Lie Groups: Fall, 2024
Lecture XI:
Linear Representations of Compact Lie Groups

December 2, 2024

1 The basics

We fix a compact, connected Lie group G. A finite-dimensional real representation of G is a finite dimensional real vector space V and a linear action $\rho\colon G \times V \to V$. Implicitly that ρ a smooth map and for every $g \in G$ the map $\rho(g)\colon V \to V$ is a linear isomorphism.

Here is the first basic result.

Theorem 1.1. *Fix a finite-dimensional real linear representation $\rho\colon G \times V \to V$. If $W \subset V$ is a linear subspace stabilized by the action, then is another subspace $W' \subset V$, stabilized by the G-action such that $V = W \oplus W'$. Thus, the G action on V is the direct sum of the G actions on W and W'.*

Proof. The left-invariant vector fields on G determine a bundle isomorphism $TG \equiv G \times \mathfrak{g}$. This isomorphism is left-invariant in the sense that under the isomorphism $d(g\cdot)\colon TG \to TG$ sends $(h,t) \mapsto (gh,t)$. Fix a positive definite, symmetric inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Using the left-invariant trivialization, we extend this inner product to a left-invariant Riemannian metric on G, left-invariant in the sense that $\langle\tau_1,\tau_2\rangle = \langle g\tau_1, g\tau_2\rangle$ for all $g, h \in G$ and all $\tau_1, \tau_2 \in T_hG$.

Choose an orientation for $\mathfrak{g}$ and hence a left-invariant orientation for TG. Let n be the dimension of G. There is a left-invariant, nowhere vanishing n-form ω on G, left-invariant in the sense that have $(g\cdot)^*(\omega) = \omega$, whose value on a $\tau_1 \wedge \cdots \wedge \tau_n$ is the signed n-dimensional volume of the parallepiped spanned by $(\tau_1, \ldots, \tau_n)$ as measured using the Riemannian metric (to give the absolute value) and the orientation (to give the sign). We define the measure of any open subset of G to be

$$\int_U \omega,$$

when U is given the induced orientation. This leads to a Borel measure on G, denoted $d\mu_G$, that is invariant under the left action of G on itself, meaning that for any open subset U of G the measure of U and gU are equal. This differential form and measure are unique up to multiplication by a positive scalar. It follows that for any measurable function f and any $g \in G$ we have

$$\int_G g^* f(h) d\mu_G = \int_G f(h) d\mu_G,$$

where by definition $g^* f(h) = f(g^{-1}h)$.

Denote by $R_g \colon G \to G$ the map sending $h \mapsto hg$. Since right and left multiplication of G on itself commute, $R_g^* \omega$ is a left-invariant and hence is a positive multiple of ω. But $\int_G R_h^* \omega = \int_G \omega$ since $R_h \colon G \to G$ is an orientation-preserving diffeomorphism. Thus, $R_h^* h\omega = \omega$ so that ω an the measure are also right-invariant.

Now let $\rho \colon G \times V \to V$ be a finite dimensional linear representation. Fix a positive definite inner product $\langle \cdot, \cdot \rangle$ on V.

Define a new inner product

$$\langle \cdot, \cdot \rangle' = \int_G \langle hv, hv \rangle d\mu_G.$$

Clearly, we have $\langle v, v \rangle' = \langle gv, gv \rangle'$ for any $g \in G$ so that the inner product is G-invariant. Being an average of positive definite inner products, it is positive definite. Using this inner product we set $W' = W^{\perp}$. Since W is stabilized by G and the inner product is stabilized by G, the subspace W' is also stabilized by G. Since the inner product is positive definite, $V = W \oplus W'$. □

Remark 1.2. On a topological group, a left-invariant, right-invariant, or bi-invariant measure that takes finite values on compact subsets is called a *left-invariant, right-invariant, or bi-invariant Haar measure.* We have just seen that any compact Lie group has a unique left-invariant Haar measure up to multiplication by a positive scalar. Furthermore, any such left-invariant Haar measure bi-invariant. The last statement holds for any compact topological group.

Definition 1.3. A finite dimensional representation is *irreducible* if the only subspaces stabilized by the action are $\{0\}$ and the entire space. A finite dimensional representation is *completely reducible* if it is a direct sum of irreducible representations.

Corollary 1.4. *Let G be a compact Lie group. Every finite-dimensional real representation of G is a direct sum of irreducible representations, i.e., every finite dimensional representation of G is completely reducible. Likewise, any finite dimensional complex representation of G is completely reducible as a complex representation*

Proof. Any one-dimensional representation is irreducible. Now we induct on the dimension of the representation. Suppose that we have an n-dimensional representation and the result is know for representations of dimension $< n$. If the n-dimensional representation is not irreducible, then it has a non-trivial invariant subspace. By Theorem 1.1 this leads to a non-trivial direct sum decomposition of the representation. Each of the summands has dimension less than n and hence is a direct sum of irreducible representations. It follows that the n-dimensional representation is completely reducible.

For a finite dimensional complex representation of G, by an analogous argument we find a G-invariant hermitian metric and use it to find a G-invariant hermitian orthogonal complement to a G-invariant complex subspace. With this, the argument above proves that the finite dimensional complex representation decomposes as a direct sum of complex representations, each of which has no non-trivial G-invariant complex subspace. □