top}$.

Proof. That f carries (X^n, X^{n-1}) into (Y^n, Y^{n-1}) is the definition of cellularity, Definition 1.7. Both the connecting homomorphism ∂_n and the map j_{n-1} induced by an inclusion of pairs are natural for maps of pairs (Proposition 6.5 of *Singular Homology*), so $f_*d_n = d_nf_*$ and f_* is a chain map; functoriality of C_*^{cw} is then that of $H_n(\cdot, \cdot)$.

The isomorphism of Theorem 4.7 is the composite of the three isomorphisms

$$H_n^{\text{cw}}(X) \cong H_n(X^n) / \operatorname{im} \partial_{n+1} \cong H_n(X^{n+1}) \cong H_n(X)$$

exhibited in its proof. The first is induced by j_n , hence natural by the naturality just used; the second and third are induced by the inclusions $X^n \subseteq X^{n+1} \subseteq X$, with which f commutes because $f(X^n) \subseteq Y^n$ and $f(X^{n+1}) \subseteq Y^{n+1}$. Each square in the resulting diagram therefore commutes, and so does the outer one. $\square$

Corollary 4.9. If no pair of cells of X have dimensions differing by one, then $H_n(X)$ is free abelian with basis the n -cells with a choice of orientation for each, for every n .

Proof. Each d_n has source or target zero, so $H_n^{\text{cw}}(X) = C_n^{\text{cw}}(X)$. $\square$

Example 4.10. Corollary 4.9 applies to the two-cell structure on S^n for $n \geq 2$, giving $H_0 = H_n = \mathbb{Z}$ and $H_k = 0$ otherwise, and to $\mathbb{C}P^k$ of Example 3.4, whose cells are all of even dimension:

$$H_i(\mathbb{C}P^k) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \dots, 2k, \\ 0, & \text{otherwise,} \end{cases} \quad H_i(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z}, & i \text{ even, } i \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It applies as well to $S^p \times S^q$ of Example 3.6 when $2 \leq p \leq q$ and $q \neq p + 1$, giving $\mathbb{Z}$ in the dimensions $0, p, q, p + q$ and 0 elsewhere; for $p = q$ the group in dimension p is $\mathbb{Z}^2$.

4.4. The boundary map and degrees. For the remaining computations one must identify d_n on the basis, and this is done by degrees of maps between spheres.

Remark 4.11. Collapsing X^{n-1} in X^n collapses each boundary sphere $S^{n-1} = \partial D_\alpha^n$ to the same single point, so directly from Definition 1.1

$$X^n/X^{n-1} = \bigvee_{\alpha \in J_n} D_\alpha^n/S^{n-1} \cong \bigvee_{\alpha \in J_n} S^n,$$

using the homeomorphisms h_n of Definition 4.2. For $\beta \in J_n$ let

$$q_\beta: X^n \longrightarrow X^n/X^{n-1} \longrightarrow S^n$$

be the quotient map followed by the projection that collapses all wedge summands but the β -th; it is a map of pairs $(X^n, X^{n-1}) \rightarrow (S^n, *)$, and $q_\beta \circ \Phi_\gamma$ is constant for $\gamma \neq \beta$ while $q_\beta \circ \Phi_\beta$ is the map $D^n \rightarrow D^n/S^{n-1} \cong S^n$ collapsing the boundary.

Definition 4.12. For $n \geq 1$ and a map $f: S^n \rightarrow S^n$, the induced endomorphism of $\tilde{H}_n(S^n) \cong \mathbb{Z}$ is multiplication by an integer $\deg f$, the *degree* of f ; it does not depend on the choice of generator. Clearly $\deg(\text{id}) = 1$, $\deg(g \circ f) = \deg g \cdot \deg f$, and a constant map has degree 0. The same definition is used for $n = 0$ with $\tilde{H}_0(S^0) \cong \mathbb{Z}$ generated by $[1] - [-1]$.

Theorem 4.13 (cellular boundary formula). *Let $n \geq 2$ and $\alpha \in J_n$. Then*

$$d_n[e_\alpha^n] = \sum_{\beta \in J_{n-1}} d_{\alpha\beta}[e_\beta^{n-1}], \quad d_{\alpha\beta} = \deg(q_\beta \circ \varphi_\alpha: S^{n-1} \rightarrow S^{n-1}),$$

where $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$ is the attaching map of the cell and $q_\beta: X^{n-1} \rightarrow S^{n-1}$ is the collapse of Remark 4.11. Only finitely many $d_{\alpha\beta}$ are non-zero. For $n = 1$,

$$d_1[e_\alpha^1] = [\varphi_\alpha(1)] - [\varphi_\alpha(-1)] \in H_0(X^0),$$

the difference of the two endpoints of the 1-cell.

Proof. The image $\varphi_\alpha(S^{n-1})$ is compact, so by Proposition 2.9 it meets only finitely many cells; for every β with e_β^{n-1} disjoint from it the map $q_\beta \varphi_\alpha$ is constant and $d_{\alpha\beta} = 0$. This proves the finiteness, and shows also that the sum on the right is meaningful.

