HEEGAARD SPLITTINGS OF PRIME 3-MANIFOLDS ARE NOT UNIQUE

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1. INTRODUCTION

Our paper demonstrates that the topology of 3-manifolds as related to Heegaard splittings is considerably more complex than previous positive results of Singer [14], Reidemeister [10], and Waldhausen [17] had indicated it to be.

A closed, orientable 3-manifold admits a Heegaard splitting of genus g if $M = X_g \cup X_g'$, where X_g and X_g' are each handlebodies of genus g, and

$$X_g \cap X_g' = \partial X_g = \partial X_g'.$$

Two such Heegaard splittings $M = X_g \cup X_g' = Y_g \cup Y_g'$ are equivalent if there is a homeomorphism $h: M \to M$ such that $h(X_g) = Y_g$ or Y_g', and $h(X_g') = Y_g'$ or Y_g. The Heegaard genus of M is the smallest integer g such that M admits such a representation. The purpose of this note is to exhibit an infinite family of prime 3-manifolds, each being a homology sphere of Heegaard genus 2, each of which exhibits at least two equivalence classes of Heegaard splittings of genus 2. Our manifolds also exhibit a second "bad" property: each may be represented as the 2-fold covering space of S^3 branched over a knot K_β, and also over a second knot K_β', where K_β and K_β' are inequivalent knot types. The nature of the examples leads us to conjecture that this phenomenon is not an isolated one, and probably happens often for 3-manifolds which are sufficiently complicated in structure.

We review briefly the historical background which motivated the question studied here. In 1933 Reidemeister [10] and Singer [14] simultaneously published proofs that all Heegaard splittings of a closed orientable 3-manifold are *stably* equivalent. (For a definition of stable equivalence, see [17].) The question of interest to us is whether the adjective "stably" can be dropped. No progress was made at all until 1968, when Waldhausen improved the Reidemeister-Singer theorem for the special cases of $M_1 = S^3$ and $M_2 = \#_n S^2 \times S^1$, establishing (in [17]) that any two Heegaard splittings of the same genus of S^3, or of $\#_n S^2 \times S^1$, are equivalent. (Note that Waldhausen's definition of equivalence of Heegaard splittings, as given in [17], is slightly stronger than ours, since he requires that h be isotopic to the identity map.) In the opposite direction, Engmann [4] found examples of connected sums of lens spaces which exhibit inequivalent Heegaard splittings (see also [1]), but the situation for prime 3-manifolds remained open. That latter question is laid to rest by our examples. The question of whether Waldhausen's positive results generalize to special classes of closed, orientable 3-manifolds other than S^3 or $\#_n S^2 \times S^1$ remains an important

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one, because for any such manifold, invariants of equivalence classes of Heegaard splittings (see, for example, [1]) would necessarily be topological invariants.

The related question of the correspondence between knot type and topological type of the associated 2-fold branched covering spaces has a similar history. If K is a tame knot in S^3, then $\pi_1(S^3 - K)$ has a unique representation onto $\mathbb{Z}_2$, which in turn defines a unique closed orientable 3-manifold $\tilde{K}$, the 2-fold covering space of S^3 branched over K. Waldhausen's positive results in [18] imply that if $\tilde{K}$ is S^3 then K is the trivial knot, while if $\tilde{K}$ is $\#^n S^2 \times S^1$, then K is the trivial link of $n + 1$ components. Again in the positive direction, Birman and Hilden proved that for the special case $g = 2$, each equivalence class of Heegaard splittings of $\tilde{K}$ determines a unique knot type K (Theorem 8 of [3]); this result will play a crucial role in our construction below. Weaker positive results of Birman and Hilden [2, 3] and of Viro [15] relate the Heegaard genus of $\tilde{K}$ to the bridge index of K. In the negative direction, however, examples had been found simultaneously by Montesinos [7, p. 113] and by Viro (Sections 3, 7 of [15]) to show that if K is allowed to be a composite link, then the topological type of $\tilde{K}$ does *not* determine the link type of K uniquely. Our present examples remove the restriction "composite" and "link" from Montesinos' and Viro's results. However, the question of whether there are special classes of manifolds $\tilde{K}$ for which the correspondence between K and $\tilde{K}$ is a bijection remains of considerable interest, because of the potential connections between invariants of K and invariants of $\tilde{K}$. We remark that, by our examples given in Section 2 below, the genus of K is not, in general, an invariant of $\tilde{K}$.

