Knots, graphs and surfaces

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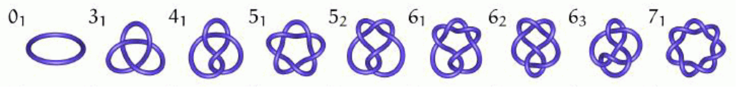
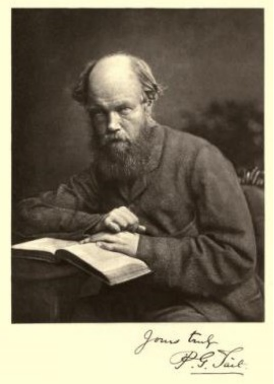
May 30, 2012

Early knot theory

Modern knot theory began in late 1800's when Tait, Little and others tried to make a periodic table of elements by tabulating knot diagrams by crossing number:





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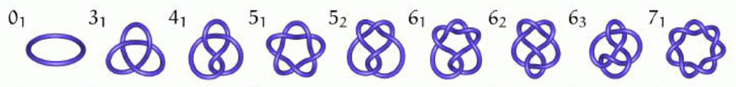


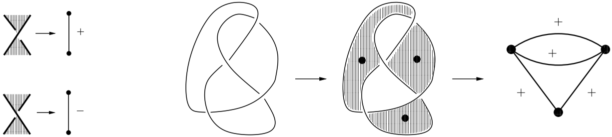
The only invariants at this time were of the form, "minimize something among all diagrams," such as crossing number, unknotting number, bridge number, etc.

Such invariants are easy to define but hard to compute: Diagrams that are minimal with respect to one property may not be minimal with respect to other properties.

Tait graph

By Jordan Curve Theorem, any link diagram can be checkerboard colored.

Thus, any link diagram corresponds to a planar graph with signed edges:



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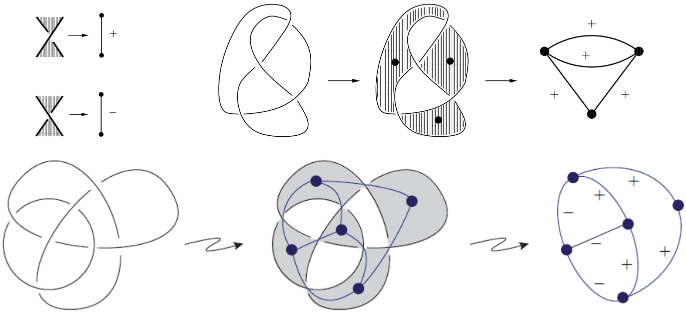

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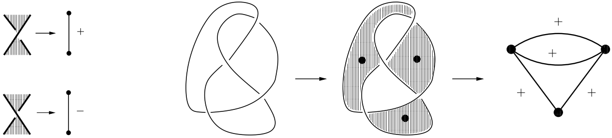

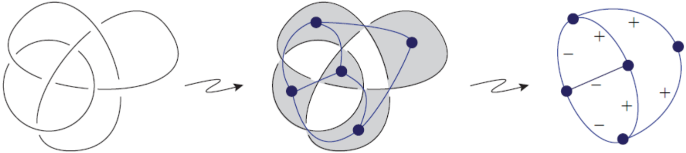

Conversely, can recover the diagram from any signed planar graph by taking its medial graph, and making crossings according to the sign on each edge:

Tait graphs for opposite checkerboard colorings are planar duals.

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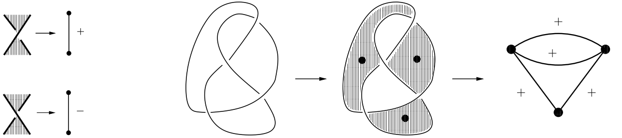

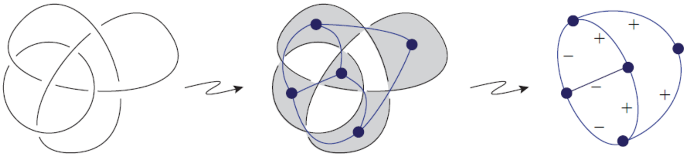

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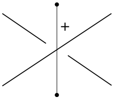

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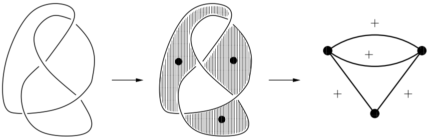
Tait emphasized the importance of alternating diagrams, for which the Tait graphs have one sign (i.e. any unsigned planar graph).



Conjecture (Tait) A reduced alternating diagram has minimal crossing number among all diagrams for that link.

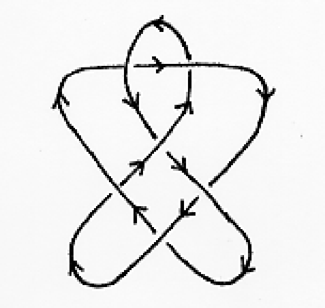
A proof had to wait about 100 years until the Jones polynomial (1984), which led to several new ideas that were used to prove Tait's conjecture.

