1 Compact Hausdorff Spaces

1.1 Basic Results

Here is basic result.

**Theorem 1.1.** Let $f : X \to Y$ be a continuous map between compact Hausdorff spaces. Then:

- If $f$ is surjective, then it is a quotient map.
- If $f$ is injective, then it is a homeomorphism onto its image.
- If $f$ is bijective, then it is a homeomorphism.

**Proof. First statement.** Recall that a surjective map $\varphi : A \to B$ of topological spaces is a *quotient map* if a subset $U \subset B$ is open if and only if $\varphi^{-1}(U)$ is open in $A$. In our case taking $U \subset Y$ to be open $f^{-1}(U)$ is open since $f$ is continuous. Conversely, if $f^{-1}(U)$ is open, the setting $F$ equal to the complement of $U$ in $Y$, we see that $f^{-1}(F)$ is the complement of $f^{-1}(U)$ and hence is closed and thus compact since $X$ is compact. It follows that $f(f^{-1}(F)) = F$ is also compact and hence closed since $Y$ is Hausdorff. Thus, if $f^{-1}(U)$ is open so is $U$.

**Second statement.** The image $f(X)$ is compact and thus a closed subset of $Y$ and hence is compact Hausdorff. Applying the first result we see that $f : X \to f(X)$ is a quotient map. Since it is one-to-one, this means it carries open sets to open sets and hence is a homeomorphism.

**Third Statement** This is immediate from the second statement.
1.2 Compactly generated Topology

**Definition 1.2.** We say a Hausdorff space $Z$ has a **compactly generated topology** or is **compactly generated** when the open subsets of $Z$ are exactly those that intersect every compact subset of $Z$ in an open subset of that compact set. Equivalently, $Z$ is compactly generated if a function $f : Z \to Y$ is continuous if and only if for every continuous map of compact Hausdorff space to $Z$, $S \to Z$, the composite with $f$ is continuous. In general a Hausdorff space $X$ there is the compactly generated topology on the set $X$, denoted $X^{\text{cpt gen}}$: namely, the open sets in the compactly generated topology are those that meet every compact subset of $X$ in an open subset. The open subsets of $X$ are open subsets in the compactly generated topology, implying that the identity set function induces a continuous map $X^{\text{cpt gen}} \to X$.

More generally, if we fix an uncountable limit cardinal $\kappa$ we say that $Z$ has a $\kappa$-**compactly generated topology** if a subset of $Z$ is open if and only if its intersection with every compact subset of $Z$ of cardinality less than $\kappa$ is open. Once again the identity set function induces a continuous map $X^{\kappa-\text{cpt gen}} \to X$. For example a a first countable space is compactly generated if and only if it is $\kappa$-compactly generated for the first uncountable cardinal.

1.3 Totally Disconnected Spaces

**Definition 1.3.** A topological space is **totally disconnected** if given any two points $x \neq y$ there is an open and closed subset $F$ containing $x$ but not $y$.

Equivalently, a space $X$ is totally disconnected, if for all pairs $x \neq y$ of points in $X$ there is a continuous function $\varphi : X \to \{0, 1\}$ with $\varphi(x) = 0$ and $\varphi(y) = 1$. Totally disconnected spaces are Hausdorff.

**Theorem 1.4.** Let $X$ be a topological space. Consider the following three statements

1) $X$ is a totally disconnected space.

2) $X$ is a Hausdorff space with a sub-basis of open sets that are also closed.

3) $X$ is a Hausdorff space with a basis of open sets that are also closed.

Then Statements 2) and 3) are equivalent and are implied by Statement 1). If $X$ is compact, then either Statement 2) or 3) implies Statement 1).

In particular, for compact spaces the three statements are equivalent.
Proof. Since a basis is obtained from a sub-basis by taking finite intersection, the second statement implies the third. Since a basis is a sub-basis, the third statement implies the second. Now let us suppose that Statement 2) holds. Since \( X \) is Hausdorff, given points \( x \neq y \) of \( X \), the subspace \( X \setminus \{y\} \) is an open subset containing \( x \) but not \( y \). Either Statement 2) or 3) then implies that there is an open and closed subset of \( X \setminus \{y\} \), establishing Statement 1) for \( X \).

Finally, suppose that \( X \) is compact and satisfies Statement 1). Clearly, \( X \) is Hausdorff. We must show that for any \( U \) open subset containing a point \( x \in X \) that there is an open and closed subset of \( U \) containing \( x \). For each \( y \in X \setminus U \) there is an open and closed subset \( V_y \) containing \( y \) and not containing \( x \). Since \( X \setminus U \) is a closed subset of the compact space \( X \), it is compact, and hence there is a finite subset \( \{y_i\}_{i=1}^n \) of points in \( X \setminus U \) such that \( V = \bigcup_{i=1}^n V_{y_i} \) covers \( X \setminus U \). Being a finite union of open and closed subspaces of \( X \), the subspace \( V \) is open and closed. By construction, it does not contain \( x \). Its complement \( X \setminus V \) is hence an open and closed subspace of \( X \) containing \( x \) and contained in \( U \). \( \square \)

1.4 Projective Limits of Sets

Some of the nicest examples of totally disconnected spaces come from projective limits of discrete sets.