By naturality of the long exact sequence (Proposition 6.5 of *Singular Homology*) applied to the map of pairs $\Phi_\alpha: (D^n, S^{n-1}) \rightarrow (X^n, X^{n-1})$,

$$\partial_n[e_\alpha^n] = \partial_n(\Phi_\alpha)_* \iota_n = (\varphi_\alpha)_*(\partial \iota_n) = (\varphi_\alpha)_* s_{n-1},$$

so that $d_n[e_\alpha^n] = j_{n-1}(\varphi_\alpha)_* s_{n-1}$.

Write $d_n[e_\alpha^n] = \sum_\gamma c_\gamma[e_\gamma^{n-1}]$ in the basis of $C_{n-1}^{\text{cw}}(X)$, and apply the homomorphism

$$(q_\beta)_*: H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow H_{n-1}(S^{n-1}, *) \cong \tilde{H}_{n-1}(S^{n-1}),$$

the last identification being Corollary 6.8 of *Singular Homology*. By Remark 4.11, $q_\beta\Phi_\gamma$ is constant for $\gamma \neq \beta$, so $(q_\beta)_*[e_\gamma^{n-1}] = (q_\beta\Phi_\gamma)_*\iota_{n-1} = 0$; and $q_\beta\Phi_\beta$ is the collapse $D^{n-1} \rightarrow D^{n-1}/S^{n-2} \cong S^{n-1}$, so $(q_\beta)_*[e_\beta^{n-1}] = s_{n-1}$ by the choice of generators in Definition 4.2. Hence the left-hand side becomes $c_\beta s_{n-1}$.

On the other hand, functoriality applied to $(X^{n-1}, \emptyset) \rightarrow (X^{n-1}, X^{n-2}) \xrightarrow{q_\beta} (S^{n-1}, *)$ gives $(q_\beta)_*j_{n-1} = (q_\beta)_*$ on $\tilde{H}_{n-1}(X^{n-1})$, so

$$c_\beta s_{n-1} = (q_\beta)_*j_{n-1}(\varphi_\alpha)_*s_{n-1} = (q_\beta\varphi_\alpha)_*s_{n-1} = \deg(q_\beta\varphi_\alpha) s_{n-1},$$

whence $c_\beta = \deg(q_\beta\varphi_\alpha)$.

For $n = 1$ the same computation applies with $S^0 = \{\pm 1\}$ and $s_0 = [1] - [-1]$: $d_1[e_\alpha^1] = j_0(\varphi_\alpha)_*([1] - [-1]) = [\varphi_\alpha(1)] - [\varphi_\alpha(-1)]$, since $C_0^{\text{cw}}(X) = H_0(X^0)$ is free on the points of X^0 and j_0 is the identity. $\square$

Remark 4.14. For a complex with cells in adjacent dimensions, where Corollary 4.9 does not apply — those of Examples 3.2, 3.3, 3.5 and 3.7, for instance, and $S^p \times S^q$ of Example 3.6 when $p = 1$ or $q = p + 1$ — Theorem 4.13 reduces the computation of H_* to the degrees of the explicit maps $q_\beta\varphi_\alpha$. There is no general direct topological mechanism for computing such a degree, but under favourable hypotheses, which can always be achieved by a small deformation of the map, there are nice descriptions. Degree is a homotopy invariant, by Theorem 4.1 of *Singular Homology*, so nothing is lost in deforming.

(i) Suppose there is a non-empty open set U of the range sphere such that f restricts to a smooth map $f^{-1}(U) \rightarrow U$ and such that $p \in U$ is a regular value of that restriction. Then $f^{-1}(p)$ is a finite set of points, and

$$\deg f = \sum_{q \in f^{-1}(p)} \text{sign det}(df_q).$$

(ii) Suppose K is a closed subset of the range sphere S^n with non-empty interior, and that we are given a triangulation of K and one of $f^{-1}(K)$ for which the restriction $f|_{f^{-1}(K)}$ is simplicial. Fix an n -simplex σ of K . Then $\deg f$ is the sum, over the n -simplices of $f^{-1}(K)$ that map onto σ , of $+1$ or -1 according as f preserves or reverses orientation on that simplex.

The smooth statement will be proved in the section of the course on manifolds; the simplicial version is covered by the homework.

5. THE EILENBERG–STEENROD AXIOMS FOR HOMOLOGY

The computations of §4 used only a handful of properties of singular homology, and they are the properties that Eilenberg and Steenrod isolated as the axioms for a homology theory. A *homology theory* assigns to every pair of spaces (X, A) a sequence of abelian groups $H_n(X, A)$, $n \in \mathbb{Z}$, and to every map of pairs $f: (X, A) \rightarrow (Y, B)$ homomorphisms $f_*: H_n(X, A) \rightarrow H_n(Y, B)$, functorially; one writes $H_n(X)$ for $H_n(X, \emptyset)$. The axioms are the following.