Aside: Seifert surface

A turning point in knot theory was the discovery of the Alexander polynomial (1920's), and its reinterpretation by Seifert (1930's).

Here is Seifert's algorithm to construct an orientable spanning surface for any knot diagram. (The checkerboard surface is a spanning surface that may not be orientable.)

1. Given a knot diagram, choose an orientation:



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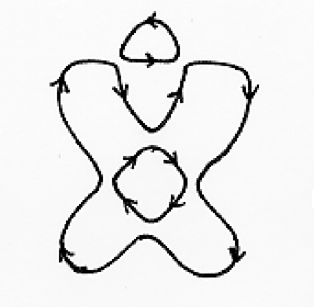
2. Splice the diagram according to the orientation:



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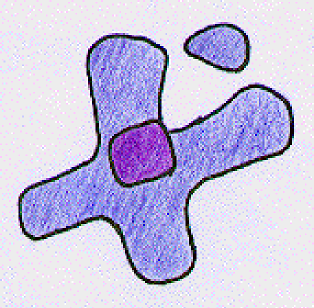
3. Put discs at different heights:



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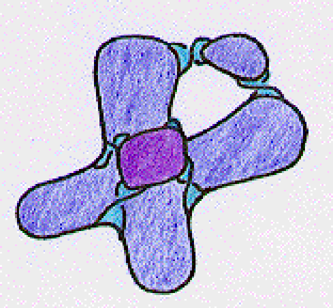
4. Connect discs with bands according to original crossings:



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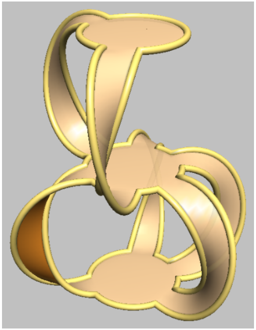

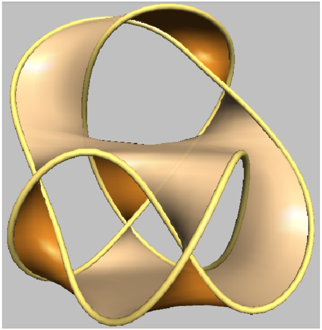

Aside: Seifert surface

The minimum genus of all Seifert surfaces for a given knot K is called the genus of K, $g(K)$.

For any alternating diagram, Seifert's algorithm produces the minimal genus Seifert surface.

Unusual property: the genus of a knot can detect Conway mutation.

From the Seifert surface, can construct the Seifert matrix to get other important invariants: determinant, signature, Alexander polynomial.

Back to our story... the Jones polynomial

In 1984, V. Jones discovered $V_L(t) \in \mathbb{Z}[t, t^{-1}]$ by studying representations of the braid group, $B_n \to A_n(t)$, with $tr : A_n(t) \to \mathbb{C}[t, t^{-1}]$.

$V_L(t)$ satisfies a skein relation:

$t^{-1}\, V$  $(t) - t\, V$  $(t) = \left(\sqrt{t} - 1/\sqrt{t}\right) V$  (t)

First polynomial link invariant to distinguish trefoils:

V  $(t) = t + t^3 - t^4$

V  $(t) = -t^{-4} + t^{-3} + t^{-1}$

Still open problem: If $V_K(t) = 1$, is $K = \bigcirc$?

Kauffman bracket polynomial

Simplest combinatorial approach to the Jones polynomial:

Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ defined recursively by

1. $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$
2. $\langle \bigcirc \; D \rangle = \delta \langle D \rangle, \quad \delta = -A^{-2} - A^{2}$
3. $\langle \bigcirc \rangle = 1$

When over-strand sweeps counterclockwise ("A–splice"), weight A
When over-strand sweeps clockwise ("B–splice"), weight A^{-1}

Example:

$$\begin{aligned} \langle \; \rangle &= A \langle \; \rangle + A^{-1} \langle \; \rangle \\ &= A \cdot \delta + A^{-1} \cdot 1 = -A^{-1} - A^{3} + A^{-1} \\ &= -A^{3} \end{aligned}$$

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Kauffman states

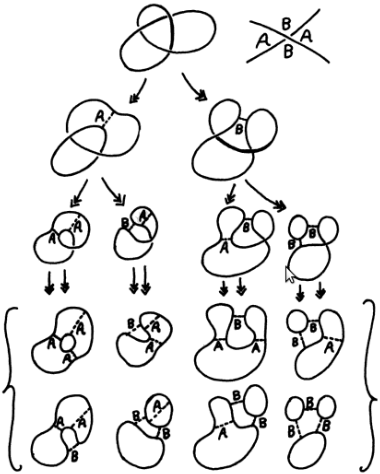
Besides the axiomatic definition, Kauffman expressed $\langle L \rangle$ as a sum of all possible *states* of L:

If L has n crossings, all possible A and B splices yield 2^n states s.

