Lie Groups: Fall, 2024
Lecture III
Local Lie Groups
Statement of Baker-Campbell-Hausdorff Formula

September 16, 2024

1 Local Lie Groups

1.1 Definition of a Local Lie Group

There is an intermediate category between the category of Lie groups and that of Lie algebras. It is the category of local Lie groups and their equivalence classes called *germs of Lie groups* (at the identity).

Definition 1.1. A *local Lie Group* consists of:

(i) a smooth manifold U

(ii) an element $e \in U$,

(iii) a diffeomorphism $\theta\colon U \to U$ fixing e with $\theta^2 = \mathrm{Id}_U$

(iv) an open subset $\Omega \subset U \times U$ and a smooth map $m\colon \Omega \to U$ called *multiplication*,

such that

(a) for every $g \in U$ the elements (e, g) and (g, e) are contained in Ω and $m(e, g) = m(g, e) = g$,

(b) for every $g \in U$ the pairs $(\theta(g), g)$ and $(g, \theta(g))$ are contained in Ω and $m(\theta(g), g) = m(g, \theta(g)) = e$,

(c) for every triple (g, h, k) of elements in G if the pairs $(g, h), (h, k), (g, m(h, k))$ and $(m(g, h), k)$ are contained in Ω then $m(g, m(h, k)) = m(m(g, h), k)$.

Clearly Properties (a), (b), and (c) are local versions of the identity law, the inverse law, and the associative law for a group. The only difference is that the domain of definition for multiplication is an open subset of $U \times U$ and the associative law only holds on a smaller open subset of $U \times U \times U$.

From now on we denote $\theta(g)$ by g^{-1}. Of course $e^{-1} = e$. We also write gh for $m(g, h)$.

The following is an elementary lemma.

Lemma 1.2. *Let $(U, e, \theta, \Omega.m)$ be a local Lie group. For $g \in U$ there are open subsets $W \subset U$ containing e and $V \subset U$ containing g such that ug is defined for every $u \in W$, and vg^{-1} is defined for every $v \in V$ and the maps $u \mapsto ug$ and $v \mapsto vg^{-1}$ are inverse diffeomorphisms between W and V. Furthermore, there is an open subset $W \subset G$ containing e such that $W^2 \times W^2 \subset \Omega$. For any $w_1, w_2, w_3 \in W$ the following two expressions $w_1(w_2w_3)$ and $(w_1w_2)w_3$ are defined and hence are equal.*

Definition 1.3. A *morphsm* of local Lie groups

$$\rho\colon (U', e', \theta', \Omega', m') \to (U, e, \theta, \Omega, m)$$

is a smooth map $\rho\colon U' \to U$ with $\rho(e') = e$ and $\rho \times \rho|_{\Omega'}\colon \Omega' \to \Omega$ such that $\rho(\theta'(x)) = \theta(\rho(x))$ for all $x \in U'$ and $m(\rho(x), \rho(y)) = \rho(m'(x, y))$ for all $(x, y) \in \Omega'$. It is clear that these morphisms can be composed and that each object has the identity morphism. Hence, we have a category of local Lie groups.

1.2 Germs of Local Lie Groups

Definition 1.4. The *germ* of a local Lie group is an equivalence class of local Lie groups for the equivalence relation generated by

$$(U, e, \theta, \Omega, m) \sim (U', e', \theta', \Omega', m')$$

if there is a morphism from the first to the second that is an embedding of U onto an open subset of U'. Such morphisms are called *elementary equivalences.*

Lemma 1.5. *Two local Lie groups $(U, e, \theta, \Omega, m)$ and $(U', e', \theta', \Omega', m')$ determine the same germ if and only if there is a third $(U'', e'', \theta'', \Omega'', m'')$ that maps to each of them by an elementary equivalence.*

The proof is left as an exercise.