2. THE CONSTRUCTION

Let β be an integer different from 0, -1, and let K_β denote the torus knot of type (3, q), where $q = |6\beta + 1|$. Let K'_β denote the knot which is illustrated in Figure 1(a) if $\beta > 0$. The portion of Figure 1 which is included in the brackets consists of β "half-twists". If $\beta < -1$ the knot diagram will be identical except that these particular crossings should be reversed.

THEOREM. (i) *The knots* K_β *and* K'_β *represent inequivalent knot types.*

(ii) *The 2-fold covering space* $\tilde{K}_\beta$ *of* S^3 *branched over* K_β *is homeomorphic to the 2-fold covering space* $\tilde{K}'_\beta$ *of* S^3 *branched over* K'_β.

(iii) *The manifold* $\tilde{K}_\beta$ *is prime.*

(iv) *The manifold* $\tilde{K}_\beta$ *has Heegaard genus* 2.

(v) *The manifold* $\tilde{K}_\beta$ *admits at least two equivalence classes of genus* 2 *Heegaard splittings.*

Proof. To see that K_β and K'_β are inequivalent knot types, note first that the genus of a torus knot of type (p, q) is $(p - 1)(q - 1)/2$ (see, for example, Theorem 1 of [9]). Thus the genus g of K_β is $|6\beta + 1| - 1$, which is greater than 5, since $\beta \neq 0$ or -1. (It is easy to see that K_{-1} and K'_{-1} are equivalent knot types.) We now show that the genus g' of K'_β is at most 5, which implies that K_β and K'_β are inequivalent knot types. If we span an orientable surface in K'_β, using the projection of Figure 1(a), by the method described in [13], or on page 140 of [5], we find that the number f of Seifert circles is $7 + |\beta|$ if β is even and $3 + |\beta|$ if β is odd. The genus g of the orientable surface is $(d - f + 1)/2$, where $d = 12 + \beta$ is the number of crossings. Thus $g' \leq 3$ if β is even and $g' \leq 5$ if β is odd, establishing (i).