Let $a(s)$ and $b(s)$ be the number of A and B splices, resp., to get s.
Let $|s|$ = number of loops in s.

$$\langle L \rangle = \sum_{\text{states } s} A^{a(s)-b(s)} \, (-A^2 - A^{-2})^{|s|-1}$$

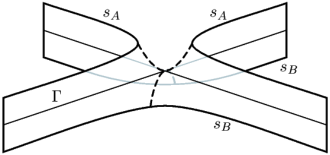
Kauffman states

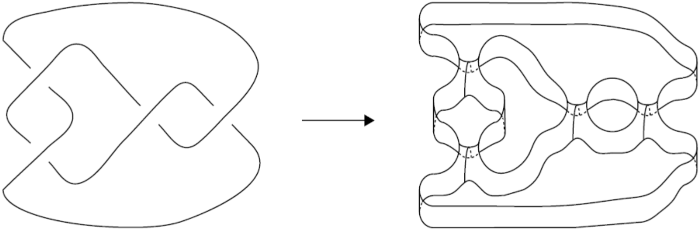


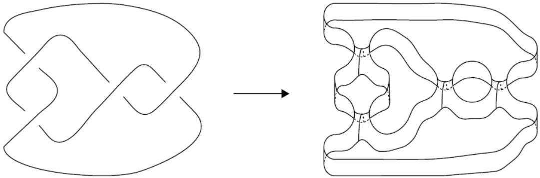
$$\left\{ \begin{array}{c|c|c|c} A^3\delta & A & A & A^{-1}\delta \\ A & A^{-1}\delta & A^{-1}\delta & A^{-3}\delta^2 \end{array} \right\}$$

Turaev surface

Let s_A and s_B be the all–A and all–B states of D.
Turaev constructed a cobordism between s_A and s_B:
Let $\Gamma \subset S^2$ be the 4–valent projection of D at height 0.
Put s_A at height 1, and s_B at height -1, joined by saddles:







Turaev surface $F(D)$: Attach $|s_A| + |s_B|$ discs to all boundary circles.

Turaev genus of D, $g_T(D) := g(F) = (c(D) + 2 - |s_A| - |s_B|)/2$.

Turaev genus of non-split link L, $g_T(L) = \min_D \; g_T(D)$.

Non-split link L is alternating iff $g_T(L) = 0$.

$g_T(L) \leq dalt(L) =$ min number of crossing changes to make L alternating.

Proof of Tait's conjecture

Conjecture (Tait) A reduced alternating diagram D has minimal crossing number among all diagrams for the alternating link L.

The proof follows from two claims:

1. Although defined for diagrams, the Jones polynomial $V_L(t)$ is a link invariant.

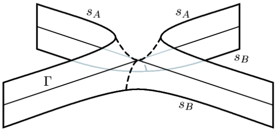
2. s_A and s_B contribute the extreme terms $\pm t^\alpha$ and $\pm t^\beta$ of $V_L(t)$.

$$\max\deg_A\langle D\rangle - \min\deg_A\langle D\rangle \leq 2(c(D)+s_A(D)+s_B(D)-2)$$

with equality if D is alternating (generally, *adequate*).

So for any link ℓ, span $V_\ell(t) = \alpha - \beta \leq c(\ell) - g_T(\ell)$, with equality if ℓ is adequate. Thus,

$$\text{span}\, V_L(t) = c(L) \quad \text{if } L \text{ is alternating.}$$



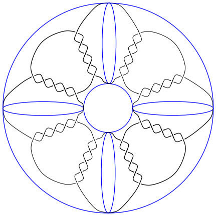
The Turaev surface $F(D)$ can be checkerboard colored with $|s_A|$ white regions (height > 0), and $|s_B|$ black regions (height < 0).

Let $\mathbb{G}_A$, $\mathbb{G}_B \subset F(D)$ be the adjacency graphs for respective regions.

$$v(\mathbb{G}_A) = |s_A|, \quad e(\mathbb{G}_A) = c(D), \quad f(\mathbb{G}_A) = |s_B|$$

If D is alternating, $\mathbb{G}_A$ and $\mathbb{G}_B$ are dual Tait graphs on $F(D) = S^2$.

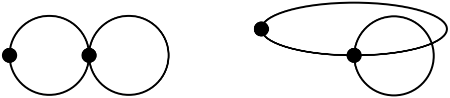
If $g_T(D) > 0$, then D is alternating on $F(D)$, $\mathbb{G}_A$, $\mathbb{G}_B$ are dual graphs embedded in $F(D)$



Ribbon graphs

An (oriented) ribbon graph $\mathbb{G}$ is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface F, such that its complement is a union of 2-cells. The genus $g(\mathbb{G}) := g(F)$.

Example



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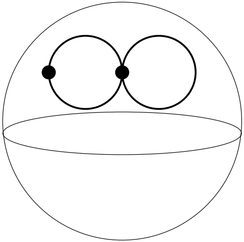

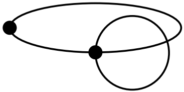

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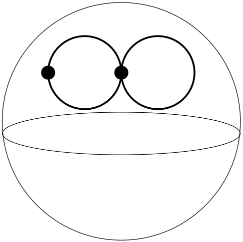

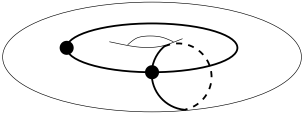

Algebraic definition

$\mathbb{G}$ can also be described by a triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ of the set $\{1, 2, \ldots, 2n\}$ such that

- σ_1 is a fixed-point-free involution.
- $\sigma_0 \circ \sigma_1 \circ \sigma_2 = \mathrm{Identity}$

This triple gives a cell complex structure for the surface of $\mathbb{G}$ such that

- Orbits of σ_0 are vertices.
- Orbits of σ_1 are edges.
- Orbits of σ_2 are faces.