Definition 1.6. Let G be a Lie group and $U \subset G$ an open neighborhood of $e \in G$ invariant under $\iota(g) = g^{-1}$. The *local Lie group determined by* G *and* U is defined to be $(U, e, \iota, \Omega, m|_{\Omega})$ where $m\colon G \times G \to G$ is the group multiplication of G and $\Omega = m^{-1}(U) \cap U \times U$.

Corollary 1.7. *Let U and U' be open neighborhoods of e in G invariant under ι. Then the local Lie subgroups of G determined by U and U' have the same germ.*

Proof. Each of these local Lie subgroups contains the local Lie subgroup determined by $U \cap U'$. □

Definition 1.8. The *germ of the Lie group* G is the germ of one (and therefore all) of its local Lie subgroups determined by an open neighborhood of e invariant under ι.

2 The Lie Algebra of a Local Lie Group

Lemma 2.1. *Any local Lie group $(U, e, \theta, \Omega, m)$ has a Lie algebra whose underlying vector space is the tangent space T_eU. The differential at e of a morphism of local Lie groups induces a homomorphism of their Lie algebras.*

Proof. The argument defining the Lie algebra structure on T_eU is exactly the same as the argument in the case of a Lie group. Given $X \in T_eU$ there is a vector field on U whose value at $g \in U$ is $g \cdot X$. (For any g multiplication by g is define on a neighborhood of e, and hence multiplication by g sends $T_eU \to T_gU$.) We call all such vector fields *left-invariant*. Then the usual argument shows that the space of these vector fields is closed under bracket and the space is identified with T_eU. Hence, there is the induced Lie algebra structure on T_eU, which is defined to be *its Lie algebra*.

The other approach to the Lie bracket also works for local Lie groups. For any $g \in U$, the element $m(ge, g^{-1}) = geg^{-1}$ is defined and hence for every $g \in U$ there is a neighborhood V of $e \in U$ such that $\rho_g(v) = gvg^{-1}$ is defined. for all $v \in V$. Thus, the differential of this map ρ_g at $v = e$ determines a map $\overline{\rho}_g\colon U \to GL(T_eU)$ and restricted to sufficiently small neighborhood of $e \in U$ this map is a local homomorphism in the sense that $\overline{\rho}_h\overline{\rho}_g = \overline{\rho}_{hg}$ for h, g sufficiently close to the identity. Define $\mathrm{ad}(X)\colon T_eU \to T_eU$ as the image of X under the differential, $d\rho_e\colon T_e(U) \to \mathrm{End}(T_eU)$. The Lie algebra of the local Lie group is then $[X, Y] = \mathrm{ad}(X)(Y)$.

The proof that these two methods define the same Lie algebra follows by the same argument as in the case of a Lie group. □

Corollary 2.2. *The germ of a local Lie group has a Lie algebra well-defined up to canonical isomorphism and a morphism of germs induces a morphism of Lie algebras.*

Proof. If μ is an embedding of one local Lie group onto an open subset of a second local Lie group, then $D\mu_e$ is an isomorphism of the tangent spaces at the identity which induces an isomorphism of Lie algebras. It follows immediately that a germ has a well-defined tangent space and Lie algebra structure up to unique isomorphism. The second statement is immediate. $\square$

3 Extending maps from local Lie subgroups to the entire group

Here is the main result.

Theorem 3.1. *Suppose that G_0 and G_1 are connected Lie groups with G_0 simply connected. Suppose given an morphism ψ from the germ of G_0 to the germ of G_1. Then there is a unique map of Lie groups $\Psi\colon G_0 \to G_1$ that agrees with ψ on the germs.*

Proof. Let $U_0 \subset G_0$ and $U_1 \subset G_1$ be open neighborhoods of the identity such that ψ is presented by a map of local Lie groups $\psi\colon (U_0, e, \iota_0, \Omega_0, m_0) \to (U_1, e, \iota_1, \Omega_1, m_1)$. Fix $W_0 \subset U_0$ an open neighborhood of e such that $W_0 \times W_0 \subset \Omega$. By replacing W_0 by $W_0 \cap W_0^{-1}$, we can also assume that W_0 is invariant under taking inverses. Then for any $w_1, w_2 \in W_0$, we have $\psi(w_1 w_2) = \psi(w_1)\psi(w_2)$.