(A1) *Dimension.* For a one-point space P ,

$$H_n(P) = \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

(A2) *Homotopy.* If $f, g: (X, A) \rightarrow (Y, B)$ are homotopic through maps of pairs then $f_* = g_*$ in every degree.

(A3) *The long exact sequence of a pair.* For every pair (X, A) there are connecting homomorphisms $\partial: H_n(X, A) \rightarrow H_{n-1}(A)$ making

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

exact, the unlabelled maps being induced by the inclusions; and ∂ is natural, in the sense that a map of pairs carries one such sequence to another commutingly.

(A4) *Excision.* If $Z \subseteq A \subseteq X$ with $\overline{Z} \subseteq \text{int } A$, then the inclusion of pairs $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \text{for every } n.$$

(A5) *Direct limits.* Let X be the union of a family $\{X_i\}_{i \in I}$ of subspaces, directed by inclusion, in such a way that every compact subset of X lies in some X_i . Then the canonical map

$$\varinjlim_i H_n(X_i) \xrightarrow{\cong} H_n(X) \quad \text{for every } n$$

is an isomorphism.

Singular homology satisfies all five: (A1) is Proposition 3.1 of *Singular Homology*, (A2) is Theorem 6.6 there, (A3) is Theorem 6.4 together with Proposition 6.5, and (A4) is Corollary 8.2. For (A5), a singular simplex has compact image and a chain is a finite sum of simplices, so $C_*(X) = \varinjlim_i C_*(X_i)$, and homology commutes with direct limits by Proposition 2.12 of *Homology*. Arbitrary disjoint unions are a special case of (A5): the finite sub-unions of $\coprod_\alpha X_\alpha$ form a family directed by inclusion in which every compact set lies, and each of them is handled by Corollary 5.2(ii) below, so that $\bigoplus_\alpha H_n(X_\alpha) \cong H_n(\coprod_\alpha X_\alpha)$. The form of excision used in practice is the one proved in *Singular Homology* as Theorem 8.1: the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms whenever $X = \text{int } A \cup \text{int } B$. It follows from (A4) on taking $Z = X \setminus B$, whose closure lies in $\text{int } A$ because $X \setminus \text{int } A \subseteq \text{int } B$.

The axioms already compute the homology of a disjoint union of two spaces.

Lemma 5.1. *Let H_* satisfy (A3) and (A4) and let $X = X_1 \sqcup X_2$ be a disjoint union, so that X_1 and X_2 are open and closed in X . Then the inclusions $i: X_1 \rightarrow X$ and $k: X_2 \rightarrow X$ induce an isomorphism*

$$(i_*, k_*): H_n(X_1) \oplus H_n(X_2) \xrightarrow{\cong} H_n(X) \quad \text{for every } n.$$

Proof. Excise $Z = X_1$ from the pair (X, X_1) . This is legitimate: X_1 is closed and open, so $\overline{Z} = X_1 = \text{int } X_1$. Since $X \setminus Z = X_2$ and $X_1 \setminus Z = \emptyset$, axiom (A4) says that the inclusion of pairs $(X_2, \emptyset) \hookrightarrow (X, X_1)$ induces isomorphisms

$$H_n(X_2) \xrightarrow{\cong} H_n(X, X_1).$$

That inclusion is the composite of k with the map of pairs $(X, \emptyset) \rightarrow (X, X_1)$, so the isomorphism is j_*k_* , where $j_*: H_n(X) \rightarrow H_n(X, X_1)$ is the map in the long exact sequence of (A3).

In particular j_* is surjective, so $\partial = 0$ in every degree, ∂j_* being zero by exactness; hence i_* is injective and the long exact sequence breaks into short exact sequences

$$0 \longrightarrow H_n(X_1) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, X_1) \longrightarrow 0.$$

The composite of $(j_*k_*)^{-1}$ with k_* is a section of j_* , so the sequence splits and $H_n(X) = i_*H_n(X_1) \oplus k_*H_n(X_2)$, which is the assertion. $\square$

Corollary 5.2. *Let H_* satisfy (A3) and (A4).*

- (i) $H_n(\emptyset) = 0$ for every n .
- (ii) *If $X = X_1 \sqcup \cdots \sqcup X_r$ is a finite disjoint union then the inclusions induce an isomorphism*

$$\bigoplus_{j=1}^r H_n(X_j) \xrightarrow{\cong} H_n(X) \quad \text{for every } n.$$

Proof. (i) Apply Lemma 5.1 to $X = \emptyset \sqcup X$ for any space X , say a point. There k is the identity of X , so $k_*H_n(X)$ is all of $H_n(X)$; as the sum $i_*H_n(\emptyset) \oplus k_*H_n(X)$ is direct, $i_*H_n(\emptyset) = 0$, and i_* is injective by the proof of the lemma. Hence $H_n(\emptyset) = 0$.