Definition 1.5. Recall that a cofiltered category is a small category \( I \) with the following properties:

- for any pair of objects \( x, y \) of \( I \) there is an object \( z \) and morphism \( z \rightarrow x \) and \( z \rightarrow y \),

- for any pair of objects \( x, y \) of \( I \) and any pair of morphisms \( f, g : x \rightarrow y \) there is an object \( z \) and a morphism \( \alpha : z \rightarrow x \) with \( f \circ \alpha = g \circ \alpha : z \rightarrow y \).

The example we encounter most often is the category of the elements of a partially ordered set \( I \) with there being a morphism from \( x \) to \( y \) iff \( x \leq y \), and in this case there is a unique such morphism. The extra condition that makes it cofiltered is that for any \( x, y \in I \) there is a \( z \in I \) with \( z \leq x \) and \( z \leq y \).

Definition 1.6. A cofiltered projective system of sets consists of a cofiltered category \( I \) and a covariant functor from \( I \) to the category of sets. In the case when the cofiltered category is the category of points in a partially ordered set \( I \) and morphism given by the \( \leq \) relation, a cofiltered projective system indexed by \( I \) is a set \( S_i \) for each \( i \in I \) and for each pair \( i \leq j \) a morphism
Remark 1.7. There are analogous notions of cofiltered projective systems of groups, abelian groups, rings, etc.

Given a cofiltered projective system of sets indexed by $I$,

$$\{S_i\}_{i \in \text{Obj}(I)}, \{f_{\varphi}: S_i \to S_j\}_{i,j,\varphi \in \text{Hom}_I(i,j)},$$

we form the limit

$$\lim_I S_i.$$ 

It consists of the subspace of the product $\prod_{i \in I} S_i$ consisting of elements $\{s_i\}_{i \in I}$ with the property that $f_{\varphi}(s_i) = s_j$ for every $i, j$ and every $\varphi \in \text{Hom}_I(i, j)$.

In the case when the $S_i$ are topological spaces the product $\prod_{i \in I} S_i$ is given the product topology of the topologies on each $S_i$. This means that a system of sub-basic open sets of the product is all sets of the form $\pi_i^{-1}(U_{\alpha_i})$ as $i$ ranges over the objects of $I$ and $U_{\alpha_i}$ ranges over the open subsets of $S_i$ with $\pi_i$ being the projection of the product onto its the $i^{th}$-factor. Thus, the basic open sets are finite intersections of these sets, and the general open set is an arbitrary union of the basic open sets. In the case that all the $S_i$ are Hausdorff spaces, the conditions $\{\varphi_{i,j}(s_i) = s_j\}_{i,j,\text{Hom}(S_i,S_j)}$ defining the limit inside the product are closed conditions, meaning that the limit is a closed subspace of the product.

1.5 Profinite Spaces

Definition 1.8. A profinite space is a cofiltered limit of finite sets. It is implicitly given topology induced from the product topology of the discrete topologies on the factors.

Theorem 1.9. A space is homeomorphic to a profinite set if and only if is a totally disconnected, compact Hausdorff space.

Proof. Let $P = \prod_{i \in I} S_i$ be an arbitrary product of finite sets indexed by a set $I$. Since the finite sets with the discrete topology are compact Hausdorff spaces, the product is also a compact Hausdorff space by Tychonov’s theorem (see below). Since the projections $\pi_i: P \to S_i$ are continuous we see...
that \( \pi_i^{-1}(s_i) \) is an open and closed subspace of \( P \). By definition these are a sub-basis for the product topology. It follows that \( P \) is totally disconnected.

Now let \( I \) be a cofiltered small category and let \( \{ S_i, f_{i,j,\varphi} \} \) be a cofiltered system of finite sets indexed by \( I \). Then \( \lim_I \{ S_i \} \subset \prod_{i \in I} S_i \), being a subspace of a totally disconnected subspace, is totally disconnected. Since the limit is a closed subspace of the product, which is compact, the limit is compact.

Conversely, suppose that \( X \) is a totally disconnected compact Hausdorff space. Consider the set \( \mathcal{P} \) of partitions \( p \) of \( X \) into finitely many disjoint, non-empty, closed and open subsets \( p = \{ U_1 \cup \ldots \cup U_k(p) \} \). Let \( X_p \) be the quotient space of \( X \) under the equivalence relation that \( x \sim_p y \) iff \( x \) and \( y \) lie in the same open and closed subset \( U_i \) in this decomposition. Let \( \pi_p : X \to X_p \) is the quotient map. The set of such partitions forms a partially ordered set by setting \( p' \leq p \) if each of the open and closed subsets of \( p' \) is contained in one of the open and closed subsets of \( p \). This makes \( \mathcal{P} \) a cofiltered set because given \( p_1 \) and \( p_2 \) we can form the non-empty intersections of all partition members of \( p_1 \) with all partition members of \( p_2 \) to construct a partition less than each of them.