Let $g \in G_0$ and choose a smooth path $\omega\colon [0,1] \to G_0$ with $\omega(0) = e$ and $\omega(1) = g$. There are points $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $\omega([t_0, t_1]) \subset W_0$ and for each $1 \leq i < n$ the path $\omega([t_i, t_{i+1}]) \subset \omega(t_i) \cdot W_0$. We define $\Psi(g)$ by defining $\Psi([t_i, t_{i+1}])$ inductively on i. For $i = 0$ we set

$$\Psi|_{[t_0,t_1]} = \psi \circ \omega|_{[t_0,t_1]}.$$

By induction on i for $t \in [t_i, t_{i+1}]$, we define

$$\Psi(t) = \Psi(t_i) \cdot \psi((\omega(t_i)^{-1}\omega(t)).$$

This completes the induction and gives us a value $\Psi(g)$, which, *a priori*, depends on the path ω from e to g, the choice of n and the points $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n$.

Claim 3.2. *Fixing the path ω, using two different divisions $0 = t_0 < \cdots < t_n = 1$ and $0 = u_0 < u_1 < \cdots < u_m = 1$, as above, to define $\Psi(g)$ give the same value.*

Proof. By taking a common refinement of two subdivisions it suffices to assume that each interval $[u_j, u_{j+1}]$ is contained in one of the intervals $[t_{i(j)}, t_{i(j)+1}]$. Let Ψ_1, resp., Ψ_2, be the function on $[0, 1]$ defined by ω and the partition $\{t_i\}_i$, resp., the partition $\{u_j\}_j$. Suppose by induction on j that $\Psi_1 = \Psi_2$ on the interval $[0, u_j]$. For $j = 0$, this is clear. Let i be such that $[u_j, u_{j+1}] \subset [t_i, t_{i+1}]$. Then for $t \in [u_j, u_{j+1}]$:

$$\Psi_2(t) = \Psi_2(u_j)\psi(\omega(u_j^{-1})\omega(t)).$$

By the inductive hypothesis

$$\Psi_2(u_j) = \Psi_1(u_j) = \Psi_1(t_i)\psi(\omega(t_i)^{-1}\omega(u_j)),$$

so that

$$\Psi_2(t) = \Psi_1(t_i)\psi(\omega(t_i)^{-1}\omega(u_j))\psi(\omega(u_j)^{-1}\omega(t)).$$

But $(\omega(t_i)^{-1}\omega(u_j))$, $(\omega(u_j)^{-1}\omega(t))$ are contained in W_0 as is their product $\omega(t_i)^{-1}\omega(t)$ so that

$$\psi(\omega(t_i)^{-1}\omega(t)) = \psi(\omega(t_i)^{-1}\omega(u_j)))\psi(\omega(u_j)^{-1}\omega(t)).$$

Thus,

$$\Psi_2(t) = \Psi_1(t_i)\psi(\omega(t_i)^{-1}\omega(t)) = \Psi_1(t),$$

for all $t \in [u_j, u_{j=1}]$. This completes the inductive proof of the claim. □

Definition 3.3. Given a path ω in G from e to an element $g \in G$, the unique map as given in Claim 3.2 is denoted by Ψ_ω. When ω is clear from context, we often drop the subscript and denote this unique path by Ψ.