(ii) Induct on r . For $r = 0$ the union is empty and the assertion is (i); for $r = 1$ there is nothing to prove. For $r \geq 2$ write $X = X_1 \sqcup Y$ with $Y = X_2 \sqcup \cdots \sqcup X_r$, which is again open and closed in X . Lemma 5.1 gives $H_n(X) \cong H_n(X_1) \oplus H_n(Y)$ by the inclusions, and the inductive hypothesis gives $H_n(Y) \cong \bigoplus_{j \geq 2} H_n(X_j)$ by the inclusions of the X_j in Y . Composing, and using that the inclusion $X_j \rightarrow X$ is the inclusion $X_j \rightarrow Y$ followed by $Y \rightarrow X$, gives the assertion. $\square$

Mayer–Vietoris is not on the list: it is a consequence of (A3) and (A4) alone, by the following diagram lemma.

Lemma 5.3 (Barratt–Whitehead). *Let*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & A_{n-1} & \longrightarrow & \cdots \\ & & a_n \downarrow & & b_n \downarrow & & c_n \downarrow & & a_{n-1} \downarrow & & \\ \cdots & \longrightarrow & A'_n & \xrightarrow{f'_n} & B'_n & \xrightarrow{g'_n} & C'_n & \xrightarrow{h'_n} & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

be a commutative diagram of abelian groups with exact rows in which every c_n is an isomorphism. Then the sequence

$$\cdots \longrightarrow A_n \xrightarrow{(a_n, f_n)} A'_n \oplus B_n \xrightarrow{f'_n - b_n} B'_n \xrightarrow{h_n c_n^{-1} g'_n} A'_{n-1} \longrightarrow \cdots$$

is exact.

Proof. Write $\partial = h_n c_n^{-1} g'_n$. Each composite of two consecutive maps vanishes: $(f'_n - b_n)(a_n, f_n) = f'_n a_n - b_n f_n = 0$ by commutativity of the left square; $\partial(f'_n - b_n)(x, y) = h_n c_n^{-1} (g'_n f'_n(x) - g'_n b_n(y)) = -h_n c_n^{-1} c_n g_n(y) = -h_n g_n(y) = 0$, using exactness of the bottom row and commutativity of the middle square; and $(a_{n-1}, f_{n-1})\partial = 0$, since $f_{n-1} h_n = 0$

by exactness of the top row and $a_{n-1}h_n c_n^{-1}g'_n = h'_n c_n c_n^{-1}g'_n = h'_n g'_n = 0$ by commutativity of the right square and exactness of the bottom row.

Exactness at $A'_n \oplus B_n$. Let $f'_n(x) = b_n(y)$. Then $c_n g_n(y) = g'_n b_n(y) = g'_n f'_n(x) = 0$, so $g_n(y) = 0$ and $y = f_n(z)$ for some $z \in A_n$. Now $f'_n(x - a_n(z)) = b_n(y) - b_n f_n(z) = 0$, so $x - a_n(z) = h'_{n+1}(w)$ for some $w \in C'_{n+1}$. Put $u = h_{n+1} c_{n+1}^{-1}(w) \in A_n$. Then $a_n(u) = h'_{n+1} c_{n+1} c_{n+1}^{-1}(w) = x - a_n(z)$, while $f_n(u) = 0$ because $f_n h_{n+1} = 0$. Hence $(x, y) = (a_n, f_n)(z + u)$.

Exactness at B'_n . Let $v \in B'_n$ with $\partial(v) = 0$. Then $c_n^{-1}g'_n(v) \in \ker h_n = \text{im } g_n$, say $c_n^{-1}g'_n(v) = g_n(y)$; so $g'_n(v - b_n(y)) = g'_n(v) - c_n g_n(y) = 0$ and $v - b_n(y) = f'_n(x)$ for some $x \in A'_n$. Thus $v = (f'_n - b_n)(x, -y)$.

Exactness at A_{n-1} . Let $t \in A_{n-1}$ with $a_{n-1}(t) = 0$ and $f_{n-1}(t) = 0$. By exactness of the top row, $t = h_n(w)$ for some $w \in C_n$, and then $h'_n c_n(w) = a_{n-1}h_n(w) = a_{n-1}(t) = 0$, so $c_n(w) \in \ker h'_n = \text{im } g'_n$, say $c_n(w) = g'_n(v)$. Then $\partial(v) = h_n c_n^{-1}g'_n(v) = h_n(w) = t$. $\square$

Corollary 5.4 (Mayer–Vietoris). *Let H_* satisfy (A3) and (A4) and let $X = \text{int } A \cup \text{int } B$. Then there is an exact sequence*

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots,$$

in which the first two maps are induced by the inclusions, with a sign in one summand.