The genus $g(\mathbb{G}) = (2 - v(\mathbb{G}) + e(\mathbb{G}) - f(\mathbb{G}))/2$.

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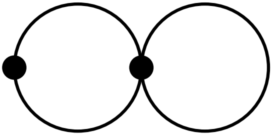

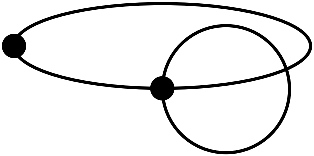

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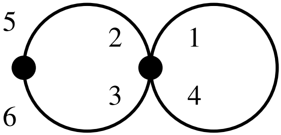

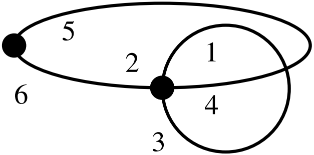

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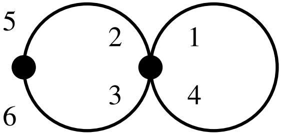

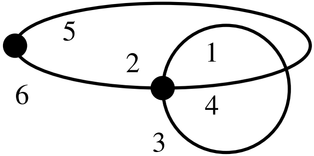

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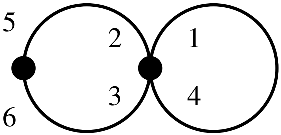

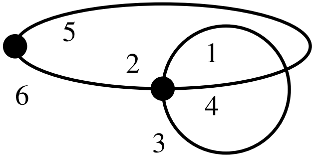

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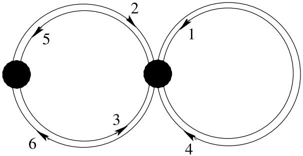
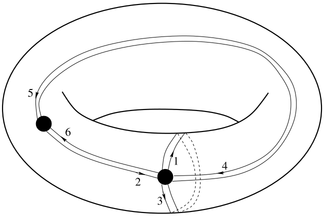


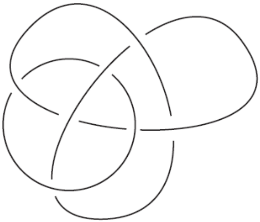


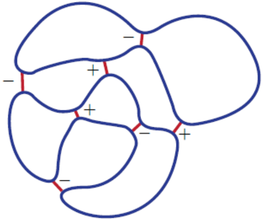
Ribbon graph from any state of a link diagram

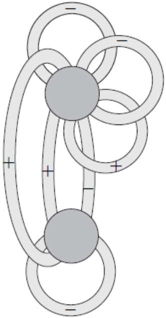
Earlier, defined $\mathbb{G}_A$, $\mathbb{G}_B$ as checkerboard graphs on Turaev surface $F(D)$.

Now, can construct the ribbon graph $\mathbb{G}_s$ directly from any state s of D:

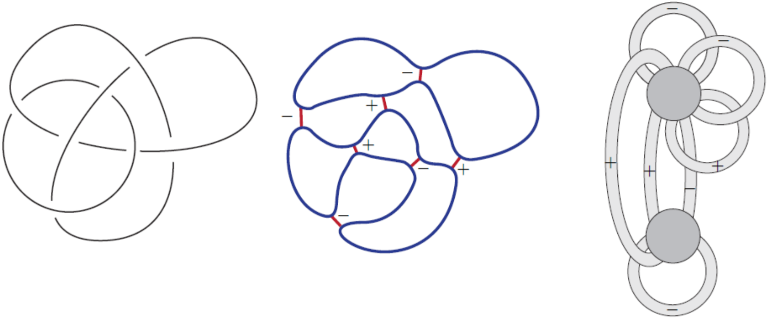
1. For each crossing of D, attach an edge between state circle(s).
2. Collapse each state circle of s to a vertex of $\mathbb{G}_s$.







Ribbon graph from any state of a link diagram



D is called A–adequate (B–adequate) if $\mathbb{G}_A$ ($\mathbb{G}_B$) has no loops.
D is adequate if it is both A–adequate and B–adequate.

Applications to geometry and topology of $S^3 - K$

We highlight some recent results by Futer, Kalfagianni, Purcell.

Main idea: Relate certain stable coefficients of colored Jones polynomials to fibering data and hyperbolic volume bounds using incompressible state surfaces.

$\mathbb{G}_s \subset F_s$, state surface constructed like Seifert surface; $\mathbb{G}_s$ is spine for F_s.
F_s is orientable (i.e. a Seifert surface) iff $\mathbb{G}_s$ is bipartite.

Thm. (Ozawa) F_A is incompressible and ∂–incompressible in $S^3 - L$ iff L is A–adequate. Similarly for F_B.

