A Little General Topology

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1 Compact Hausdorff Spaces

1.1 Basic Results

Here is basic result.

Theorem 1.1. *Let $f\colon X \to Y$ be a continuous map between compact Hausdorff spaces. Then:*

- *If f is surjective, then it is a quotient map.*
- *If f is injective, then it is a homeomorphism onto its image.*
- *If f is bijective, then it is a homeomorphism.*

Proof. **First statement.** Recall that a surjective map $\varphi\colon A \to B$ of topological spaces is a *quotient map* if a subset $U \subset B$ is open if and only if $\varphi^{-1}(U)$ is open in A. In our case taking $U \subset Y$ to be open $f^{-1}(U)$ is open since f is continuous. Conversely, if $f^{-1}(U)$ is open, the setting F equal to the complement of U in Y, we see that $f^{-1}(F)$ is the complement of $f^{-1}(U)$ and hence is closed and thus compact since X is compact. It follows that $f(f^{-1}(F)) = F$ is also compact and hence closed since Y is Hausdorff. Thus, if $f^{-1}(U)$ is open so is U.

Second statement. The image $f(X)$ is compact and thus a closed subset of Y and hence is compact Hausdorff. Applying the first result we see that $f\colon X \to f(X)$ is a quotient map. Since it is one-to-one, this means it carries open sets to open sets and hence is a homeomorphism.

Third Statement This is immediate from the second statement. □

1.2 Compactly generated Topology

Definition 1.2. We say a Hausdorff space Z *has a compactly generated topology* or is *compactly generated* when the open subsets of Z are exactly those that intersect every compact subset of Z in an open subset of that compact set. Equivalently, Z is compactly generated if a function $f\colon Z \to Y$ is continuous if and only if for every continuous map of compact Hausdorff space to Z, $S \to Z$, the composite with f is continuous. In general a Hausdorff space X there is the compactly generated topology on the set X, denoted $X^{\text{cpt gen}}$: namely, the open sets in the compactly generated topology are those that meet every compact subset of X in an open subset. The open subsets of X are open subsets in the compactly generated topology, implying that the identity set function induces a continuous map $X^{\text{cpt gen}} \to X$.

More generally, if we fix an uncountable limit cardinal κ we say that Z has a κ-compactly generated topology if a subset of Z is open if and only if its intersection with every compact subset of Z of cardinality less than κ is open. Once again the identity set function induces a continuous map $X^{\kappa-\text{cpt gen}} \to X$. For example a a first countable space is compactly generated if and only if it is κ-compactly generated for the first uncountable cardinal.

1.3 Totally Disconnected Spaces

Definition 1.3. A topological space is *totally disconnected* if given any two points $x \neq y$ there is an open and closed subset F containing x but not y.

Equivalently, a space X is totally disconnected, if for all pairs $x \neq y$ of points in X there is a continuous function $\varphi\colon X \to \{0, 1\}$ with $\varphi(x) = 0$ and $\varphi(y) = 1$. Totally disconnected spaces are Hausdorff.

Theorem 1.4. *Let X be a topological space. Consider the following three statements*

1) X is a totally disconnected space.

2) X is a Hausdorff space with a sub-basis of open sets that are also closed.

3) X is a Hausdorff space with a basis of open sets that are also closed.

Then Statements 2) and 3) are equivalent and are implied by Statement 1). If X is compact, then either Statement 2) or 3) implies Statement 1).

In particular, for compact spaces the three statements are equivalent.

Proof. Since a basis is obtained from a sub-basis by taking finite intersection, the second statment implies the third. Since a basis is a sub-basis, the third statement implies the second. Now let us suppose that Statement 2) holds. Since X is Hausdorff, given points $x \neq y$ of X, the subspace $X \setminus \{y\}$ is an open subset containing x but not y. Either Statement 2) or 3) then implies that there is an open and closed subset of $X \setminus \{y\}$, establishing Statement 1) for X.

Finally, suppose that X is compact and satisfies Statement 1). Clearly, X is Hausdorff. We must show that for any U open subset containing a point $x \in X$ that there is an open and closed subset of U containing x. For each $y \in X \setminus U$ there is an open and closed subset V_y containing y and not containing x. Since $X \setminus U$ is a closed subset of the compact space X, it is compact, and hence there is a finite subset $\{y_i\}_{i=1}^n$ of points in $X \setminus U$ such that $V = \cup_{I=1}^n V_{y_i}$ covers $X \setminus U$. Being a finite union of open and closed subspaces of X, the subspace V is open and closed. By construction, it does not contain x. Its complement $X \setminus V$ is hence an open and closed subspace of X containing x and contained in U. $\square$

1.4 Projective Limits of Sets

Some of the nicest examples of totally disconnected spaces come from projective limits of discrete sets.