Corollary 3.4. *Given a path ω from e to some $g \in G$, the resulting function $\Psi_\omega \colon [0, 1] \to G$ as in the previous theorem is the unique function that satisfies:*

- $\Psi_\omega(0) = e$, *and*
- *for each $t \in [0, 1]$ there is a neighborhood V of t such that for all $s \in V$, $\Psi_\omega(t)^{-1}\Psi_\omega(s) \in W_0$ and $\Psi_\omega(s) = \Psi_\omega(t)\psi(\Psi_\omega(t)^{-1}\Psi_\omega(s))$.*

Proof. Suppose that Ψ' is a function satisfying the two conditions in the statement of the corollary. Then, it is easy to find a partition $0 = t_0 < \cdots < t_n = 1$ as in the proof of Theorem 3.1 for Ψ', which, by the above claim, proves that $\Psi' = \Psi_\omega$. $\square$

Now let ω_s be a one-parameter family of paths in G_0 from e to g wth $\omega_0 = \omega$. For each s we construct, as above a map $\Psi_{\omega_s}\colon I \to G_1$. For ω_0 we have a decomposition $0 = t_0 < t_1 < \cdots < t_n = 1$ with $\omega_0(t_i)^{-1}\omega_0(t) \in W_0$ for all $i = 0, \ldots, n$ and all $t \in [t_i, t_{i+1}]$.

From now on, we denote Ψ_{ω_s} by Ψ_s.

Claim 3.5. *There is $\epsilon > 0$, such that for all $|s| < \epsilon$, for all $i \leq n-1$, and for all $t \in [t_i, .t_{i+1}]$ we have*

$$\Psi_s(t) = \Psi_0(t_i)\psi(\omega_0(t_i)^{-1}\omega_s(t)).$$

Proof. The proof is by induction on i. For $i = 0$ it is clear since $\omega_s(0) = \omega_0(0) = e \in G_0$. Suppose that we have established the result for $i-1$ and we consider $t \in [t_{i-1}, t_i]$. By definition

$$\Psi_s(t) = \Psi_s(t_{i-1})\psi(\omega_s(t_{i-1})^{-1}\omega_s(t)).$$

By induction there is $\epsilon > 0$, such that for all $|s| < \epsilon$

$$\Psi_s(t_{i-1}) = \Psi_0(t_{i-1})\psi(\omega_0(t_{i-1})^{-1}\omega_s(t)).$$

Thus,

$$\Psi_s(t) = \Psi_0(t_{i-1})\psi(\omega_0(t_{i-1})^{-1}\omega_s(t_{i-1})) \cdot \psi(\omega_s(t_{i-1}^{-1})\omega_s(t)).$$

By a standard compactness argument, there is $0 < \epsilon' \leq \epsilon$ such that for any s with $|s| < \epsilon'$, for every $i = 0, \ldots, n$, and for every $t \in [t_i, t_{i+1}]$ we have $(\omega_0(t_i)^{-1}\omega_s(t_i)$, $\omega_s(t_i)^{-1}\omega_s(t)$, and $\omega_0(t_i)^{-1}\omega_s(t)$ are all in W_0. Hence, for all i and all $t \in [t_i, t_{i+1}]$ and all s with $|s| < \epsilon$, we have

$$\psi(\omega_0(t_{i-1})^{-1}\omega_s(t)) = \psi(\omega_0(t_{i-1})^{-1}\omega_s(t_{i-1}))\psi(\omega_s(t_{i-1})^{-1}\omega_s(t)).$$

Thus:

$$\Psi_s(t) = \Psi_0(t_{i-1})\psi(\omega_0(t_{i-1})^{-1}\omega_s(t)).$$

Now replace ϵ by ϵ'. This completes the inductive proof of the claim. $\square$

Applying this claim with $t = t_n$, we see that for $|s| < \epsilon$

$$\Psi_s(t_n) = \Psi_0(t_n)\psi(\omega_0(t_n)^{-1}\omega_s(t_n)) = \Psi_0(t_n)\psi(g^{-1}g) = \Psi_0(t_n).$$

This shows that $\Psi_s(g)$ is a locally constant function of s.