Proof. Write $C = A \cap B$ and apply Lemma 5.3 to the map of long exact sequences of (A3) induced by the inclusion of pairs $(A, C) \hookrightarrow (X, B)$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(C) & \longrightarrow & H_n(A) & \longrightarrow & H_n(A, C) \longrightarrow H_{n-1}(C) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, B) \longrightarrow H_{n-1}(B) \longrightarrow \cdots \end{array}$$

It commutes by the naturality in (A3), and the third vertical map is an isomorphism by excision, in the form recorded above with the roles of A and B interchanged. The lemma therefore gives an exact sequence

$$\cdots \longrightarrow H_n(C) \longrightarrow H_n(B) \oplus H_n(A) \longrightarrow H_n(X) \longrightarrow H_{n-1}(C) \longrightarrow \cdots,$$

whose first map is $x \mapsto (x, x)$ and whose second is $(u, w) \mapsto u - w$, both induced by inclusions. Interchanging the two summands and changing the sign in one of them turns these into the maps $x \mapsto (x, -x)$ and $(a, b) \mapsto a + b$ of Corollary 7.20 of *Singular Homology*. $\square$

5.1. Uniqueness. Everything computed in §4 used only the axioms, and the computation was determined by the cells and their attaching maps. That is the content of the theorem towards which the section is aimed.

Theorem 5.5 (uniqueness of homology on CW complexes). *Let h_* be a homology theory satisfying (A1)–(A5), let H_* denote singular homology, and fix a generator ε of $h_0(P) \cong \mathbb{Z}$ for a one-point space P . Then for every CW complex X and every n there is an isomorphism*

$$\Theta_n = \Theta_n(X): h_n(X) \xrightarrow{\cong} H_n(X),$$

carrying ε to the class of the point in $H_0(P)$, and natural for morphisms of CW complexes: for every morphism $f: X \rightarrow Y$ of Definition 1.7 the square

$$\begin{array}{ccc} h_n(X) & \xrightarrow{\Theta_n(X)} & H_n(X) \\ h_n(f) \downarrow & & \downarrow H_n(f) \\ h_n(Y) & \xrightarrow{\Theta_n(Y)} & H_n(Y) \end{array}$$

commutes.

The two theories are therefore the same on the spaces of this chapter, and any construction satisfying the axioms computes singular homology.

Proof. The dimension axiom gets us started: by (A1) the groups $h_n(P)$ are those of singular homology, and ε corresponds to the class of the point.

Spheres and balls. As in §4, everything rests on the homology of the pairs (D^n, S^{n-1}) , and that is computed by induction from the axioms alone. We use throughout that for a non-empty Y with base point the constant map $Y \rightarrow P$ splits the sequence (A3) of the pair $(Y, \{*\})$, so that $h_k(Y) \cong h_k(P) \oplus h_k(Y, \{*\})$ naturally. For $n \geq 1$ the ball D^n is contractible, so by (A2) the groups $h_k(D^n, \{*\})$ vanish, and the sequence (A3) of the pair (D^n, S^{n-1}) gives

$$\partial: h_k(D^n, S^{n-1}) \xrightarrow{\cong} h_{k-1}(S^{n-1}, \{*\}),$$

the base point being taken in S^{n-1} . Covering S^n by the complements of the two poles and applying Corollary 5.4 gives $h_k(S^n, \{*\}) \cong h_{k-1}(S^{n-1}, \{*\})$, exactly as in the proof of Theorem 2.5 of *Further Properties of Singular Homology*. Starting from (A1) and Lemma 5.1, which give $h_k(S^0, \{*\}) \cong h_k(P)$, induction gives

$$h_k(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & k = n, \\ 0, & k \neq n, \end{cases}$$

with a generator ι_n^h determined by ε through the displayed isomorphisms, and corresponding under them to the generator ι_n of Definition 4.2.

The skeletal pairs. The proof of Lemma 4.1 used only homotopy invariance, excision and the homology of disjoint unions, so it applies verbatim to h_* : the characteristic maps induce an isomorphism $\bigoplus_{\alpha \in J_n} h_k(D_\alpha^n, S^{n-1}) \cong h_k(X^n, X^{n-1})$, the direct sum decomposition of the disjoint union of the open cells being the special case of (A5) recorded after the axioms. With the previous step, $h_k(X^n, X^{n-1}) = 0$ for $k \neq n$ while $h_n(X^n, X^{n-1})$ is free abelian on the n -cells, the basis being the classes $(\Phi_\alpha)_* \iota_n^h$.

Passing to the skeleta. Every compact subset of X lies in a finite subcomplex by Theorem 2.11, hence in some skeleton, so (A5) applied to the filtration by skeleta gives $h_n(X) = \varinjlim_N h_n(X^N)$; this is what the argument of Lemma 4.3 gave for singular homology. With it, the proofs of Proposition 4.4 and Theorem 4.7 apply word for word to h_* , since they use nothing but the long exact sequences of the skeletal pairs and that statement; so $h_n(X)$ is the homology of the cellular complex $C_n^h(X) = h_n(X^n, X^{n-1})$ with differential $j_{n-1} \circ \partial_n$.