Definition 1.5. Recall that a *cofiltered category* is a small category I with the following properties:

- for any pair of objects x, y of I there is an object z and morphism $z \to x$ and $z \to y$,
- for any pair of objects x, y of I and any pair of morphisms $f, g\colon x \to y$ there is an object z and a morphism $\alpha\colon z \to x$ with $f \circ \alpha = g \circ \alpha\colon z \to y$.

The example we encounter most often is the category of the elements of a partially set I with there being a morphism from x to y iff $x \leq y$, and in this case there is a unique such morphism. The extra condition that makes it cofiltered is that for any $x, y \in I$ there is $z \in I$ with $z \leq x$ and $z \leq y$.

Definition 1.6. A *cofiltered projective system of sets* consists of a cofiltered category I and a covariant functor from I to the category of sets. In the case when the cofiltered category is the category of points in a partially ordered set I and morphism given by the $\leq$ relation, a cofiltered projective system indexed by I is a set S_i for each $i \in I$ and for each pair $i \leq j$ a morphism

$S_i \to S_j$ that compose correctly, including that the map associated with $i \leq i$ is the identity of S_i. The system is said to be *indexed* by I.

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Remark 1.7. There are analogous notions of cofiltered projective systems of groups, abelian groups, rings, etc.

Given a cofiltered projective system of sets indexed by I,

$$\{S_i\}_{i\in \mathrm{Obj}(I)}, \{f_\varphi \colon S_i \to S_j\}_{i,j,\varphi\in \mathrm{Hom}_I(i,j)},$$

we form the limit

$$\lim{}_I S_i.$$

It consists of the subspace of the product $\prod_{i\in I} S_i$ consisting of elements $\{s_i\}_{i\in I}$ with the property that $f_\varphi(s_i) = s_j$ for every i, j and every $\varphi \in \mathrm{Hom}_I(i,j)$.

In the case when the S_i are topological spaces the product $\prod_{i\in I} S_i$ is given the product topology of the topologies on each S_i. This means that a system of sub-basic open sets of the product is all sets of the form $\pi_i^{-1}(U_{\alpha_i})$ as i ranges over the objects of I and U_{α_i} ranges over the open subsets of S_i with π_i being the projection of the product onto its the i^{th}-factor. Thus, the basic open sets are finite intersections of these sets, and the general open set is an arbitrary union of the basic open sets. In the case that all the S_i are Hausdorff spaces, the conditions $\{\varphi_{i,j}(s_i) = s_j\}_{i,j,\mathrm{Hom}(S_i,S_j)}$ defining the limit inside the product are closed conditions, meaning that the limit is a closed subspace of the product.

1.5 Profinite Spaces

Definition 1.8. A *profinite space.* is a cofiltered limit of finite sets. It is implicitly given topology induced from the product topology of the discrete topologies on the factors.

Theorem 1.9. *A space is homeomorphic to a profinite set if and only if is a totally disconnected, compact Hausdorff space.*

Proof. Let $P = \prod_{i\in I} S_i$ be an arbitrary product of finite sets indexed by a set I. Since the finite sets with the discrete topology are compact Hausdorff spaces, the product is also a compact Hausdorff space by Tychonov's theorem (see below). Since the projections $\pi_i \colon P \to S_i$ are continuous we see that $\pi_i^{-1}(s_i)$ is an open and closed subspace of P. By definition these are a sub-basis for the product topology. It follows that P is totally disconnected.

Now let I be a cofiltered small category and let $\{S_i, f_{i,j,\varphi}\}$ be a cofiltered system of finite sets indexed by I. Then $\lim_I\{S_i\} \subset \prod_{i\in I} S_i$, being a subspace of a totally disconnected subspace, is totally disconnected. Since the limit is a closed subspace of the product, which is compact, the limit is compact.

Conversely, suppose that X is a totally disconnected compact Hausdorff space. Consider the set $\mathcal{P}$ of partitions p of X into finitely many disjoint, non-empty, closed and open subsets $p = \{U_1 \coprod \ldots \coprod U_{k(p)}\}$. Let X_p be the quotient space of X under the equivalence relation that $x \sim_p y$ iff x and y lie in the same open and closed subset U_i in this decomposiion. Let $\pi_p \colon X \to X_p$ is the quotient map. The set of such partitions forms a partially ordered set by setting $p' \leq p$ if each of the open and closed subsets of p' is contained in one of the open and closed subsets of p. This makes $\mathcal{P}$ a cofiltered set because given p_1 and p_2 we can form the non-empty intersections of all partition members of p_1 with all partition members of p_2 to construct a partition less than each of them.