Since G_0 is simply connected, any two paths ω and ω' from e to g are connected by a one-parameter family $\{\omega_s\}_{0\leq s\leq 1}$ of paths from e to g. We have just seen that the value of $\Psi_s(g)$ is locally constant in s, and hence is the same at the end points $s = 0$ and $s = 1$. Thus, this process yields a well-defined function $\Psi\colon G_0 \to G_1$ agreeing with ψ on W_0.

Furthermore, it is immediate from the construction that for any $g \in G_0$ and any u in the connected component of the identity of W_0, we have $\Psi(gu) = \Psi(g)\psi(u)$. Thus, Ψ is a smooth map.

Let ω be a path from e to g. Then the path μ defined by $\mu(t) = \omega(t)^{-1}$ is a path from e to g^{-1}. Since W_0 is closed under taking inverses and $\psi(u^{-1}) = \psi(u)^{-1}$, it is clear that $\Psi_\omega(t)^{-1} = \Psi_\mu(t)$ for all t, and in particular $\Psi(g)^{-1} = \Psi(g^{-1})$.

Lastly, we show that Ψ sends multiplication in G_0 to multiplication in G_1. Fix $g \in G_0$ and consider the subset $X_g \subset G_0$ of $g' \in G_0$ such that $\Psi(g)\Psi(g') = \Psi(gg')$. Clearly, since multiplication in G_0 and G_1 and Ψ are continuous, X_g is a topologically closed subset of $G.$. For any w in the component of the identity of W_0, we have $\Psi(gg'w) = \Psi(g)\Psi(g'w) = \Psi(g)\Psi(g')w$ and composing in the other order $\Psi(gg'w) = \Psi(gg')\psi(w)$. Thus, if $g' \in X_g$ the a neighborhood of $g' \in G_0$ is also contained in X_g, proving that X_g is open. Clearly, $e \in X_g$. Since G_0 is connected, $X_g = G_0$. Since this is true for every $g \in G_0$. the map Ψ preserves the multiplications. $\square$

Proposition 3.6. *If G_0 and G_1 are connected Lie groups with the same germs and G_0 is simply connected, then there is a unique Lie group homomorphism $\Psi\colon G_0 \to G_1$ extending the identification of their germs. This map is a covering map. If G_1 is also simply connected, then this map is an isomorphism of Lie groups*

Proof. By the previous result there is a unique Lie group homomorphism $\Psi\colon G_0 \to G_1$ that identifies the germs of G_0 and G_1. In particular Ψ is a local diffeomorphism from an open neighborhood W_0 of $e \in G_0$ to an open neighborhood U_1 of $e \in G_1$. By equivariance, Ψ is a local diffeomorphism from G_0 to G_1. The kernel K of Ψ is a discrete normal subgroup. We have a covering map $G_0 \to G_0/K$, and the map Ψ factors to give a map $\overline{\Psi}\colon G_0/K \to G_1$ that is one-to-one local diffeomorphism whose image contains a neighborhood U_1 of the identity. Since the image of $\overline{\Psi}$ contains a neighborhood of the identity if $\{x_m\}$ is a sequence in $\mathrm{Im}(\overline{\Psi})$ and $x_m \mapsto x$ as $m \mapsto \infty$, then $x_m^{-1}x \in U_1$ for all m sufficiently large. Hence x_m and $x_m^{-1}x$ are contained in the image of $\overline{\Psi}$. It follows that $x \in \mathrm{Im}(\overline{\Psi})$. This shows that $\mathrm{Im}(\overline{\Psi})$ is closed. Since G_1 is connected, $\mathrm{Im}(\overline{\Psi}) = G_1$. □

Corollary 3.7. *There is at most one simply connected Lie group up to canonical isomorphism with a given germ.*

Remark 3.8. We shall eventually see that every germ of a local Lie group extends to Lie group.