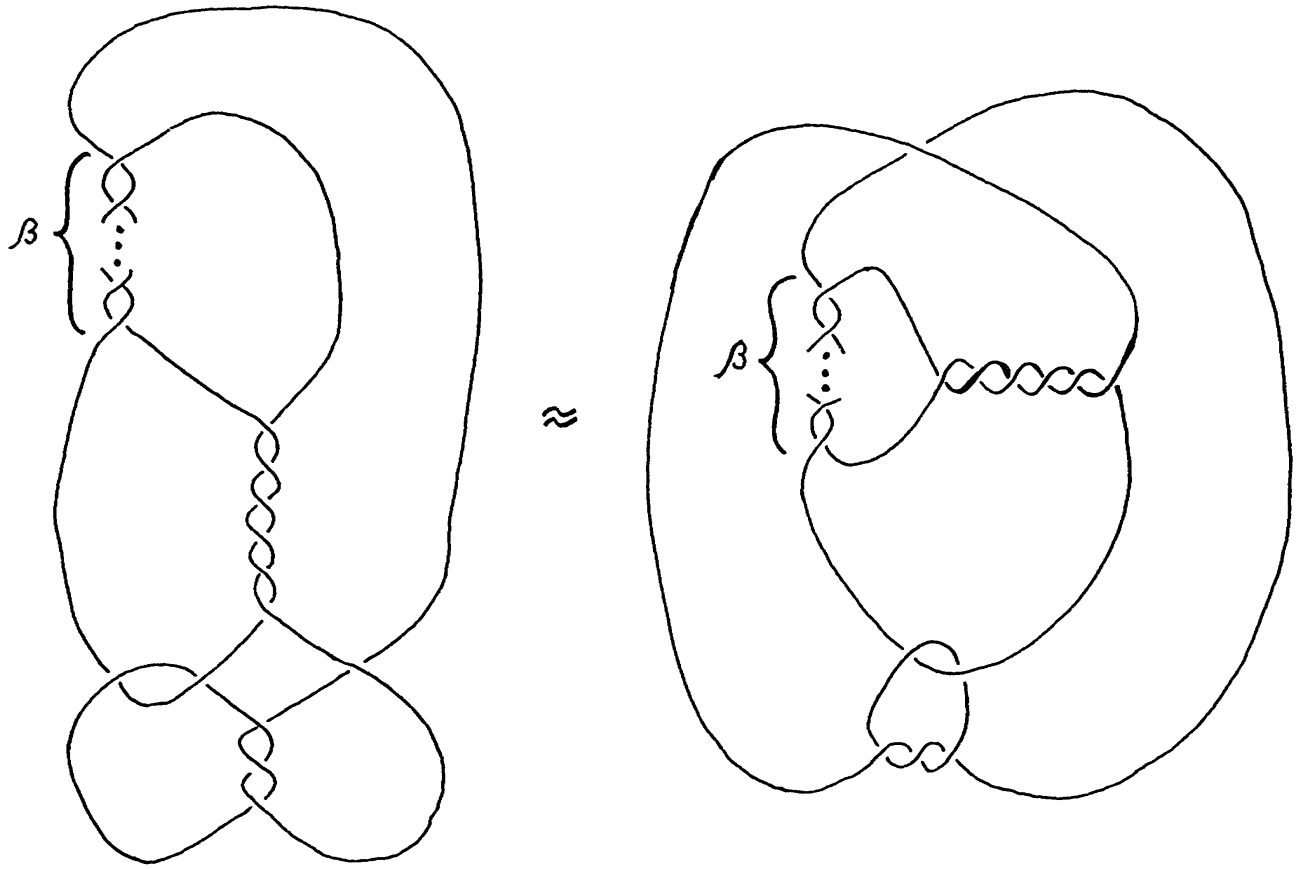


Figure 1.

To establish (iii), we note first that if K is a torus knot of type (p, q) with p and q both odd (and relatively prime), then $\tilde{K}$ is the orientable Seifert fiber space with base S^2 and three exceptional fibers of multiplicities 2, p, q and which is a homology sphere (Zusatz zu Satz 17 of [12]). In the case $p = 3$ and $q = |6\beta + 1|$ this manifold may be described by the symbol $(0 \circ 0 \mid -1; (2, 1), (3, 1), (|6\beta + 1|, |\beta|))$ in the notation of [12]. By Theorem 7.1 and Lemma 10.2 of [16], it then follows that $\tilde{K}$ is a prime 3-manifold.

To prove (ii), we note that $\tilde{K}'_\beta$ belongs to a class of manifolds studied by Montesinos in [8]. Using the projection of Figure 1(b), we may recognize K'_β as the knot defined by the schematic diagram on page 6 of [8], with $(1, b) = (1, -1)$, $(\alpha_1, \beta_1) = (2, 1)$, $(\alpha_2, \beta_2) = (3, 1)$, and $(\alpha_3, \beta_3) = (|6\beta + 1|, |\beta|)$. It then follows from the theorem in Section 2 of [8] that $\tilde{K}'_\beta$ is also the Seifert fiber space $(0 \circ 0 \mid -1; (2, 1), (3, 1), (|6\beta + 1|, |\beta|))$, which establishes part (ii) of our theorem.

To prove (iv), note that the torus knot of type (3, q) has bridge index 3 (Theorem 10 of [11]); hence by Lemma 2 of [3] it has plat index 6. Therefore, by Theorem 5 of [3], $\tilde{K}_\beta$ has Heegaard genus ≤ 2. Since $\tilde{K}_\beta$ has a noncyclic fundamental group [12], its Heegaard genus cannot be less than 2, which establishes (iv).

Finally, we prove (v). Note that the algorithm given in Section 5 of [3] allows one to find in a natural way a Heegaard splitting of genus g - 1 for the 2-fold covering space $\tilde{K}$ of a knot K, whenever K is presented as a 2g-plat. Thus we may find a Heegaard splitting of genus 2 for $\tilde{K}_\beta$ from any 6-plat presentation of K_β,

using that algorithm. A second Heegaard splitting of genus 2 may be found for $\tilde{K}_\beta = \tilde{K}'_\beta$ from the presentation of K'_β in Figure 1(a), which is easily altered to a 6-plat presentation. By Theorem 8 of [3], these Heegaard splittings are equivalent only if K_β and K'_β are equivalent knot types. Since by (i) above the knots K_β and K'_β are *inequivalent*, assertion (v) is established, and the proof of the theorem is complete.