There is a natural continuous map \( X \to \prod_{p \in \mathcal{P}} X_p \) given by the product of the maps \( \pi_p \). Since \( X \) is totally disconnected, given a pair of distinct points \( x, y \) of \( X \) there are disjoint open and closed subsets each containing one of the points. Thus, for each pair \( x \neq y \) of points of \( X \), there is a partition \( p \) with \( \pi_p(x) \neq \pi_p(y) \). Hence, the map \( X \to \prod_{p \in \mathcal{P}} X_p \) is injective.

The maps \( \pi_p \) are compatible with the inclusion maps between partitions. Thus, image of the map is contained in \( \lim_{\mathcal{P}} X_p \). This defines a continuous one-to-one map from \( X \to \lim_{\mathcal{P}} X_p \). We claim that the map is onto. Suppose that \( \{ U_i(p) \}_{p \in \mathcal{P}} \) is a compatible system, meaning that \( U_i(p) \) is one of the elements of the partition \( p \) and if \( p' \leq p \) then \( U_i(p') \subset U_i(p) \). Since \( X \) is compact and the \( U_i(p) \) are closed in \( X \), they are compact. We must show that \( \cap_{p \in \mathcal{P}} U_i(p) \) is non-empty. But each finite intersection \( U_{i(p_1)} \cap \cdots \cap U_{i(p_k)} \) is non-empty since there is \( p \leq p_1, \ldots, p_k \) and for any such \( p \) we have \( \emptyset \neq U_i(p) \subset \cap_{j=1}^k U_{i(j)}(p_j) \). Thus, we have a cofiltered system of non-empty compact subsets under inclusion. The limit of such a system of compact sets is non-empty (and compact). Any point of \( X \) in this limit maps to the point given by the \( \{ U_i(p) \} \) in \( \lim_{\mathcal{P}} X_p \) and hence the image of the map is exactly \( \lim_{\mathcal{P}} X_p \).

This proves that the map \( X \to \prod_{\mathcal{P}} X_p \) is a continuous bijection \( X \to \lim_{\mathcal{P}} X_p \). Since both spaces are compact Hausdorff spaces, this map is a homeomorphism.
2 Filters and Ultrafilters

2.1 Basic Definitions and Results

Definition 2.1. Let \( X \) be a set. A \textit{filter} on \( X \) is a subset \( \mathcal{F} \) of the power set, \( 2^X \), with the following properties:

- If subsets \( A \) and \( B \) of \( X \) are contained in \( \mathcal{F} \) then so is \( A \cap B \), and more generally finite intersections of elements of \( \mathcal{F} \) are elements of \( \mathcal{F} \).
- If \( A \in \mathcal{F} \) and \( A \subset B \), then \( B \in \mathcal{F} \).
- \( \emptyset \notin \mathcal{F} \) and \( X \in \mathcal{F} \).

A collection of subsets \( \{A_i\} \) of \( X \) satisfies \textit{finite intersection property} if every finite intersection of the \( A_i \) is non-empty. Thus, the subsets of \( X \) belonging to a filter satisfy the finite intersection property. The second condition in the definition is called the \textit{superset property}. The subsets of \( X \) belonging to \( \mathcal{F} \) are called the \textit{elements} of \( \mathcal{F} \). Filters are partially ordered under inclusion \( \mathcal{F}' \leq \mathcal{F} \) iff every element of \( \mathcal{F}' \) is also an element of \( \mathcal{F} \). An \textit{ultrafilter} is a filter that is maximal with respect to this order, i.e., not less than any other filter.

Remark 2.2. What I call a filter is sometimes called a \textit{proper filter} meaning that the empty set is not an element of the filter.

Claim 2.3. Let \( X \) be a set and let \( \{A_i\}_{i \in I} \) be a family of subsets of \( X \) satisfying the finite intersection property. Let \( \mathcal{F} \) be the set of all subsets of \( X \) that contain at least one finite intersection of the \( A_i \). Then \( \mathcal{F} \) is a filter.

Proof. Since the \( A_i \) has the finite intersection property, \( \mathcal{F} \) does not have the empty set as an element. It obviously satisfies the superset property. Suppose \( V_1 \) and \( V_2 \) are elements of \( \mathcal{F} \). Say \( A_{i_1} \cap \cdots \cap A_{i_k} \subset V_1 \) and \( A_{i_{r+1}} \cap \cdots \cap A_{i_{r+s}} \subset V_2 \). Then \( \bigcap_{j=1}^{r+s} A_{i_j} \subset V_1 \cap V_2 \). Applying a direct induction we see that \( \mathcal{F} \) satisfies the finite intersection property.