4 Extending local Lie Subgroups of a Lie group

Let's begin by showing how to enhance a local Lie sub group of a Lie group G to a Lie group that is one-to-one immersed in G.

Theorem 4.1. *1.) Let G be a Lie group and let $(U, e, \theta, \Omega, m)$ be a local Lie sub group of G. (This means that there is a morphism of local Lie groups $(U, e, \theta, \Omega, m) \to G$ which is a locally closed embedding on U.) Then there is a Lie group N and an identification of $(U, e, \theta, \Omega, m)$ with the local Lie subgroup of N determined by a neighborhood U_0 of the identity in N. Furthermore, there is a one-to-one immersion of $N \to G$ whose restriction to U is the given identification of U with U_0. The subgroup of G generated by U is an open subgroup of N. If U is connected, then this subgroup is the connected component of the identity, N_0, of N.*
2.) The universal covering of N_0 is a simply connected Lie group with the given germ.

Proof. Let $N \subset G$ be the set of elements $g \in G$ such that gUg^{-1} contains an open neighborhood of the identity in U.

Claim 4.2. *For any $g \in N$, there is an open neighborhood V of the identity in U such that the map $V \to G$ given by $v \mapsto gvg^{-1}$ is a diffeomorphism of V onto an open subset $V' \subset U$ containing the identity.*

Proof. By the definition of N there is an open subset $V' \subset U$ containing the identity with $V' \subset gUg^{-1}$. Thus, $V = g^{-1}V'g \subset U$. Since conjugation by g^{-1} is a diffeomorphism of G, the map $v' \mapsto g^{-1}v'g$ is a smooth map from $V' \to U$ whose image is V. Being the restriction of a diffeomorphism to a smooth submanifold of G, this map is one-to-one has injective differential at each point. It follows that as a map $V' \to U$ it has surjective differential at each point and hence is a diffeomorphism onto an open subset V of U.

This shows that V is an open subset of U containing e. Conjugation by g is the inverse diffeomorphism from $V \subset U$ to $V' \subset U$. $\square$

Claim 4.3. *N is a subgroup of G.*

Proof. Suppose that $g \in N$. Then according to the previous claim there is an open neighborhood $V \subset U$ of e such that conjugation by g maps it diffeomorphically onto an open neighborhood $V' \subset U$ of e. Of course, conjugation by g^{-1} takes V' to V, establishing that $g^{-1} \in N$.

Now suppose that $g_1, g_2 \in N$. For $i = 1, 2$, let $V_i \subset U$ be an open neighborhood of the identity with the property that $V_i' = g_i V_i g_i^{-1}$ is also an open neighborhood of e in U. Then $V_1 \cap V_2'$ is an open neighborhood of the identity in U and $g_2^{-1}(V_1 \cap V_2')g_2 \subset V_2$ is an open neighborhood of the identity in U. Analogously,

$$(g_1 g_2)[g_2^{-1}(V_1 \cap V_2')g_2](g_2^{-1}g_1^{-1}) = g_1(V_1 \cap V_2')g_1^{-1} \subset V_1'$$

is also a neighborhood of the identity in U, proving that $g_1 g_2 \in N$. $\square$

There is an open neighborhood $V \subset U$ of e such that $V \times V \subset \Omega$, and, there is an open neighborhood $W \subset V$ of e such that $m(W, W) \subset V$. Now we define a topology on N that makes it a smooth manifold of the dimension of U. Namely, for any $g \in N$ we define gW to be an open neighborhood of $g \in N$ with the topology and smooth structure it inherits from $W \subset U$ translated by left multiplication by g.