There is a natural continuous map $X \to \prod_{p\in\mathcal{P}} X_p$ given by the product of the maps π_p. Since X is totally disconnected, given a pair of distinct points x, y of X there are disjoint open and closed subsets each containing one of the points. Thus, for each pair $x \neq y$ of points of X, there is a partition p with $\pi_p(x) \neq \pi_p(y)$. Hence, the map $X \to \prod_{p\in\mathcal{P}} X_p$ is injective.

The maps π_p are compatible with the inclusion maps between partitions. Thus, image of the map is contained in $\lim_{\mathcal{P}} X_p$. This defines a continuous one-to-one map from $X \to \lim_{\mathcal{P}} X_p$. We claim that the map is onto. Suppose that $\{U_{i(p)}\}_{p\in\mathcal{P}}$ is a compatible system, meaning that $U_{i(p)}$ is one of the elements of the partition p and if $p' \leq p$ then $U_{i_{p'}} \subset U_{i_p}$. Since X is compact and the $U_{i(p)}$ are closed in X, they are compact. We must show that $\cap_{p\in\mathcal{P}} U_{i(p)}$ is non-empty. But each finite intersection $U_{i(p_1)} \cap \cdots \cap U_{i(p_k)}$ is non-empty since there is $p \leq p_1, \ldots, p_k$ and for any such p we have $\emptyset \neq U_{i(p)} \subset \cap_{j=1}^k U_{i(p_j)}$. Thus, we have a cofiltered system of non-empty compact subsets under inclusion. The limit of such a system of compact sets is non-empty (and compact). Any point of X in this limit maps to the point given by the $\{U_{i(p)}$ in $\lim_{\mathcal{P}} X_p$ and hence the image of the map is exactly $\lim_{\mathcal{P}} X_p$.

This proves that the map $X \to \prod_{\mathcal{P}} X_p$ is a continuous bijectiom $X \to \lim_{\mathcal{P}} X_p$. Since both spaces are compact Hausdorff spaces, this map is a homeomorphism. □

2 Filters and Ultrafilters

2.1 Basic Definitions and Results

Definition 2.1. Let X be a set. A *filter* on X is a subset $\mathcal{F}$ of the power set, 2^X, with the following properties:

- If subsets A and B of X are contained in $\mathcal{F}$ then so is $A \cap B$, and more generally finite intersections of elements of $\mathcal{F}$ are elements of $\mathcal{F}$.
- If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$.
- $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.

A collection of subsets $\{A_i\}$ of X satisfies *finite intersection property* if every finite intersection of the A_i is non-empty. Thus, the subsets of X belonging to a filter satisfy the finite intersection property. The second condition in the definition is called the *superset property.* The subsets of X belonging to $\mathcal{F}$ are called the *elements* of $\mathcal{F}$. Filters are partially ordered under inclusion $\mathcal{F}' \leq \mathcal{F}$ iff every element of $\mathcal{F}'$ is also an element of $\mathcal{F}$. An *ultrafilter* is a filter that is maximal with respect to this order, i.e., not less than any other filter.

Remark 2.2. What I call a filter is sometimes called a *proper filter* meaning that the empty set is not an element of the filter.

Claim 2.3. *Let X be a set and let $\{A_i\}_{i \in I}$ be a family of subsets of X satisfying the finite intersection property. Let $\mathcal{F}$ be the set of all subsets of X that contain at least one finite intersection of the A_i. Then $\mathcal{F}$ is a filter.*

Proof. Since the A_i has the finite intersection property, $\mathcal{F}$ does not have the empty set as an element. It obviously satisfies the superset property. Suppose V_1 and V_2 are elements of $\mathcal{F}$. Say $A_{i_1} \cap \cdots A_{i_r} \subset V_1$ and $A_{i_{r+1}} \cap \cdots \cap A_{i_{r+s}} \subset V_2$. Then $\cap_{j=1}^{r+s} A_{i_j} \subset V_1 \cap V_2$. Applying a direct induction we see that $\mathcal{F}$ satisfies the finite intersection property. $\square$

Definition 2.4. A *principal* ultrafilter for X is one for which there is a point $x \in X$ such that the elements of the ultrafilter are all $A \subset X$ with $x \in A$. It is the principal ultrafilter *generated* by x. Associating to $x \in X$ the principal ultrafilter it defines an injection from X to the set of ultrafilters on X. It is easy to see that a principal ultrafilter is in fact maximal in the partial order so that the notation is consistent: A principal ultrafilter is an ultrafilter.