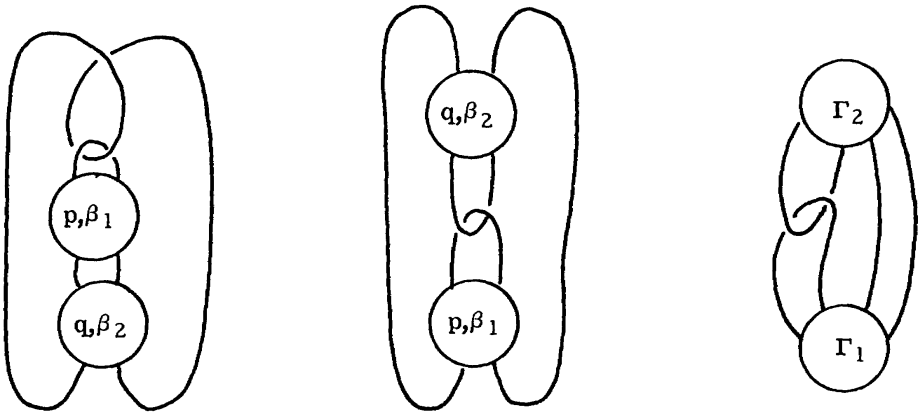
One may generalize (i), (ii), (iii), (iv) and show that the homology sphere $\Sigma_{p,q}$, which has a Seifert fibration with exceptional fibers of multiplicities 2, p, q, is a 2-fold covering of S^3 branched over the torus knot of type (p, q) and also over a second knot with 3 bridges. We justify this assuming, as we may, that $3 < p < q$, since the case $p = 3$ was treated earlier.

Let b, β_1, β_2 be integers such that

$$2bpq + pq + 2\beta_1 q + 2\beta_2 p = \pm 1 \qquad (0 < \beta_1 < p,\ 0 < \beta_2 < q).$$

Necessarily $b = -1$ or -2. If we replace b by $-3-b$, β_1 by $p-\beta_1$, and β_2 by $q-\beta_2$, the above equality still holds, so that we may assume $b = -1$. Then $\Sigma_{p,q}$ is the manifold $(0 \circ 0 \mid -1; (2, 1), (p, \beta_1), (q, \beta_2))$ in Seifert's notation (see page 208 of [12]).

By Section 2 of [8] this manifold is the 2-fold covering of S^3 branched over the knot $K_{p,q}$ of Figure 2(a). (The portion of the diagram which is inside the circles will be given explicitly below.) That same knot is represented in Figures 2(b) and 2(c), where the portion labeled Γ_1 (respectively, Γ_2) in Figure 2(c) is obtained from the portion labeled (p, β_1) (respectively, (q, β_2)) in Figure 2(b) by a 90° counterclockwise (respectively, clockwise) rotation about an axis perpendicular to the plane of projection.



(a) (b) (c)

Figure 2.

We examine next the details of the schematic of Figure 2(c). A circle labeled "m_i" or "$-m_i$" will be used to denote a portion of the knot diagram which contains a 2-braid with $+m_i$ or $-m_i$ crossings, in the manner indicated in Figure 3. Note that there are two possible orientations of the braid, depending on whether the symbol m_i is printed horizontally, or rotated through 90°. Details of the portion of the knot diagram which is labeled Γ_1 in Figure 2(c) are given in Figure 4. The integers $m_1, \cdots, m_r$ are obtained from a continued fraction expansion of p/β_1:

$$p/\beta_1 = m_1 + \frac{1}{m_2 + \cdots + \frac{1}{m_r}}.$$

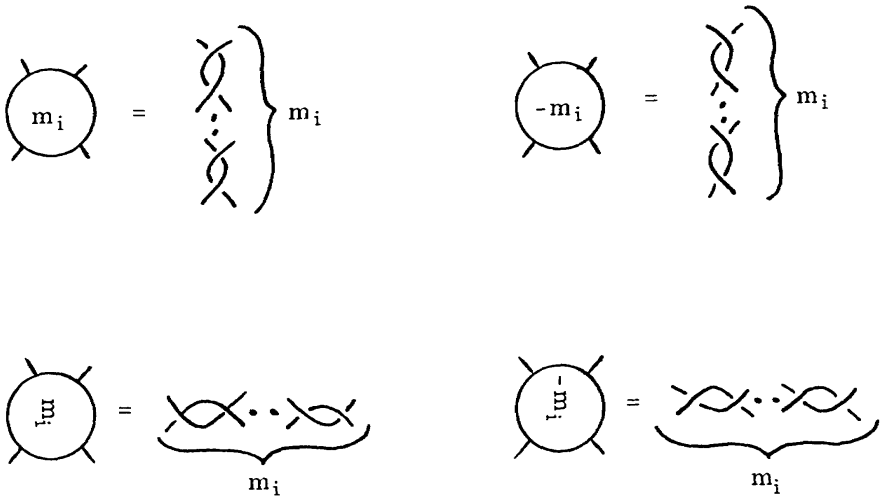


Figure 3.