Comparison. The bases produced above correspond: sending $(\Phi_\alpha)_* \iota_n^h$ to $(\Phi_\alpha)_* \iota_n = [e_\alpha^n]$ defines an isomorphism $C_*^h(X) \rightarrow C_*^{\text{cw}}(X)$, and it carries one differential to the other,

both being the composite $j_{n-1} \circ \partial_n$ of maps that are natural in the pair, computed on a basis element through the same characteristic map. Taking homology and composing with the identifications of the previous step gives Θ_n . A morphism $f: X \rightarrow Y$ is a map of pairs $(X^n, X^{n-1}) \rightarrow (Y^n, Y^{n-1})$ for every n , and every map used above is natural for such maps, by (A3) and Addendum 4.8; so the square of the statement commutes. Finally Θ_0 carries ε to the class of the point, that being the normalization of the generators in the first two steps. $\square$

APPENDIX A. NORMALITY AND LOCAL CONTRACTIBILITY

This appendix proves Theorem 2.6: a CW complex is T_1 and normal, hence Hausdorff, and it is locally contractible. The notation is that of §2.

Proposition A.1. *A CW complex X is T_1 and normal; in particular it is Hausdorff.*

Proof. Points are closed. We show by induction on n that every point of X^n is a closed subset of X^n ; the weak topology, Definition 1.2(iii), then gives that $\{p\}$ is closed in X for every $p \in X$, since $\{p\} \cap X^n$ is either empty or the point itself.

For $n = 0$ the assertion holds because X^0 is discrete. Let $n \geq 1$, assume it in degree $n - 1$, and let $p \in X^n$. We apply Lemma 2.2(ii) to $X^n = X^{n-1} \cup_{\{\varphi_\alpha\}} \coprod_{\alpha \in J_n} D_\alpha^n$. First, $\{p\} \cap X^{n-1}$ is either empty or the point p itself, and in the latter case it is closed in X^{n-1} by the inductive hypothesis. Second, let $\alpha \in J_n$. If $p \notin \Phi_\alpha(D_\alpha^n)$ then $\Phi_\alpha^{-1}(p) = \emptyset$; if $p \in e_\alpha^n$ then $\Phi_\alpha^{-1}(p)$ is a single point of $\text{int } D_\alpha^n$; and otherwise $p \in X^{n-1}$ and $\Phi_\alpha^{-1}(p) = \varphi_\alpha^{-1}(p)$, which is closed in S^{n-1} because φ_α is continuous and $\{p\}$ is closed in X^{n-1} by the inductive hypothesis. In all cases $\Phi_\alpha^{-1}(p)$ is closed, so $\{p\}$ is closed in X^n , completing the induction.

Normality. Let C_0, C_1 be disjoint closed subsets of X . We construct by induction on n disjoint open subsets V_0^n, V_1^n of X^n with

$$C_i \cap X^n \subseteq V_i^n \quad (i = 0, 1), \quad \text{and} \quad V_i^n \cap X^{n-1} = V_i^{n-1} \quad \text{for } n \geq 1.$$

Granting this, put $V_i = \bigcup_n V_i^n$. Then $V_i \cap X^n = V_i^n$ is open in X^n for every n , so V_i is open by the weak topology, Definition 1.2(iii); the V_i are disjoint because the V_i^n are; and $C_i \subseteq V_i$. That is normality.

For $n = 0$ set $V_i^0 = C_i \cap X^0$, which is open because X^0 is discrete; the two are disjoint because the C_i are. Let $n \geq 1$ and suppose V_0^{n-1} and V_1^{n-1} have been constructed. Fix $\alpha \in J_n$ and work in D_α^n ; since α stays fixed until the last paragraph, we leave it off the sets P_i, A_i, W_i introduced below. Radial coordinates identify $D_\alpha^n \setminus \{0\}$ with a half-open collar on its boundary:

$$S^{n-1} \times [0, 1) \xrightarrow{\cong} D_\alpha^n \setminus \{0\}, \quad (y, t) \mapsto (1 - t)y,$$

a homeomorphism restricting to the identity on $S^{n-1} = S^{n-1} \times \{0\}$. Put

$$P_i = \varphi_\alpha^{-1}(V_i^{n-1}) \subseteq S^{n-1}, \quad A_i = \Phi_\alpha^{-1}(C_i) \subseteq D_\alpha^n.$$

The P_i are disjoint open subsets of S^{n-1} and the A_i are disjoint closed subsets of D_α^n ; moreover $A_i \cap S^{n-1} = \varphi_\alpha^{-1}(C_i)$ is contained in P_i , because $C_i \cap X^{n-1} \subseteq V_i^{n-1}$.

We claim that for all sufficiently small $\varepsilon > 0$, writing $U_\varepsilon(A)$ for the ε -neighbourhood of A in D_α^n :

- (1) $U_\varepsilon(A_0)$ and $U_\varepsilon(A_1)$ are disjoint;

(2) $U_\varepsilon(A_i) \cap (S^{n-1} \times [0, \varepsilon)) \subseteq P_i \times [0, \varepsilon)$ for $i = 0, 1$.