2.2 Ultrafilters

A standard application of Zorn's lemma shows:

Theorem 2.5. *Every filter is a subfilter of an ultrafilter. In particular, any collection of subsets of X that satisfies the finite intersection property are all elements of some ultrafilter.*

Proof. Given a totally ordered increasing family $\{\mathcal{F}_\alpha\}_{\alpha\in A}$ of filters, the union $\mathcal{F} = \cup_{\alpha\in A}\mathcal{F}_\alpha$ is easily seen to be a filter that is an upper bound for all the $\mathcal{F}_\alpha$. Thus, by Zorn's lemma there is a maximal filter, and in fact any given filter is a subfilter of an ultrafilter. The second part of the theorem follows immediately from this and Claim 2.3. □

Lemma 2.6. *A filter $\mathcal{F}$ on X is an ultrafilter if and only if for every $A \subset X$ exactly one of A and its complement $X \setminus A$ is an element of $\mathcal{F}$.*

Proof. Fix an ultrafilter $\mathcal{F}$. Let us first show

Claim 2.7. *For $A \subset X$ if $A \in \mathcal{F}$ and $A = A_1 \cup \cdots \cup A_k$, then at least one of A_i is an element of $\mathcal{F}$. If the A_i are disjoint then exactly one of then is an element of $\mathcal{F}$.*

Proof. First we consider the case when $k = 2$. Suppose that $A_1, A_2 \notin \mathcal{F}$. Then define $\mathcal{M} = \{U \in 2^X \mid A_1 \cup U \in \mathcal{F}\}$. Since $A_1 \notin \mathcal{F}$, it follows that $\mathcal{M}$ has the finite intersection property. Obviously, since $\mathcal{F}$ has the superset property, so does $\mathcal{M}$. It follows that $\mathcal{M}$ is a filter. Clearly, since $\mathcal{F}$ has the superset property, $\mathcal{F} \leq \mathcal{M}$. But $A_2 \in \mathcal{M}$ and is not in $\mathcal{F}$. This contradicts the maximality of $\mathcal{F}$. Hence one of A_1 and A_2 is an element of $\mathcal{F}$. The general case of the first statement then follows immediately by induction. The second follows from the first since no pair of disjoint sets are simultaneously elements of an ultrafilter. □

Returning to the proof of lemma, we suppose $A \in 2^X$. Since $X \in \mathcal{F}$, applying the claim immediately above, we see that exactly one of A and $\overline{A} = X \setminus A$ is an element of $\mathcal{F}$.

Conversely, if $\mathcal{F}$ is a filter with the property that for every $A \in 2^X$ either A or $\overline{A}$ is an element of $\mathcal{F}$ then $\mathcal{F}$ is maximal. For adding any other set not already in $\mathcal{F}$ to it would give a set of subsets of X containing both a set and its complement. Such a subset of 2^X cannot be contained in a filter. □

Maybe the best way to think about an ultrafilter $\mathcal{F}$ is as a measure on the subsets of X that assigns only values 1 and 0. It takes value 1 on those sets that are elements of $\mathcal{F}$ and zero on those that are not. It is a finitely additive measure.

2.3 Pushforwards of Ultrafilters and Limits of Ultrafilters

Lemma 2.8. *Let $f\colon X \to Y$ be a set function and let $\mathcal{F}$ be an ultrafilter on X. We define the* pushforward *$f_*(\mathcal{F})$ to be the collection of subsets U of Y with the property that $f^{-1}(U)$ is an element of $\mathcal{F}$. Then $f_*(\mathcal{F})$ is an ultrafilter.*

Proof. The finite additivity and superset property are clear for $f_*\mathcal{F}$. It is also clear that for each $U \in 2^Y$ either U or $Y \setminus U$ is contained in $f_*\mathcal{F}$, but not both. Since $Y \in f_*(\mathcal{F})$, it follows that $\emptyset \notin f_*(\mathcal{F})$. □

Definition 2.9. Suppose that $\mathcal{F}$ is an ultrafilter for the set underlying a topological space X. A point $x \in X$ is said to be a *limit point* of $\mathcal{F}$ if every open neighborhood of x is an element of $\mathcal{F}$.