To show that these choices define a topology and a smooth manifold structure on N we need only show that on two-fold overlaps the smooth structures are compatible, meaning that the overlap function from one neighborhood to the other is a diffeomorphism. So let gW and $g'W$ be two smooth patches with $gW \cap g'W \neq \emptyset$. Take a point x in the intersection. Then there are $w, w' \in W$ with $x = gw = g'w'$. It follows that $g^{-1}g' = w(w')^{-1}$, and hence $g^{-1}g' \in W^2 \subset V$. This means that multiplication by $g^{-1}g' \colon W \to W$ is a multiplication in the local Lie group, and hence multiplication by $g^{-1}g'$ is a smooth map from $W \to U$. This smooth map carries the open subset $(g')^{-1}(g'W \cap gW) \subset W$ to the open subset $g^{-1}(gW \cap g'W) \subset W$ and is exactly the overlap transformation in one direction. The symmetric argument show that the inverse overlap function is also smooth. Since these maps are inverses of each other, each is a diffeomorphism.

This completes the proof that we have defined a smooth manifold structure on N. It has the property that the restriction of this smooth structure to $U \subset N$ agrees with the smooth structure U already has. Thus, U is a neighborhood of e in N. Notice that the inclusion map $N \to G$ is smooth immersion and is one-to-one. It follows that since G is a Hausdorff space, so is N.

Next we show that with this smooth structure the group multiplication on N inherited from the G-multiplication is smooth. We fix $g_1, g_2 \in N$ and consider the product map $g_1W \times g_2W \to N$. By restricting to a smaller neighborhood of the identity $W' \subset W$ we can suppose that the image of multiplication $g_1W' \times g_2W'$ lies in g_1g_2W. The map is given by

$$(g_1w)(g_2w') = (g_1g_2)(g_2^{-1}wg_2))w'.$$

Since $g_2 \in N$, if we restrict to a sufficiently small neighborhood T of e in W conjugation by g_2^{-1} sends T diffeomorphically onto $T' \subset W$. Since the product $W \times W \to U$ is smooth, it follows that

$$(g_1w)g_2(w') \mapsto g_1g_2(g_2^{-1}wg_2)w'$$

is a smooth map in some neighborhood of (e, e), and consequently that multiplication on N is smooth.

Lastly, we show that $g \mapsto g^{-1}$ is a smooth map $N \to N$. We fix $g \in N$ and consider the inverse map from gW to N. The map sends gw to $g^{-1}gw^{-1}g^{-1}$. As before, since $g \in N$, restricting w to lie in a smaller neighborhood of e in W conjugation of g sends that neighborhood $V \subset W$ of e diffeomorphically onto another neighborhood of e in W. Then since the inverse in W is given by θ, it is also smooth, showing that $w \to gw^{-1}g^{-1}$ is a smooth map of V to N. By the definition of the topology on N, this means that the map $gw \to g^{-1}(gwg^{-1})$ is a smooth map of $gV \to N$. This completes the proof of the first statement of the theorem.

The second statement of the theorem is clear. □

While N may not be second countable, if U is second countable, then the subgroup of N generated by U is second countable. In particular, the connected component of the identity $N_0 \subset N$ is a second countable Lie group.

Remark 4.4. We shall give a sketch of a proof in a later in the course that every local Lie group is a sub local Lie group of a Lie group. As a consequence, the above theorem holds for all local Lie groups, meaning that the germ of every local Lie group is the germ of a simply connected Lie group.

Example. Let G be a Lie group and $U = \{e\}$. It is a sub local Lie group. Then $N = G$ and the topology on N is the discrete topology. The immersion $N \to G$ is the identity map, which is a surjective, one-to-one immersion, but far from a diffeomorphism. Any time the local Lie group has a positive dimensional normalizer in G, then the Lie algebra N will have uncountably many components.

5 From Lie Algebras to Local Lie Groups

The purpose of the rest of this lecture is to state a theorem defining a map from Lie algebras to local Lie groups. This theorem will be proved in the following lectures.