For (1) we may assume both A_i non-empty; they are disjoint closed subsets of the compact metric space D_α^n , so $d(A_0, A_1) > 0$ and any $\varepsilon \leq \frac{1}{2}d(A_0, A_1)$ will do. For (2), if the inclusion failed for every $\varepsilon > 0$ we could choose $\varepsilon_k \rightarrow 0$ and points $x_k = (y_k, t_k) \in S^{n-1} \times [0, 1)$ with

$$d(x_k, A_i) < \varepsilon_k, \quad t_k < \varepsilon_k, \quad y_k \notin P_i.$$

Passing to a subsequence, $x_k \rightarrow x_\infty$ by compactness of D_α^n ; since $d(x_k, A_i) \rightarrow 0$ and A_i is closed, $x_\infty \in A_i$, and since $t_k \rightarrow 0$, $x_\infty \in S^{n-1}$. So $x_\infty \in A_i \cap S^{n-1} \subseteq P_i$. But $y_k \rightarrow x_\infty$ as well, again because $t_k \rightarrow 0$, and the y_k lie in the closed set $S^{n-1} \setminus P_i$, so $x_\infty \notin P_i$, a contradiction.

Fix $\varepsilon < 1$ satisfying (1) and (2), and set

$$W_i = (P_i \times [0, \varepsilon)) \cup U_\varepsilon(A_i) \subseteq D_\alpha^n.$$

These are open, and they are disjoint: the two collars meet in $(P_0 \cap P_1) \times [0, \varepsilon) = \emptyset$, the two ε -neighbourhoods are disjoint by (1), and a cross term such as $(P_0 \times [0, \varepsilon)) \cap U_\varepsilon(A_1)$ lies by (2) in $(P_0 \cap P_1) \times [0, \varepsilon) = \emptyset$. Moreover $A_i \subseteq W_i$, and $W_i \cap S^{n-1} = P_i$, the inclusion $\subseteq$ again by (2).

Restoring the index: for each $\alpha \in J_n$ we have produced disjoint open subsets $W_{0,\alpha}$, $W_{1,\alpha}$ of D_α^n with

$$\Phi_\alpha^{-1}(C_i) \subseteq W_{i,\alpha}, \quad W_{i,\alpha} \cap S^{n-1} = \varphi_\alpha^{-1}(V_i^{n-1}).$$

Let $V_i^n \subseteq X^n$ be the union of V_i^{n-1} with the sets $\Phi_\alpha(W_{i,\alpha} \cap \text{int } D_\alpha^n)$ for $\alpha \in J_n$. Then $V_i^n \cap X^{n-1} = V_i^{n-1}$, and $\Phi_\alpha^{-1}(V_i^n) = W_{i,\alpha}$ for every $\alpha \in J_n$ by the second of these and the injectivity of Φ_α on $\text{int } D_\alpha^n$, so V_i^n is open in X^n by Lemma 2.2(ii). The V_0^n and V_1^n are disjoint because the V_i^{n-1} are, because the $W_{i,\alpha}$ are, and because distinct open cells are disjoint; and $C_i \cap X^n \subseteq V_i^n$ by the first of these together with $C_i \cap X^{n-1} \subseteq V_i^{n-1}$. This completes the induction.

A T_1 normal space is Hausdorff. □

This is the first assertion of Theorem 2.6.

We turn to local contractibility. The construction produces, inside a given neighbourhood of a point x , another neighbourhood of x built up out of collars in successive cells.

Theorem A.2. *A CW complex X is locally contractible; that is, for every $x \in X$ and every open $U \ni x$ there is an open V with $x \in V \subseteq U$ such that V is contractible.*

Proof. Let $x \in e_{\alpha_0}^k$. We construct open sets $V_n \subseteq U \cap X^n$ for $n \geq k$, each containing x , with V_n open in X^n and, for $n > k$, with $V_n \cap X^{n-1} = V_{n-1}$ and with V_n strong deformation retracting to V_{n-1} .

For $n = k$: the cell $e_{\alpha_0}^k$ is open in X^k , and Φ_{α_0} maps $\text{int } D^k$ homeomorphically onto it by Lemma 2.2(iii). Write $z = \Phi_{\alpha_0}^{-1}(x)$. The set $\Phi_{\alpha_0}^{-1}(U) \cap \text{int } D^k$ is an open neighbourhood of z ; choose inside it an open round ball B about z , and set $V_k = \Phi_{\alpha_0}(B)$, an open subset of X^k contained in U .