Definition 2.4. A \textit{principal} ultrafilter for \( X \) is one for which there is a point \( x \in X \) such that the elements of the ultrafilter are all \( A \subset X \) with \( x \in A \). It is the principal ultrafilter \textit{generated} by \( x \). Associating to \( x \in X \) the principal ultrafilter it defines an injection from \( X \) to the set of ultrafilters on \( X \). It is easy to see that a principal ultrafilter is in fact maximal in the partial order so that the notation is consistent: A principal ultrafilter is an ultrafilter.
2.2 Ultrafilters

A standard application of Zorn’s lemma shows:

**Theorem 2.5.** Every filter is a subfilter of an ultrafilter. In particular, any collection of subsets of \(X\) that satisfies the finite intersection property are all elements of some ultrafilter.

**Proof.** Given a totally ordered increasing family \(\{\mathcal{F}_\alpha\}_{\alpha \in A}\) of filters, the union \(\mathcal{F} = \bigcup_{\alpha \in A} \mathcal{F}_\alpha\) is easily seen to be a filter that is an upper bound for all the \(\mathcal{F}_\alpha\). Thus, by Zorn’s lemma there is a maximal filter, and in fact any given filter is a subfilter of an ultrafilter. The second part of the theorem follows immediately from this and Claim 2.3. □

**Lemma 2.6.** A filter \(\mathcal{F}\) on \(X\) is an ultrafilter if and only if for every \(A \subset X\) exactly one of \(A\) and its complement \(X \setminus A\) is an element of \(\mathcal{F}\).

**Proof.** Fix an ultrafilter \(\mathcal{F}\). Let us first show

**Claim 2.7.** For \(A \subset X\) if \(A \in \mathcal{F}\) and \(A = A_1 \cup \cdots \cup A_k\), then at least one of the \(A_i\) is an element of \(\mathcal{F}\). If the \(A_i\) are disjoint then exactly one of them is an element of \(\mathcal{F}\).

**Proof.** First we consider the case when \(k = 2\). Suppose that \(A_1, A_2 \notin \mathcal{F}\). Then define \(\mathcal{M} = \{U \in 2^X \mid A_1 \cup U \in \mathcal{F}\}\). Since \(A_1 \notin \mathcal{F}\), it follows that \(\mathcal{M}\) has the finite intersection property. Obviously, since \(\mathcal{F}\) has the superset property, so does \(\mathcal{M}\). It follows that \(\mathcal{M}\) is a filter. Clearly, since \(\mathcal{F}\) has the superset property, \(\mathcal{F} \leq \mathcal{M}\). But \(A_2 \in \mathcal{M}\) and is not in \(\mathcal{F}\). This contradicts the maximality of \(\mathcal{F}\). Hence one of \(A_1\) and \(A_2\) is an element of \(\mathcal{F}\). The general case of the first statement then follows immediately by induction. The second follows from the first since no pair of disjoint sets are simultaneously elements of an ultrafilter. □

Returning to the proof of lemma, we suppose \(A \in 2^X\). Since \(X \in \mathcal{F}\), applying the claim immediately above, we see that exactly one of \(A\) and \(\overline{A} = X \setminus A\) is an element of \(\mathcal{F}\).

Conversely, if \(\mathcal{F}\) is a filter with the property that for every \(A \in 2^X\) either \(A\) or \(\overline{A}\) is an element of \(\mathcal{F}\) then \(\mathcal{F}\) is maximal. For adding any other set not already in \(\mathcal{F}\) to it would give a set of subsets of \(X\) containing both a set and its complement. Such a subset of \(2^X\) cannot be contained in a filter. □

Maybe the best way to think about an ultrafilter \(\mathcal{F}\) is as a measure on the subsets of \(X\) that assigns only values 1 and 0. It takes value 1 on those
sets that are elements of $\mathcal{F}$ and zero on those that are not. It is a finitely additive measure.

### 2.3 Pushforwards of Ultrafilters and Limits of Ultrafilters

**Lemma 2.8.** Let $f: X \to Y$ be a set function and let $\mathcal{F}$ be an ultrafilter on $X$. We define the pushforward $f_*(\mathcal{F})$ to be the collection of subsets $U$ of $Y$ with the property that $f^{-1}(U)$ is an element of $\mathcal{F}$. Then $f_*(\mathcal{F})$ is an ultrafilter.

*Proof.* The finite additivity and superset property are clear for $f_*(\mathcal{F})$. It is also clear that for each $U \in 2^Y$ either $U$ or $Y \setminus U$ is contained in $f_*(\mathcal{F})$, but not both. Since $Y \in f_*(\mathcal{F})$, it follows that $\emptyset \notin f_*(\mathcal{F})$. \hfill $\Box$

**Definition 2.9.** Suppose that $\mathcal{F}$ is an ultrafilter for the set underlying a topological space $X$. A point $x \in X$ is said to be a limit point of $\mathcal{F}$ if every open neighborhood of $x$ is an element of $\mathcal{F}$.

**Proposition 2.10.** Suppose that $X$ is a space and $\mathcal{F}$ is an ultrafilter for the set underlying $X$. If $X$ is compact, then there is a limit point for $\mathcal{F}$. If $X$ is Hausdorff, then there is at most one limit point for $\mathcal{F}$. If $X$ is a compact Hausdorff space then $\mathcal{F}$ has exactly one limit point. Any limit point of an ultrafilter on a topological space $X$ is contained in the closure of any element of the ultrafilter.