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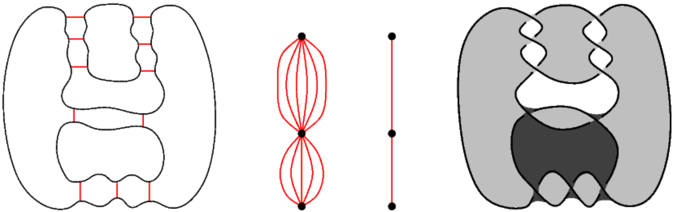
Let $\mathbb{G}'_A = \mathbb{G}_A$ with all duplicate edges removed, similarly for $\mathbb{G}'_B$.



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Thm. If D is an A–adequate diagram of a hyperbolic link K,

$$\mathrm{vol}(S^3 - K) \geq v_8(\chi_-(\mathbb{G}'_A) - E(D))$$

Thm. $S^3 - K$ fibers over S^1 with fiber F_A iff $\mathbb{G}'_A$ is a tree.

If K is A–adequate, let $\beta_K =$ penultimate coefficient of $J^n_K(t)$, which stabilizes for $n > 1$.

Cor. $S^3 - K$ fibers over S^1 with fiber F_A iff $\beta_K = 0$.

Related polynomial invariants

1. (1954) Tutte polynomial for graphs given by spanning tree expansion:

$$T_G(x, y) = \sum_T x^{i(T)} y^{j(T)}$$

where $i(T)$ is the number of internally active edges and $j(T)$ is the number of externally active edges of G for a given spanning tree T.

2. (1987) Applying Tutte's results, Thistlethwaite defined a spanning tree expansion for the Jones polynomial of links. If L is alternating, $V_L(t) \doteq T_G(-t, -1/t)$, where G is the Tait graph of L.

3. (2001) Bollobás and Riordan extended the Tutte polynomial to an invariant of oriented ribbon graphs, Bollobás–Riordan–Tutte polynomial.

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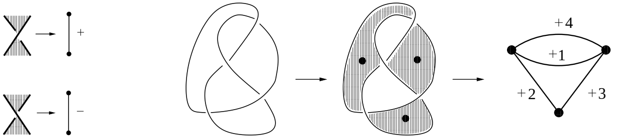
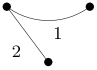
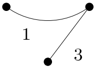
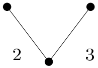
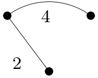
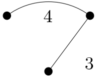
Spanning tree model for the Jones polynomial

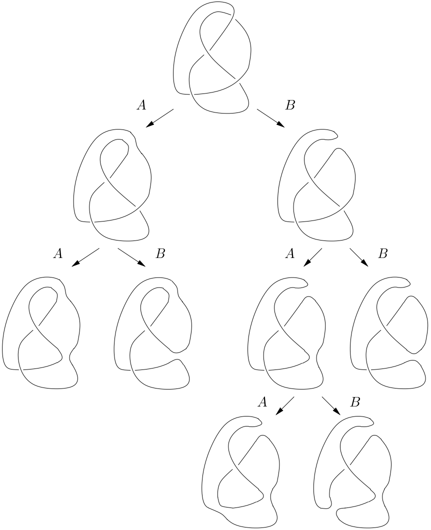
Choose any order on the crossings, hence on edges of the Tait graph:



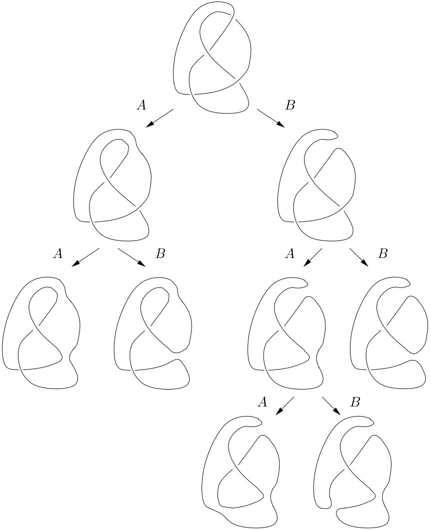
Spanning trees					
Activities	$LLdd$	$LdDd$	ℓDDd	ℓLdD	$\ell\ell DD$
Weights	A^{-8}	$-A^{-4}$	$-A^{4}$	1	A^{8}

$$\langle D\rangle = A^{-8} - A^{-4} + 1 - A^{4} + A^{8}, \text{ and writhe } w(D) = 0$$

Let $t = A^{-4}$: $\quad V_K(t) = t^{-2} - t^{-1} + 1 - t + t^{2}$



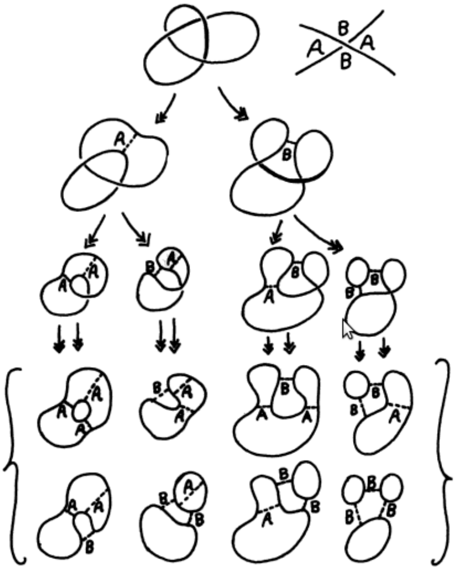
The activity word for T determines the twisted unknot $U(T)$ as a partial splicing of the link diagram.