For $n > k$, suppose V_{n-1} is given. Fix $\alpha \in J_n$ and write $P_\alpha = \varphi_\alpha^{-1}(V_{n-1})$, an open subset of S^{n-1} . Define $\delta_\alpha: S^{n-1} \rightarrow [0, 1]$ by

$$\delta_\alpha(y) = \frac{1}{2} \min\{1, d(y, S^{n-1} \setminus P_\alpha), d(y, D_\alpha^n \setminus \Phi_\alpha^{-1}(U))\},$$

with the convention $d(y, \emptyset) = 1$; it is continuous, and $\delta_\alpha(y) > 0$ exactly when $y \in P_\alpha$, because $\Phi_\alpha^{-1}(U)$ is open and contains P_α . Put

$$W_\alpha = \left\{ z \in D_\alpha^n : z \neq 0, \frac{z}{|z|} \in P_\alpha, 1 - |z| < \delta_\alpha\left(\frac{z}{|z|}\right) \right\}.$$

This is open in D_α^n , being defined by an open condition on the continuous functions $z \mapsto z/|z|$ and $z \mapsto |z|$ on $D_\alpha^n \setminus \{0\}$; it satisfies $W_\alpha \cap S^{n-1} = P_\alpha$; and $W_\alpha \subseteq \Phi_\alpha^{-1}(U)$, since for $z \in W_\alpha$ with $y = z/|z|$ we have $|z - y| = 1 - |z| < d(y, D_\alpha^n \setminus \Phi_\alpha^{-1}(U))$ and $y \in \Phi_\alpha^{-1}(U)$. Set

$$V_n = V_{n-1} \cup \bigcup_{\alpha \in J_n} \Phi_\alpha(W_\alpha \cap \text{int } D_\alpha^n).$$

Then $V_n \cap X^{n-1} = V_{n-1}$ and $\Phi_\alpha^{-1}(V_n) = W_\alpha$ for every $\alpha \in J_n$, because Φ_α is injective on $\text{int } D_\alpha^n$ and distinct cells are disjoint; so V_n is open in X^n by Lemma 2.2(ii), and $V_n \subseteq U$.

Let $V = \bigcup_{n \geq k} V_n$. Since $V \cap X^n$ is open in X^n for every n (it is empty for $n < k$), V is open in X by the weak topology, Definition 1.2(iii), and $x \in V \subseteq U$. Moreover V carries the weak topology determined by the subspaces $V_n = V \cap X^n$: if $C \subseteq V$ has $C \cap V_n$ closed in V_n for all n , then $(V \setminus C) \cap X^n$ is open in X^n for all n , so $V \setminus C$ is open in X and C is closed in V . Since $I = [0, 1]$ is compact, $V \times I$ likewise carries the weak topology determined by the $V_n \times I$ — the product of a quotient map with the identity map of a locally compact space is again a quotient map — so a map on $V \times I$ is continuous as soon as its restriction to each $V_n \times I$ is continuous.

Each V_n , $n > k$, strong deformation retracts to V_{n-1} by pushing radially outwards in the collars W_α . Explicitly, define $D^{(n)}: V_n \times I \rightarrow V_n$ by

$$D^{(n)}(v, s) = v \quad (v \in V_{n-1}), \quad D^{(n)}(\Phi_\alpha(z), s) = \Phi_\alpha\left((1-s)z + s\frac{z}{|z|}\right) \quad (z \in W_\alpha \cap \text{int } D_\alpha^n).$$

The segment from z to $z/|z|$ consists of points $tz/|z|$ with $|z| \leq t \leq 1$, and these lie in W_α because $1 - t \leq 1 - |z| < \delta_\alpha(z/|z|)$; so $D^{(n)}$ takes values in V_n . It is continuous, being continuous on $V_{n-1} \times I$ and on each $W_\alpha \times I$ and compatible on the overlaps. It is the identity at $s = 0$, it fixes V_{n-1} throughout, and at $s = 1$ it retracts V_n onto V_{n-1} by $\rho_n(\Phi_\alpha(z)) = \varphi_\alpha(z/|z|)$.

Now define $H: V \times I \rightarrow V$ by performing $D^{(n)}$ during the interval $[2^{-(n-k)}, 2^{-(n-k)+1}]$, for each $n > k$, and letting H be stationary on $[0, 2^{-(n-k)}]$ on points of V_n . Concretely, for $v \in V_n$ and $t \in I$,

$$H(v, t) = \begin{cases} v, & t \leq 2^{-(n-k)}, \\ D^{(m)}(\rho_{m+1} \circ \cdots \circ \rho_n(v), 2^{m-k}t - 1), & 2^{-(m-k)} \leq t \leq 2^{-(m-k)+1}, \\ \rho_{k+1} \circ \cdots \circ \rho_n(v), & t = 1, \end{cases}$$

the middle case being taken for $k < m \leq n$. For fixed n this is a finite composite of continuous maps, agreeing at the endpoints of the intervals, so $H|_{V_n \times I}$ is continuous; and the definitions for n and $n+1$ agree on $V_n \times I$, since $D^{(n+1)}$ and ρ_{n+1} are the identity on V_n and points of V_n are stationary before time $2^{-(n-k)}$. Hence H is continuous. It satisfies $H_0 = \text{id}_V$, $H_t|_{V_k} = \text{id}$ for all t , and $H_1(V) \subseteq V_k$: a point of V_n has been pushed into V_k by time 1. Thus V deformation retracts onto V_k . Since $V_k = \Phi_{\alpha_0}(B)$ is homeomorphic to an open ball, it is contractible, and therefore so is V . $\square$

This is the second assertion of Theorem 2.6, and completes its proof.