*Proof.* If $\mathcal{F}$ has no limit point, then for every $x \in X$ there is an open neighborhood $U_x$ that is not a member of $\mathcal{F}$. If $X$ is compact, then the open covering $\{U_x\}_{x \in X}$ has a finite sub-cover, $U_{x_1}, \ldots, U_{x_k}$. Since $U_{x_1} \cup \cdots \cup U_{x_k} = X$ and $X \in \mathcal{F}$, it follows from Claim 2.7 that one of the $U_{x_i}$ is an element of $\mathcal{F}$. This is a contradiction and shows that there is at least one limit point for $\mathcal{F}$.

If $X$ is Hausdorff and $x \neq y$ are limits of $\mathcal{F}$, then there are disjoint neighborhoods of $x$ and $y$ in $\mathcal{F}$ contradicting the fact that $\mathcal{F}$ has finite intersection property.

Putting the two statements together, establishes the result for compact Hausdorff spaces.

Suppose that $\mathcal{F}$ is an ultrafilter on $X$ with $A$ an element of $\mathcal{F}$. If $x$ is not in the closure of $A$, then there is an open neighborhood $V$ of $x$ disjoint from $A$. Thus, $V$ is not an element of $\mathcal{F}$, which implies that $x$ is not the limit point of $\mathcal{F}$.

\hfill $\Box$
2.4 The Stone Topology on the Space of Ultrafilters

**Definition 2.11.** Let $X$ be a set and let $\mathcal{U}(X)$ be the set of ultrafilters on $X$. For any subset $A \subset X$ we define $U_A \subset \mathcal{U}(X)$ to be the set of ultrafilters having $A$ as an element.

**Lemma 2.12.** With notation as above

1. $U_{\emptyset} = \emptyset$.
2. $U_X = \mathcal{U}(X)$.
3. $U_{A_1} \cap \cdots \cap U_{A_k} = U_{A_1 \cap \cdots \cap A_k}$.
4. $U_{A_1} \cup \cdots \cup U_{A_k} = U_{A_1 \cup \cdots \cup A_k}$.
5. Using overline to denote complement, we have $\overline{U_A} = U_{\overline{A}}$.

**Proof.** Items 1 and 2 are immediate. For the third, $U_{A_1} \cap U_{A_2}$ consists of all ultrafilters that have both $A_1$ and $A_2$ as elements. If $A_1 \cap A_2 = \emptyset$, then $U_{A_1} \cap U_{A_2} = \emptyset$. Otherwise, using Claim 2.7 we see that $A_1 \cap A_2$ is an element of any $\mathcal{F} \in U_{A_1} \cap U_{A_2}$, and hence $U_{A_1} \cap U_{A_2} \subset U_{A_1} \cap U_{A_2}$. The opposite inclusion is clear from the superset property of ultrafilters. Induction allows us to pass from the case $k = 2$ for this statement to the case of general finite $k$. The proof of Item 4 is analogous. The last item is clear since $A \bigcup \overline{A} = X$ every ultrafilter on $X$ either has $A$ or $\overline{A}$ as an element but not both. \[\square\]

**Definition 2.13.** We define a topology on $\mathcal{U}(X)$, the *Stone Topology*, by defining the sub-basic open sets to be indexed by subsets of $X$. The sub-basic open set associated to $A \subset X$, denoted $U_A$ is the set of all ultrafilters on $X$ that have $A$ as an element. Then the basic open sets are subsets of ultrafilters than have as elements all members of a given finite family subsets of $X$. We define $\beta X$ to be the set $\mathcal{U}(X)$ with the Stone Topology.

It follows immediately from Proposition 2.12 that the collection of subsets $\{U_A\}_{A \subset X}$ of $\mathcal{U}(X)$ are closed under finite intersections. Since they are defined to be a sub-basis for the Stone Topology, they are in fact a basis. A subset of $\beta X$ is open if and only if it is a union of sets of the form $U_A$ for $A \subset X$.

**Corollary 2.14.** The basic open sets of $\beta X$ are also closed. Also, $\beta X$ is Hausdorff. Thus, $\beta X$ is completely disconnected.
Proof. By Property 5 in Proposition 2.12 the basic open sets of $\beta X$ are also closed. If $F$ and $G$ are distinct ultrafilters, then there is a subset $A \subset X$ that is contained in $F$ but not in $G$. Hence, $F \in U_A$ and $G \in U_{\neg A}$. This proves that $\beta X$ is Hausdorff. It follows from Theorem 1.4 that $\beta X$ is totally disconnected.

**Definition 2.15.** Given a set $X$, for $x \in X$ we set $u(x) \in \beta X$ equal to the principal ultrafilter generated by $x$. Recall that its elements are the subsets of $X$ that contain $x$. This defines a function $u: X \to \beta X$.