Proposition 2.10. *Suppose that X is a space and $\mathcal{F}$ is an ultrafilter for the set underlying X. If X is compact, then there is a limit point for $\mathcal{F}$. If X is Hausdorff, then there is at most one limit point for $\mathcal{F}$. If X is a compact Hausdorff space then $\mathcal{F}$ has exactly one limit point. Any limit point of an ultrafilter on a topological space X is contained in the closure of any element of the ultrafilter.*

Proof. If $\mathcal{F}$ has no limit point, then for every $x \in X$ there is an open neighborhood U_x that is not a member of $\mathcal{F}$. If X is compact, then the open covering $\{U_x\}_{x\in X}$ has a finite sub-cover, $U_{x_1}, \ldots, U_{x_k}$. Since $U_{x_1}\cup\cdots\cup U_{x_k} = X$ and $X \in \mathcal{F}$, it follows from Claim 2.7 that one of the U_{x_i} is an element of $\mathcal{F}$. This is a contradiction and shows that there is at least one limit point for $\mathcal{F}$.

If X is Hausdorff and $x \neq y$ are limits of $\mathcal{F}$, then there are disjoint neighborhoods of x and y in $\mathcal{F}$ contradicting the fact that $\mathcal{F}$ has finite intersection property.

Putting the two statements together, establishes the result for compact Hausdorff spaces.

Suppose that $\mathcal{F}$ is an ultrafilter on X with A an element of $\mathcal{F}$. If x is not in the closure of A, then there is an open neighborhood V of x disjoint from A. Thus, V is not an element of of $\mathcal{F}$, which implies that x is not the limit point of $\mathcal{F}$. □

2.4 The Stone Topology on the Space of Ultrafilters

Definition 2.11. Let X be a set and let $\mathcal{U}(X)$ be the set of ultrafilters on X. For any subset $A \subset X$ we define $U_A \subset \mathcal{U}(X)$ to be the set of ultrafilters having A as an element.

Lemma 2.12. *With notation as above*

1. $U_\emptyset = \emptyset$.
2. $U_X = \mathcal{U}(X)$.
3. $U_{A_1} \cap \cdots \cap U_{A_k} = U_{A_1 \cap \cdots \cap A_k}$.
4. $U_{A_1} \cup \cdots \cup U_{A_k} = U_{A_1 \cup \cdots \cup A_k}$..
5. *Using overline to denote complement, we have* $\overline{U_A} = U_{\overline{A}}$.

Proof. Items 1 and 2 are immediate. For the third, $U_{A_1} \cap U_{A_2}$ consists of all ultrafilters that have both A_1 and A_2 as elements. If $A_1 \cap A_2 = \emptyset$, then $U_{A_1} \cap U_{A_2} = \emptyset$. Otherwise, using Claim 2.7 we see that $A_1 \cap A_2$ is an element of any $\mathcal{F} \in U_{A_1} \cap U_{A_2}$, and hence $U_{A_1} \cap U_{A_2} \subset U_{A_1} \cap U_{A_2}$. The opposite inclusion is clear from the superset property of ultrafilters. Induction allows us to pass from the case $k = 2$ for this statement to the case of general finite k. The proof of Item 4 is analogous. The last item is clear since $A \coprod \overline{A} = X$ every ultrafilter on X either has A or $\overline{A}$ as an element but not both. □

Definition 2.13. We define a topology on $\mathcal{U}(X)$, the *Stone Topology*, by defining the sub-basic open sets to be indexed by subsets of X. The sub-basic open set associated to $A \subset X$, denoted U_A is the set of all ultrafilters on X that have A as an element. Then the basic open sets are subsets of ultrafilters than have as elements all members of a given finite family subsets of X. We define βX to be the set $\mathcal{U}(X)$ with the Stone Topology.

It follows immediately from Proposition 2.12 that the collection of subsets $\{U_A\}_{A \subset X}$ of $\mathcal{U}(X)$ are closed under finite intersections. Since they are defined to be a sub-basis for the Stone Topology, they are in fact a basis. A subset of βX is open if and only if it is a union of sets of the form U_A for $A \subset X$.

Corollary 2.14. *The basic open sets of βX are also closed. Also, βX is Hausdorff. Thus, βX is completely disconnected.*

Proof. By Property 5 in Proposition 2.12 the basic open sets of βX are also closed. If $\mathcal{F}$ and $\mathcal{G}$ are distinct ultrafilters, then there is a subset $A \subset X$ that is contained in $\mathcal{F}$ but not in $\mathcal{G}$. Hence, $\mathcal{F} \in U_A$ and $\mathcal{G} \in U_{\overline{A}}$. This proves that βX is Hausdorff. It follows from Theorem 1.4 that βX is totally disconnected. □

Definition 2.15. Given a set X, for $x \in X$ we set $u(x) \in \beta X$ equal to the principal ultrafilter generated by x. Recall that its elements are the subsets of X that contain x. This defines a function $u\colon X \to \beta X$.