The schematic is slightly different in the cases $r = 2s$ and $r = 2s - 1$, as indicated in Figure 4. One may now deform Figure 4 and put the horizontal 2-braids into a vertical position, as indicated in Figure 5. The 4-braid which is associated with the diagram in Figure 5 is:

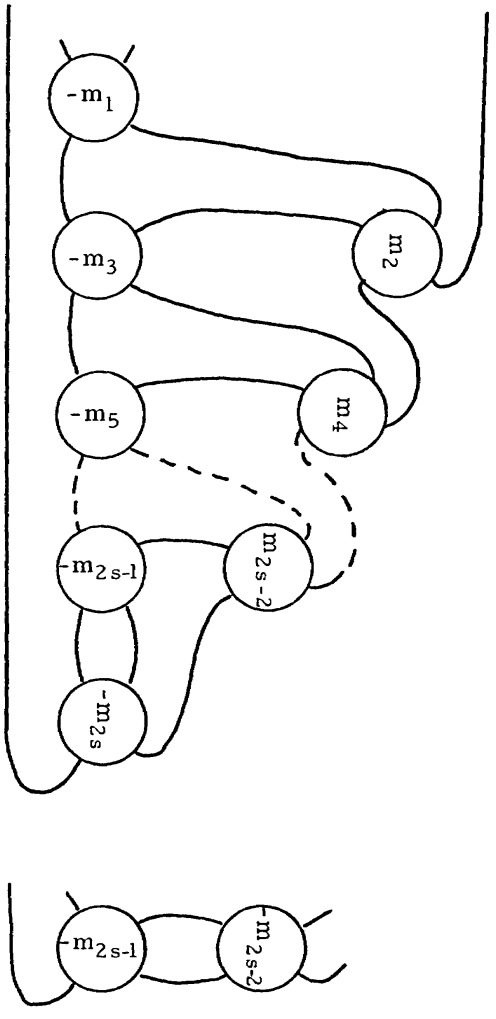


Figure 4.

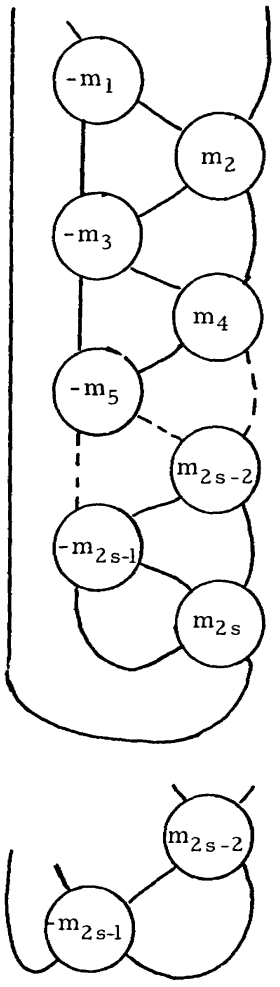


Figure 5.

$$\sigma_3^{m_r}\,\sigma_2^{-m_{r-1}}\cdots\sigma_3^{m_2}\,\sigma_2^{-m_1} \qquad \text{if } r \text{ is even,}$$

$$\sigma_2^{-m_r}\,\sigma_3^{m_{r-1}}\cdots\sigma_3^{m_2}\,\sigma_2^{-m_1} \qquad \text{if } r \text{ is odd.}$$

This projection has precisely 2 local minima and no local maxima. Similarly, one shows that the part of the knot containing Γ_2 can be deformed so that it has precisely 2 local maxima and no local minima. It follows that $K_{p,q}$ has a projection with exactly 3 local maxima, so that $K_{p,q}$ has bridge index ≤ 3 [6].

The manifold $\Sigma_{p,q}$ is also a 2-fold covering of S^3 branched over the torus knot of type (p, q) (see [12, p. 222]). However, this knot has bridge index p (Theorem 10 of [11]), which we are assuming is greater than 3, and it is therefore not equivalent to $K_{p,q}$.

Since $K_{p,q}$ has bridge index ≤ 3, by [3] it follows that, as in the case $p = 3$, the manifold $\Sigma_{p,q}$ has Heegaard genus 2.

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