Each unknot $U(T)$ contributes a monomial to $\langle D \rangle$: Let $\sigma(U) = \#A - \#B$,

$$\langle D \rangle = \sum_U (-A)^{3w(U)} \cdot A^{\sigma(U)} = \sum_T \mu(T)$$

From Kauffman states ...



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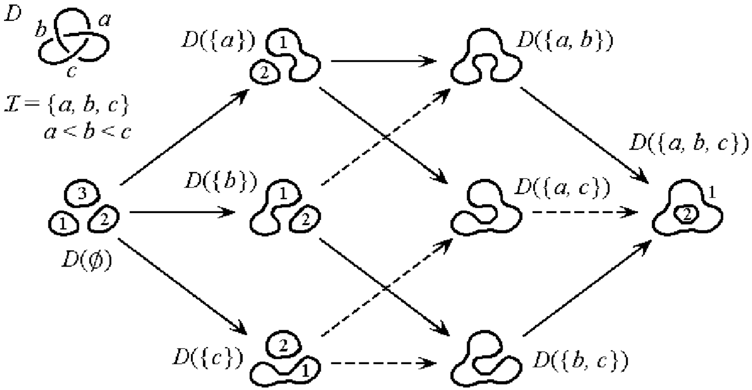

Khovanov homology $\{H^{i,j}(D), \partial\}$

$$\chi(H^{i,j}) = \sum_{i,j} (-1)^i q^j \mathrm{rank}(H^{i,j}) = (q + q^{-1}) V_L(q^2)$$

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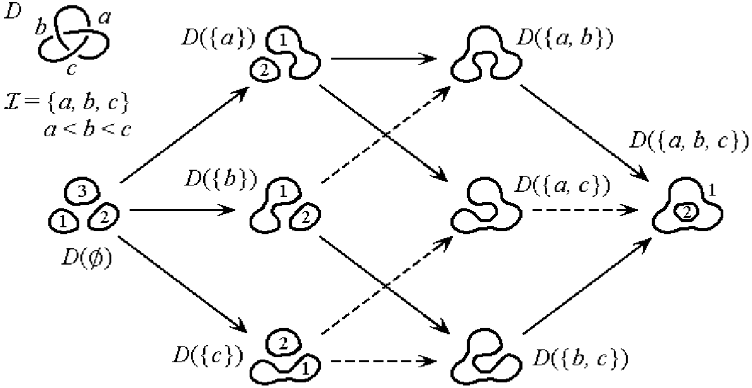

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Skein exact sequences

From the Kauffman bracket skein relation, there is an exact sequence for Khovanov homology.

Using Rasmussen's notation,

$$\xrightarrow{\cdot u} q^{2+3\epsilon}u^{1+\epsilon}\, H(\asymp) \longrightarrow H(\nwarrow\!\!\!\nearrow) \longrightarrow q\, H(\circlearrowleft\circlearrowright) \xrightarrow{\cdot u}$$

$$\xrightarrow{\cdot u} q^{-1}\, H(\circlearrowleft\circlearrowright) \longrightarrow H(\nwarrow\!\!\!\nearrow) \longrightarrow q^{1+3\epsilon}u^{\epsilon}\, H(\asymp)\, q^{-1} \xrightarrow{\cdot u}$$

Notation like $q\,H(\asymp)$ means the complex $H(\asymp)$ is shifted such that its Poincaré polynomial is multiplied by q. The arrow marked $\cdot u$ is the boundary map in the long exact sequence, raising the homological grading.

This skein exact sequence is similar to one for Heegaard-Floer homology.

Spanning tree model for Khovanov homology

Every spanning tree T of G (with ordered edges) corresponds to an activity word, which gives the twisted unknot $U(T)$.

Define a bigrading on spanning trees: $u(T) = -w(U)$ and $v(T) = E_+(T)$

Define $\mathcal{C}(D) = \bigoplus_{u,v} \mathcal{C}_v^u(D)$, where $\mathcal{C}_v^u(D) = \mathbb{Z}\langle T \subset G \mid u(T) = u,\ v(T) = v\rangle$

Thm. (Champanerkar-K) For a knot diagram D, there exists a spanning tree complex $\mathcal{C}(D) = \{\mathcal{C}_v^u(D), \partial\}$ with $\partial : \mathcal{C}_v^u \to \mathcal{C}_{v-1}^{u-1}$ that is a deformation retract of the reduced Khovanov complex,

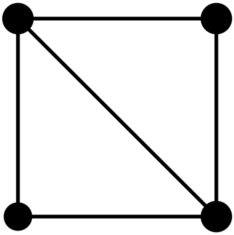
$$\widetilde{H}^{i,j}(D;\mathbb{Z}) \cong H_v^u(\mathcal{C}(D);\mathbb{Z})$$

with $u = j - i + k_1$ and $v = j/2 - i + k_2$.