Theorem 2.16. *Let X be a set. Then the space of ultrafilters on X, βX, is a compact, totally disconnected Hausdorff space and $u\colon X \to \beta X$ is a continuous one-to-one map from X with the discrete topology to βX. Its image $u(X)$ is dense in βX and the subspace topology on $u(X)$ is the discrete topology so that u is a homeomorphism onto its image.*

Proof. From the definition of $u\colon X \to \beta X$, it is clear that u is one-to-one. It is continuous since X is given the discrete topology. The open subset $U_{\{x\}}$ contains the principal ultrafilter generated by x but no other principal ultrafilter. Thus, $u\colon X \to u(X)$ is a homeomorphism. Since every basic set U_A contains $u(a)$ for every $a \in A$, the image $u(X)$ meets every open set and hence $u(X)$ is dense in βX.

To see that βX is compact we must show that any open covering $\{U_i\}_{i\in I}$ with index set I has a finite sub-covering. Since each open set is a union of basic sets, we can arrange (possibly by changing the index set I) that for each of the open sets $\{U_i\}_{i\in I}$ are of the form $U_i = U(A_i)$ for a subset $A_i \in X$.

We assume that this cover has no finite sub-cover. This means that for every finite subset $F \subset I$

$$V(F) = \cap_{i\in F}\overline{U_i} = \cap_{i\in F}\overline{U(A_i)} \neq \emptyset.$$

By the same reasoning, as F ranges over the finite subsets of I, we obtain a collection of closed (and open) subsets $V(F)$ of $\mathcal{U}(X)$ with the finite intersection property. Thus, applying Proposition 2.12 gives us that for any finite subset F of I, setting $\overline{A_F} = \cap_{i\in F}\overline{A_i}$, the $\{\overline{A_F}\}_{\{F\}}$ is a collection of sets in X with the finite intersection property. By Claim 2.3 this collection defines a filter on X consisting of all subsets of X that contain one of the finite intersections of the $\overline{A_F}$. This filter is contained in an ultrafilter u_0, which then also has as elements all the subsets $\overline{A_i}$ of X. Hence, for each $i \in I$, since $\overline{A_i}$ is an element of u_0 we have $u_0 \notin U_i = U(A_i)$. Thus, $u_0 \notin \cup_{i\in I}U_i$.

This contradicts the fact that the $\{U_i\}_{i\in I}$ are a covering of $\mathcal{U}(X)$, proving that the assumption that there is no finite sub-cover of the original cover is false and establishing the compactness of βX. □

Corollary 2.17. *For any set X, the space βX is profinite.*

Proof. This is immediate from Theorems 2.16 and 1.9. □

2.5 Extending maps of X to a compact set to all of βX

Theorem 2.18. *Let X be a set with the discrete topology and $i\colon X \to \beta X$ as above. Then if Y is a compact Hausdorff space and $f\colon X \to Y$ is a set function (which is a continuous map with the given topologies) then there is exactly one continuous map $\varphi\colon \beta X \to Y$ satisfying $\varphi \circ i = f$.*

Proof. Since $i(X)$ is dense in βX there is at most one continuous extension of f to all of βX. The definition of φ is more or less obvious: for each ullrafilter $\mathcal{F}$ on X the push forward $f_*(\mathcal{F})$ is an ultrafilter on Y. Siince Y is compact Hausdorff, $f_*(\mathcal{F})$ has a unique limit point. We define the image under φ of the point $\mathcal{F} \in \beta X$ to be the limit of $f_*(\mathcal{F})$. It is clear that if the principal ulterfilter generated by $x \in X$, then $f_*(\mathcal{F})$ is principal ultrafilter on Y generated by $f(x)$. Of course the limit of this ultrafilter is $f(x)$. This shows that $\varphi \circ i = f$.

It remains only to show that φ is continuous. Let $\varphi(\mathcal{F}) = y \in Y$. Let $V \subset Y$ be an open neighborhood of y. To show the continuity of φ, we construct a basic open set of βX, U_A for an appropriate $A \subset X$ containing $\mathcal{F}$ and whose image under φ is contained in V. Since Y is compact Hausdorff, the frontier of V, which is defined as the closure of V minus V, is a compact set disjoint from y. Thus, there is an open subset W of Y containing the frontier of V but disjoint from an open neighborhood $V' \subset V$ of y. The closure of V' is contained in V. Let $A = f^{-1}(V')$. Clearly, $\mathcal{F} \in U_A$. The open set U_A is as required as is shown by the next claim.

