Calculus III, Midterm Exam 2

Calculus III, Math V1201, Section 001, Spring 2018
Instructor: Ilya Kofman
Date: April 4, 2018

Name: __________________________

Show all work. To receive full credit, you must justify your answers.
Use the back side of the page if you need more space to do a problem.

No calculators or electronic devices of any type are allowed.

Useful curvature formulas:

\[ \kappa(t) = \frac{||r''(t)||}{||r'(t)||^3} \quad \text{and} \quad \kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} \]

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>26</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
</tr>
</tbody>
</table>
**Problem 1.** Let \( \mathbf{r}(t) = (2t - 1, 3 \cos(2t), -3 \sin(2t)) \) for \( 0 \leq t \leq 2\pi \).

(a) Find the arclength for \( 0 \leq t \leq 2\pi \).

(b) Find the arclength parametrization.

(c) Precisely describe the trajectory of the particle as a helix, including its height, base circle, number of revolutions, and direction.
Problem 2. Let \( \mathbf{r}(t) = (2 - \cos(t), \sqrt{2}\sin(t), \cos(t)) \).

(a) Show that \( \mathbf{r}'(t) \times \mathbf{r}''(t) \) is a constant vector.

(b) Find the unit vectors \( \mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t) \).

(c) Find the equation of the osculating plane at \( \mathbf{r}(\pi/2) \).

(d) Find the curvature \( \kappa(t) \).

(e) Use your answers above to precisely describe this curve in \( \mathbb{R}^3 \).
Problem 3. Let \( \mathbf{r}(t) = (2t, t^2, \frac{1}{3}t^3) \). For \( t = 1 \), find \( a_T, a_N, T, N \), so \( \mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \).
Problem 4. Evaluate the limits below or write DNE. In either case, justify.

(a) \( \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^4 + y^2} \) \hspace{1cm} (b) \( \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6 + 2y^2} \)

(c) Let \( f(x, y) = \frac{x^ay^b}{x^2 + y^2} \) for \( (x, y) \neq (0, 0) \), and \( f(0, 0) = 0 \).

Show that \( f(x, y) \) is continuous at \( (0, 0) \) if \( a, b \geq 0 \) and \( a + b > 2 \).
Problem 5. Let \( f(x, y) = x^2 + y^2 + xy - x + 4y \).

(a) Why is \( f(x, y) \) differentiable for all \( (x, y) \)?

(b) Find the equation of the tangent plane to the graph of \( f(x, y) \) at \( P(1, 2) \).

(c) For which \( Q(x, y) \) is the tangent plane to the graph of \( f(x, y) \) horizontal? Justify.

(d) Use a linear approximation at \( (1, 2) \) to approximate \( f(1.1, 1.9) \).

(e) Suppose \( x = 3s - t \) and \( y = s^2t \). Compute \( \frac{\partial f}{\partial s} \) at the point \( (s, t) = (1, 1) \).