5.1 The Baker-Campbell-Hausdorff Formula

We have shown that for any Lie group G there is a local Lie group that is a neighborhood of the identity in G and whose underlying submanifold U is the diffeomorphic image of an open subset in the Lie algebra under the exponential mapping. The question naturally arises as to whether the multiplication in a local Lie group that is a sufficiently small neighborhood of the identity in G is determined by the Lie bracket (and the linear structure) on the Lie algebra. The answer is 'yes,' and in fact the multiplication for the local Lie group structure is given by the *Baker-Campbel-Hausdorff formula.*

One way to view the question is to consider two elements e^A and e^B in G for $A, B \in \mathfrak{g}$ sufficiently close to zero. The goal is to write the product $e^A e^B$ as $e^{H(A,B)}$ where $H(A,B)$ is a convergent power series (with some positive radius of convergence) whose n^{th} order terms are universal linear combinations of all possible brackets of A and B of order n, that is to say linear combinations of brackets of n terms each of which is either A or B.

Let us examine the first two terms in the case of $GL_n(\mathbb{R})$ to see how this would work. We write

$$\begin{aligned} e^A e^B = \sum_{n,m} \frac{A^n B^m}{n!m!} &= 1 + (A+B) + (A^2/2 + AB + B^2/2) \\ &+ (A^3/6 + A^2B/2 + AB^2/2 + B^3/6) + \cdots . \end{aligned}$$

Thus, the power series for $H(A,B)$ begins

$$H(A,B) = (A+B) + \cdots .$$

Let us compute the quadratic term $Q(A,B)$ in $H(A,B)$. It must satisfy the equation

$$A^2/2 + AB + B^2/2 = (A+B)^2/2 + Q(A,B).$$

Thus,

$$Q(A,B) = AB - (AB+BA)/2 = (AB-BA)/2 = \frac{1}{2}[A,B].$$

A homework problem is to evaluate the cubic and quartic terms in a similar manner.

Theorem 5.1. *(Baker-Campbell-Hausdorff Formula) Let L be the free Lie algebra generated by X and Y. There is a formal infinite sum $H(X,Y)$ in two variables where the n^{th} term in the sum is a linear combination with rational coefficients of the Lie brackets of order n of X and Y*

$$[Z_1, [Z_2, \cdots, [Z_{n-1}, Z_n]] \cdots]$$

where the Z_i range over X and Y, such that there is an equality of formal power series

$$\log(\exp(X)\exp(Y)) = H(X,Y).$$

For any finite dimensional real Lie algebra L, fixing a positive definite inner product $\langle\cdot,\cdot\rangle$ on L with associated norm $|\cdot|$, there is $r > 0$ so that defining $U \subset L$ by $U = \{X \in L \,|\, |X| < r\}$, the power series $H(A,B)$ converges absolutely for $(A,B) \in U \times U$ and defines an analytic function $H\colon U \times U \to L$. The open set U is invariant under $X \mapsto -X$. Let $\Omega \subset U \times U$ be $H^{-1}(U) \cap (U \times U)$. Defining $\theta(A) = -A$ and $m(A,B) = H(A,B)$, makes $(U, 0, \theta, \Omega, m)$ is a local Lie group. If $L = \mathfrak{g}$ for a Lie group G, and possibly replacing r by r' with $0 < r' < r$ so that $\exp|_U$ is a diffeomorphism onto an open subset of G, the restriction of the exponential mapping to U defines an embedding of $(U, 0, \theta, \Omega, m)$ onto a local Lie sub group of G that is a neighborhood of the identity.

We call any such local Lie group defined on an open set $U \subset L$ of 0, invariant under $X \mapsto -X$, sufficiently small so that the BCH series converges and *a local Lie group determined by the Lie algebra L*. Any two such have the same germ.

There is an explicit formula due to Hausdorff. But the actually coefficients are not important. The only important thing is that such a series exists and has a positive radius of convergence. We shall give the proof in the next lecture. It uses the Poincaré-Birkhoff-Witt Theorem. The convergence is a direct computation that we leave to the exercises.