Claim 2.19. *$\varphi(U_A) \subset V$.*

Proof. Let $\mathcal{G} \in U_A$, meaning that $\mathcal{G}$ is an ultrafilter having A as an element. Since A is an element of $\mathcal{G}$, the set $f^{-1}(f(A))$ contains A and is also an element of $\mathcal{G}$. Thus, $f(A)$ is an element of $f_*(\mathcal{G})$ and hence by Proposition 2.10, the limit of $f_*(\mathcal{G})$ is contained in the closure of $f(A)$. Since $f(A) \subset V'$, the closure of $f(A)$ is contained in the closure of V' which is contained in V. This shows that $\varphi(\mathcal{G})$, which is the limit of $f_*(\mathcal{G})$, is contained in V □

This completes the proof of the continuity of φ. □

Remark 2.20. A very similar argument shows that a function $f\colon Y \to Z$ between compact Hausdorff spaces is continuous if and only if it preserves limits of ultrafilters. That is to say $f_*(\lim(\mathcal{F})) = \lim(f_*(\mathcal{F}))$ for every ultrafilter in Y.

Corollary 2.21. *Every compact Hausdorff space is the quotient of a totally disconnected space.*

Proof. Let C be a compact Hausdorff space and C' the same set with the discrete topology. Then the map $C' \to C$ given by the identity map is continuous and extends to a continuous map $\beta C" \to C$ which obviously is surjective. Since $\beta C'$ and C are compact Hausdorff spaces this map is a quotient map. □

2.6 Tychonov's Theorem

An application of the theory of ultrafilters is Tychonov's Theorem.

Theorem 2.22. *(Tychonov's Theorem) Let $\{X_i\}_{i\in I}$ be a family of compact spaces indexed by a set I. Then $\prod_{i\in I} X_i$ with the product topology is compact.*

Remark 2.23. For a product of two compact spaces (or indeed any finite collection of compact spaces) one can give a direct argument. The proof in the case of infinite products in more indirect.

Proof. Let $P = \prod_{i\in I} X_i$. Fix an open covering $\{U_j\}_{j\in J}$. We must show that this covering has a finite sub-covering. Suppose that there is no finite subcovering. Then the complementary closed sets $F_j = P \backslash U_j$ have the finite intersection property. By Claim 2.3 there is an ultrafilter $\mathcal{F}$ containing all finite intersections of the F_j. For each $i \in I$ let $\pi_i\colon P \to X_i$ be the projection and let $(\pi_i)_*(\mathcal{F})$ be the pushforwrad ultrafilter. Since X_i is compact, this ultrafilter has a limit x_i.

We claim $p = \{x_i\}_{i\in I} \in P$ is not in $\cup_{j\in J} U_j$, which is a contradiction since by assumption these open sets cover P. If the point p were in this union, then there would be a basic open set of P containing p and contained in one of the U_i. Any basic open set containing p is of the form $\prod_{k=1}^r W_{i_k} \times \prod_{i\in I'} X_i$, where $W_{i_k} \subset X_{i_k}$ is an open subset containing x_{i_k} and $I' = I \setminus \{i_1, \ldots, i_k\}$. But for each $1 \le n \le r$, the set W_{i_n} is an open neighborhood of x_{i_n} which is a limit point of $(\pi_{i_n})_*(\mathcal{F})$. It follows that

$$Z_{i_n} = \pi_{i_n}^{-1}(W_{i_n}) = W_{i_n} \times \prod_{i\in I\backslash\{i_n\}} X_i$$

is an element of $\mathcal{F}$. By the finite intersection property,

$$\prod_{k=1}^{r} W_{i_k} \times \prod_{i \in I'} X_i = \cap_{1 \le n \le k} Z_{i_n}$$

is also an element of $\mathcal{F}$. Thus, it meets every element of $\mathcal{F}$, and in particular, it meets each F_i. Thus, this open subset is not contained in any of the U_i. □

2.7 The Stone-Cech Compactification of a Space.

For any topological space X the Stone-Cech compactification is defined to be a compact space βX together with a continuous map $i\colon X \to \beta X$ that satisfies this universal mapping property. That is to say given any compact space Y and a continuous map $f\colon X \to Y$ there is a unique (continous) map $\varphi_f\colon \beta X \to Y$ with $\varphi_f \circ i = f$. Since it satisfies the universal mapping problem, if βX exists then it is unique up to unique homeomorphism.