**Theorem 2.16.** Let $X$ be a set. Then the space of ultrafilters on $X$, $\beta X$, is a compact, totally disconnected Hausdorff space and $u: X \to \beta X$ is a continuous one-to-one map from $X$ with the discrete topology to $\beta X$. Its image $u(X)$ is dense in $\beta X$ and the subspace topology on $u(X)$ is the discrete topology so that $u$ is a homeomorphism onto its image.

Proof. From the definition of $u: X \to \beta X$, it is clear that $u$ is one-to-one. It is continuous since $X$ is given the discrete topology. The open subset $U_{\{x\}}$ contains the principal ultrafilter generated by $x$ but no other principal ultrafilter. Thus, $u: X \to u(X)$ is a homeomorphism. Since every basic set $U_A$ contains $u(a)$ for every $a \in A$, the image $u(X)$ meets every open set and hence $u(X)$ is dense in $\beta X$.

To see that $\beta X$ is compact we must show that any open covering $\{U_i\}_{i \in I}$ with index set $I$ has a finite sub-covering. Since each open set is a union of basic sets, we can arrange (possibly by changing the index set $I$) that for each of the open sets $\{U_i\}_{i \in I}$ are of the form $U_i = U(A_i)$ for a subset $A_i \in X$.

We assume that this cover has no finite sub-cover. This means that for every finite subset $F \subset I$

$$V(F) = \cap_{i \in F} U_i = \cap_{i \in F} U(A_i) \neq \emptyset.$$ 

By the same reasoning, as $F$ ranges over the finite subsets of $I$, we obtain a collection of closed (and open) subsets $V(F)$ of $U(X)$ with the finite intersection property. Thus, applying Proposition 2.12 gives us that for any finite subset $F$ of $I$, setting $A_F = \cap_{i \in F} A_i$, the $\{A_F\}_{F}$ is a collection of sets in $X$ with the finite intersection property. By Claim 2.3 this collection defines a filter on $X$ consisting of all subsets of $X$ that contain one of the finite intersections of the $A_F$. This filter is contained in an ultrafilter $u_0$, which then also has as elements all the subsets $A_i$ of $X$. Hence, for each $i \in I$, since $A_i$ is an element of $u_0$ we have $u_0 \notin U_i = U(A_i)$. Thus, $u_0 \notin \cup_{i \in I} U_i$. 

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This contradicts the fact that the \( \{U_i\}_{i \in I} \) are a covering of \( U(X) \), proving that the assumption that there is no finite sub-cover of the original cover is false and establishing the compactness of \( \beta X \).

**Corollary 2.17.** For any set \( X \), the space \( \beta X \) is profinite.

**Proof.** This is immediate from Theorems 2.16 and 1.9.

### 2.5 Extending maps of \( X \) to a compact set to all of \( \beta X \)

**Theorem 2.18.** Let \( X \) be a set with the discrete topology and \( i: X \to \beta X \) as above. Then if \( Y \) is a compact Hausdorff space and \( f: X \to Y \) is a set function (which is a continuous map with the given topologies) then there is exactly one continuous map \( \varphi: \beta X \to Y \) satisfying \( \varphi \circ i = f \).

**Proof.** Since \( i(X) \) is dense in \( \beta X \) there is at most one continuous extension of \( f \) to all of \( \beta X \). The definition of \( \varphi \) is more or less obvious: for each ultrafilter \( \mathcal{F} \) on \( X \) the push forward \( f_*(\mathcal{F}) \) is an ultrafilter on \( Y \). Since \( Y \) is compact Hausdorff, \( f_*(\mathcal{F}) \) has a unique limit point. We define the image under \( \varphi \) of the point \( \mathcal{F} \in \beta X \) to be the limit of \( f_*(\mathcal{F}) \). It is clear that if the principal ultrafilter generated by \( x \in X \), then \( f_*(\mathcal{F}) \) is principal ultrafilter on \( Y \) generated by \( f(x) \). Of course the limit of this ultrafilter is \( f(x) \). This shows that \( \varphi \circ i = f \).

It remains only to show that \( \varphi \) is continuous. Let \( \varphi(\mathcal{F}) = y \in Y \). Let \( V \subset Y \) be an open neighborhood of \( y \). To show the continuity of \( \varphi \), we construct a basic open set of \( \beta X \), \( U_A \) for an appropriate \( A \subset X \) containing \( \mathcal{F} \) and whose image under \( \varphi \) is contained in \( V \). Since \( Y \) is compact Hausdorff, the frontier of \( V \), which is defined as the closure of \( V \) minus \( V \), is a compact set disjoint from \( y \). Thus, there is an open subset \( W \) of \( Y \) containing the frontier of \( V \) but disjoint from an open neighborhood \( V' \subset V \) of \( y \). The closure of \( V' \) is contained in \( V \). Let \( A = f^{-1}(V') \). Clearly, \( \mathcal{F} \in U_A \). The open set \( U_A \) is as required as is shown by the next claim.