From spanning trees to quasi-trees

For a planar graph, a spanning tree is a spanning subgraph whose regular neighbourhood has one boundary component.

Example



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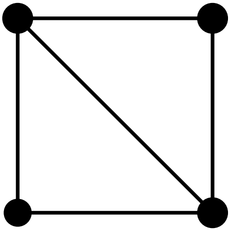

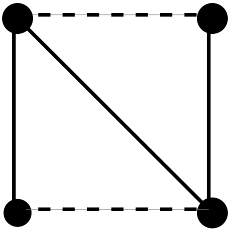

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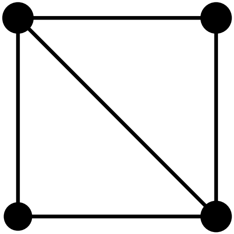

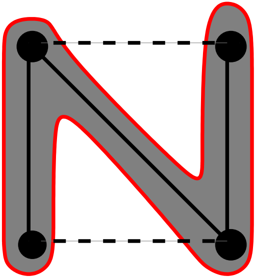

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A quasi-tree of a ribbon graph is a spanning ribbon subgraph with one face.

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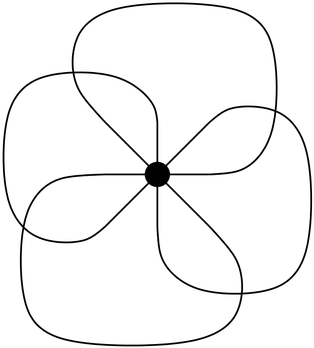

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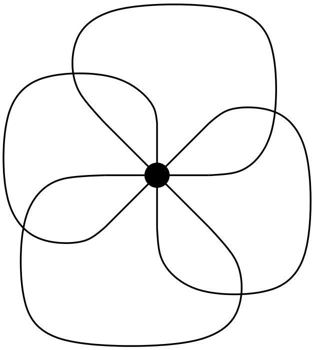

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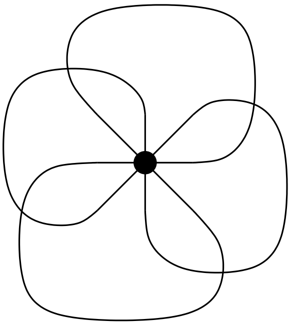
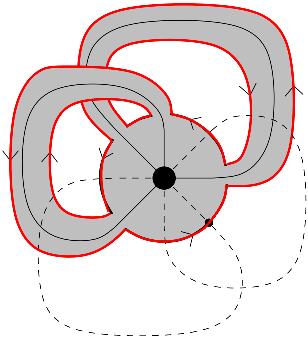
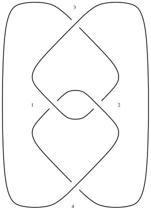




Example: From diagram to Tait graph and ribbon graph



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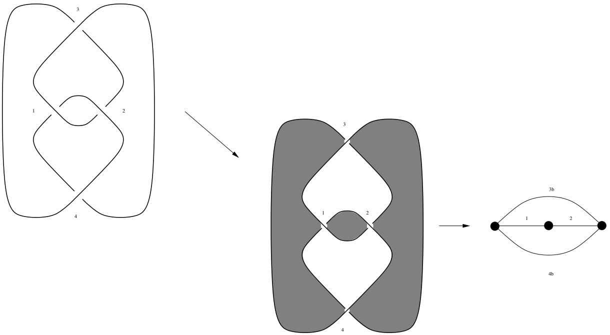

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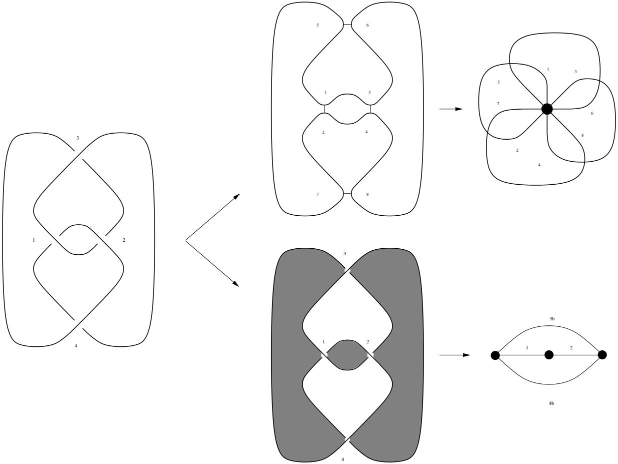