One constructs it rather tautologically. Consider all maps $f_Y\colon X \to Y$ from X to a compact space Y and form the map $X \to \prod_{f_Y} Y$ given by the product of the f_Y. Of course, it is not piossible to do this since we are dealing with classes rather than sets of spaces and maps. But, it suffices to consider only maps $X \to Y$ with dense image. Then any compact space Y that contains a continuous dense image of X is up to homeomorphism given by a topology on some subset of the power set of the power set of X, and the space given by topologies on subsets of this fixed set for a set. Using all such compact Y and maps $f_Y\colon X \to Y$with dense image gives us a set of possibilites and with these restrictions we form $\prod_{f_Y} f_Y\colon X \to \prod_{f_Y} Y$. By Tychonov's theorem the range is compact and hence the closure of the image of X under this map is a compact subset βX equipped with a map $X \to \beta X$. For any compact set Z and a map with dense image $f\colon X \to Z$, we find f_Y in our collection such that up to a homeomorphism $h\colon Z \to Y$, $f = h^{-1} \circ f_Y$. Thus, the projection onto this factor restricts to βX to give an extension of $f\colon X \to Y$, which followed by h^{-1} is the required extension of $f\colon X \to Z$ to a map on $\beta X \to Z$. In case $f\colon X \to Z$ does not have dense image one replaces Z by the compact subspace $Z_0 \subset Z$ that is the closure of the image $f(X)$ and argues as before.

In case of a regular Hausdorff space X, one in which a closed set and a disjoint point can be separated by a continuous function to the unit interval I one can use the product over all continuous functions $X \to I$ of I and make the same type of argument.

For discrete X we can do much better.

Theorem 2.24. *For a discrete space X the inclusion map $u\colon X \to \beta X$ is the Stone-Cech compactification.*

Proof. We have seen that for any compact space Y any map $f\colon X \to Y$ (automatically continuous since X has the discrete topology) there is a unique continuous extension of f over βX. This is the defining property of the Stone-Cech compactification. $\square$

3 Extremally Disconnected Spaces

Definition 3.1. A space X is *extremely disconnected* if it is a compact Hausdorff space and if every surjection $C \to X$ from a compact Hausdorff space C splits, i.e., has a section.

Lemma 3.2. *Let X be an extremally disconnected Hausdorff space, then X is totally disconnected.*

Proof. Let $x \neq y$ be two points of X. Since X is Hausdorff there are disjoint open sets U, V with $x \in U$ and $y \in V$. Let $A = X \setminus V$ and $B = X \setminus U$. These are compact subsets that cover X. Since X is extremely disconnected, there is a continuous section $s\colon X \to A \coprod B$ for the natural projection $A \coprod B \to X$. The image of x lies in A and the image of y lies in B, so that $s^{-1}(A)$ is an open and closed subset of X containing x but not y. $\square$

Corollary 3.3. *Suppose that $f\colon Y \to Z$ is a surjection between compact Hausdorff spaces and let X be an extremely disconnected space. Then any continuous $\alpha\colon X \to Z$ lifts to Y; i.e., there is a continuous map $\beta\colon X \to Y$ with $\alpha = f \circ \beta$.*

Proof. We form the fibered product $X \times_Z Y$. Since $Y \to Z$ is surjective, the projection $X \times_Z Y \to X$ is also surjective. Since X is extremely disconnected, there is a section $X \to X \times_Z Y$. The composition of this section followed by the projection to Y is the required lift. $\square$

Proposition 3.4. *For any discrete set X the Stone-Cech compactification βX of X is extremally disconnected.*

Proof. Let A be a compact Hausdorff space and $\rho\colon A \to \beta X$ a continuous surjection. Since subspace $X \subset \beta X$ is discrete, there is a a section $i\colon X \to A$ for ρ over X. By the universal property of βX the map $i\colon X \to A$ extends to a continuous map $\hat{i}\colon \beta X \to A$. The composition $\rho \circ \hat{i}\colon \beta X \to \beta X$ is the identity on X and since X is dense in βX, it follows that $\hat{i}\colon \beta X \to C$ is a section. □

Corollary 3.5. *Every compact Hausdorff space is the quotient of an extremely disconnected space. If the set underlying the compact space has cardinality less than κ then one can choose the extremely disconnected set to have cardianlity less than κ.*

Proof. Let C be a compact Hausdorff space of cardinality less than κ. Let C' be C with the discrete topology. Then $\beta C'$ has cardinality at most $2^{2^{\mathrm{card}(C)}}$. Since κ is a limit cardinal, this cardinality is also less than κ. Then the identity map $C' \to C$ is continuous and extends to a map $\beta C' \to C$ that is surjective. □

Remark 3.6. Every extremally disconnected space is a retract of the Stone-Cech compactification of some discrete set.