**Claim 2.19.** \( \varphi(U_A) \subset V \).

**Proof.** Let \( \mathcal{G} \in U_A \), meaning that \( \mathcal{G} \) is an ultrafilter having \( A \) as an element. Since \( A \) is an element of \( \mathcal{G} \), the set \( f^{-1}(f(A)) \) contains \( A \) and is also an element of \( \mathcal{G} \). Thus, \( f(A) \) is an element of \( f_*(\mathcal{G}) \) and hence by Proposition 2.10, the limit of \( f_*(\mathcal{G}) \) is contained in the closure of \( f(A) \). Since \( f(A) \subset V' \), the closure of \( f(A) \) is contained in the closure of \( V' \), which is contained in \( V \). This shows that \( \varphi(\mathcal{G}) \), which is the limit of \( f_*(\mathcal{G}) \), is contained in \( V \).

This completes the proof of the continuity of \( \varphi \).
Remark 2.20. A very similar argument shows that a function \( f: Y \to Z \) between compact Hausdorff spaces is continuous if and only if it preserves limits of ultrafilters. That is to say \( f_*(\text{lim}(\mathcal{F})) = \text{lim}(f_*(\mathcal{F})) \) for every ultrafilter in \( Y \).

Corollary 2.21. Every compact Hausdorff space is the quotient of a totally disconnected space.

Proof. Let \( C \) be a compact Hausdorff space and \( C' \) the same set with the discrete topology. Then the map \( C' \to C \) given by the identity map is continuous and extends to a continuous map \( \beta C' \to C \) which obviously is surjective. Since \( \beta C' \) and \( C \) are compact Hausdorff spaces this map is a quotient map. \( \square \)

2.6 Tychonov’s Theorem

An application of the theory of ultrafilters is Tychonov’s Theorem.

Theorem 2.22. (Tychonov’s Theorem) Let \( \{X_i\}_{i \in I} \) be a family of compact spaces indexed by a set \( I \). Then \( \prod_{i \in I} X_i \) with the product topology is compact.

Remark 2.23. For a product of two compact spaces (or indeed any finite collection of compact spaces) one can give a direct argument. The proof in the case of infinite products is more indirect.

Proof. Let \( P = \prod_{i \in I} X_i \). Fix an open covering \( \{U_j\}_{j \in J} \). We must show that this covering has a finite sub-covering. Suppose that there is no finite subcovering. Then the complementary closed sets \( F_j = P \setminus U_j \) have the finite intersection property. By Claim 2.3 there is an ultrafilter \( \mathcal{F} \) containing all finite intersections of the \( F_j \). For each \( i \in I \) let \( \pi_i: P \to X_i \) be the projection and let \( (\pi_i)_*(\mathcal{F}) \) be the pushforward ultrafilter. Since \( X_i \) is compact, this ultrafilter has a limit \( x_i \).

We claim \( p = \{x_i\}_{i \in I} \in P \) is not in \( \cup_{j \in J} U_j \), which is a contradiction since by assumption these open sets cover \( P \). If the point \( p \) were in this union, then there would be a basic open set of \( P \) containing \( p \) and contained in one of the \( U_i \). Any basic open set containing \( p \) is of the form \( \prod_{i=1}^r W_{ik} \times \prod_{i \in P \setminus I} X_i \), where \( W_{ik} \subset X_{ik} \) is an open subset containing \( x_{ik} \) and \( I' = I \setminus \{i_1, \ldots, i_k\} \). But for each \( 1 \leq n \leq r \), the set \( W_{in} \) is an open neighborhood of \( x_{in} \) which is a limit point of \( (\pi_{in})_*(\mathcal{F}) \). It follows that

\[
Z_{in} = \pi_{in}^{-1}(W_{in}) = W_{in} \times \prod_{i \in P \setminus \{i_n\}} X_i
\]
is an element of $\mathcal{F}$. By the finite intersection property,

$$\prod_{k=1}^{r} W_k \times \prod_{i \in I} X_i = \bigcap_{1 \leq n \leq k} Z_{i_n}$$

is also an element of $\mathcal{F}$. Thus, it meets every element of $\mathcal{F}$, and in particular, it meets each $F_i$. Thus, this open subset is not contained in any of the $U_i$. \qed

2.7 The Stone-Cech Compactification of a Space.

For any topological space $X$ the Stone-Cech compactification is defined to be a compact space $\beta X$ together with a continuous map $i: X \to \beta X$ that satisfies this universal mapping property. That is to say given any compact space $Y$ and a continuous map $f: X \to Y$ there is a unique (continous) map $\varphi_f: \beta X \to Y$ with $\varphi_f \circ i = f$. Since it satisfies the universal mapping problem, if $\beta X$ exists then it is unique up to unique homeomorphism.