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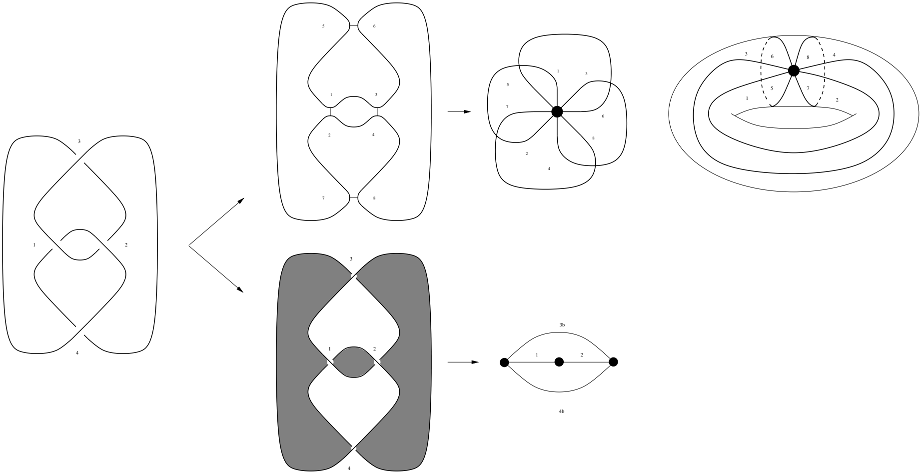

D connected link diagram, G its Tait graph, $\mathbb{G}_A$ its all–A ribbon graph.

Thm. (Champanerkar-K-Stoltzfus) Quasi-trees of $\mathbb{G}_A$ are in one-one correspondence with spanning trees of G:

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Moreover, every $\mathbb{Q}$ corresponds to an ordered chord diagram, which we used to define Tutte-like activities for edges of $\mathbb{G}_A$ with respect to $\mathbb{Q}$.

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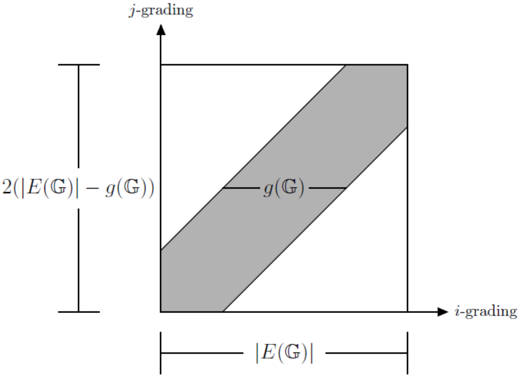
Turaev genus and homological width

Cor. (Champanerkar-K-Stoltzfus) For any knot K, the width of its reduced Khovanov homology $w_{KH}(K) \leq 1 + g_T(K)$.

Proof. For any ribbon graph $\mathbb{G}$, $g(\mathbb{G}) = \max_{\mathbb{Q} \subset \mathbb{G}} g(\mathbb{Q})$. Therefore, the quasi-tree complex $\mathcal{C}(\mathbb{G}_A)$ has at most $1 + g(\mathbb{G}_A)$ rows.



Turaev genus and homological width

For an adequate knot K with an adequate diagram D, T. Abe showed

$$g_T(K) = g_T(D) = w_{KH}(K) - 1 = c(K) - \mathrm{span}\, V_K(t)$$

Similar bounds for homological width of knot Floer homology in terms of $g_T(K)$ were obtained by Adam Lowrance.

Dasbach and Lowrance also proved bounds in terms of $g_T(K)$ for the Ozsváth-Szabó τ invariant and the Rasmussen s invariant.

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Related open problems

1. Is there a homologically thin knot with $g_T(K) > 1$?
   Generally, are there any lower bounds independent of knot homology?

2. Which operations on knots preserve or increase Turaev genus?
   By Abe's result, for adequate knots $g_T(K\#K') = g_T(K) + g_T(K')$ and g_T is preserved under mutation. How about non-adequate knots?

3. Do the results by Futer, Kalfagianni, and Purcell for adequate knots extend to all knots?

4. Krushkal defined a 4-variable polynomial invariant $P_{\mathbb{G}}$ that generalizes Tutte's duality for graphs, $T_G(X, Y) = T_{G^*}(Y, X)$, and specializes to Kauffman bracket. Do the Kauffman bracket and BRT polynomials determine $P_{\mathbb{G}}$? (See ArXiv: 0903.5312v3)

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   By Abe's result, for adequate knots $g_T(K \# K') = g_T(K) + g_T(K')$ and g_T is preserved under mutation. How about non-adequate knots?

3. Do the results by Futer, Kalfagianni, and Purcell for adequate knots extend to all knots?

4. Krushkal defined a 4-variable polynomial invariant $P_{\mathbb{G}}$ that generalizes Tutte's duality for graphs, $T_G(X, Y) = T_{G^*}(Y, X)$, and specializes to Kauffman bracket. Do the Kauffman bracket and BRT polynomials determine $P_{\mathbb{G}}$? (See ArXiv: 0903.5312v3)

Further reading from recent accessible papers

1. **Jones polynomials, volume and essential knot surfaces: A survey** by Futer, Kalfagianni, and Purcell. ArXiv: 1110.6388 (2011).

2. **Partials duals of plane graphs, separability and the graphs of knots** by Moffatt. *AGT* 12 (2012) 1099-1136. ArXiv: 1007.4219 (2012).

3. **A Turaev surface approach to Khovanov homology** by Dasbach and Lowrance. ArXiv: 1107.2344 (2011).

Thank you!

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Thank you!

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