One constructs it rather tautologically. Consider all maps $f_Y: X \to Y$ from $X$ to a compact space $Y$ and form the map $X \to \prod f_Y Y$ given by the product of the $f_Y$. Of course, it is not piosssible to do this since we are dealing with classes rather than sets of spaces and maps. But, it suffices to consider only maps $X \to Y$ with dense image. Then any compact space $Y$ that contains a continuous dense image of $X$ is up to homeomorphism given by a topology on some subset of the power set of the power set of $X$, and the space given by topologies on subsets of this fixed set for a set. Using all such compact $Y$ and maps $f_Y: X \to Y$ with dense image gives us a set of possibilites and with these restrictions we form $\prod f_Y: X \to \prod f_Y Y$. By Tychoonov’s theorem the range is compact and hence the closure of the image of $X$ under this map is a compact subset $\beta X$ equipped with a map $X \to \beta X$. For any compact set $Z$ and a map with dense image $f: X \to Z$, we find $f_Y$ in our collection such that up to a homeomorphism $h: Z \to Y$, $f = h^{-1} \circ f_Y$. Thus, the projection onto this factor restricts to $\beta X$ to give an extension of $f: X \to Y$, which followed by $h^{-1}$ is the required extension of $f: X \to Z$ to a map on $\beta X \to Z$. In case $f: X \to Z$ does not have dense image one replaces $Z$ by the compact subspace $Z_0 \subset Z$ that is the closure of the image $f(X)$ and argues as before.

In case of a regular Hausdorff space $X$, one in which a closed set and a disjoint point can be separated by a continuous function to the unit interval $I$ one can use the product over all continuous functions $X \to I$ of $I$ and make the same type of argument.

For discrete $X$ we can do much better.
Theorem 2.24. For a discrete space \( X \) the inclusion map \( u: X \to \beta X \) is the Stone-Cech compactification.

Proof. We have seen that for any compact space \( Y \) any map \( f: X \to Y \) (automatically continuous since \( X \) has the discrete topology) there is a unique continuous extension of \( f \) over \( \beta X \). This is the defining property of the Stone-Cech compactification. \( \square \)

3 Extremally Disconnected Spaces

Definition 3.1. A space \( X \) is extremally disconnected if it is a compact Hausdorff space and if every surjection \( C \to X \) from a compact Hausdorff space \( C \) splits, i.e., has a section.

Lemma 3.2. Let \( X \) be an extremally disconnected Hausdorff space, then \( X \) is totally disconnected.

Proof. Let \( x \neq y \) be two points of \( X \). Since \( X \) is Hausdorff there are disjoint open sets \( U, V \) with \( x \in U \) and \( y \in V \). Let \( A = X \setminus V \) and \( B = X \setminus U \). These are compact subsets that cover \( X \). Since \( X \) is extremely disconnected, there is a continuous section \( s: X \to A \bigsqcup B \) for the natural projection \( A \bigsqcup B \to X \). The image of \( x \) lies in \( A \) and the image of \( y \) lies in \( B \), so that \( s^{-1}(A) \) is an open and closed subset of \( X \) containing \( x \) but not \( y \). \( \square \)

Corollary 3.3. Suppose that \( f: Y \to Z \) is a surjection between compact Hausdorff spaces and let \( X \) be an extremally disconnected space. Then any continuous \( \alpha: X \to Z \) lifts to \( Y \); i.e., there is a continuous map \( \beta: X \to Y \) with \( \alpha = f \circ \beta \).

Proof. We form the fibered product \( X \times_Z Y \). Since \( Y \to Z \) is surjective, the projection \( X \times_Z Y \to X \) is also surjective. Since \( X \) is extremely disconnected, there is a section \( X \to X \times_Z Y \). The composition of this section followed by the projection to \( Y \) is the required lift. \( \square \)

Proposition 3.4. For any discrete set \( X \) the Stone-Cech compactification \( \beta X \) of \( X \) is extremally disconnected.

Proof. Let \( A \) be a compact Hausdorff space and \( \rho: A \to \beta X \) a continuous surjection. Since subspace \( X \subset \beta X \) is discrete, there is a section \( i: X \to A \) for \( \rho \) over \( X \). By the universal property of \( \beta X \) the map \( i: X \to A \) extends to a continuous map \( \hat{i}: \beta X \to A \). The composition \( \rho \circ \hat{i}: \beta X \to \beta X \) is the
identity on $X$ and since $X$ is dense in $\beta X$, it follows that $\hat{i}: \beta X \to C$ is a section.

Corollary 3.5. Every compact Hausdorff space is the quotient of an extremely disconnected space. If the set underlying the compact space has cardinality less than $\kappa$ then one can choose the extremely disconnected set to have cardinality less than $\kappa$.

Proof. Let $C$ be a compact Hausdorff space of cardinality less than $\kappa$. Let $C'$ be $C$ with the discrete topology. Then $\beta C'$ has cardinality at most $2^{2^{\text{card}(C)}}$. Since $\kappa$ is a limit cardinal, this cardinality is also less than $\kappa$. Then the identity map $C' \to C$ is continuous and extends to a map $\beta C' \to C$ that is surjective.

Remark 3.6. Every extremally disconnected space is a retract of the Stone-Cech compactification of some discrete set.