Columbia University in the City of New York | *New York, N.Y. 10027*

DEPARTMENT OF MATHEMATICS

508 Mathematics Building
2990 Broadway

Fall Semester 2005

Professor Ioannis Karatzas

W4061: MODERN ANALYSIS

Description

The algebra of sets; ordered sets, the real number system, Euclidean space. Finite, countable, and uncountable sets. Elements of general topology: metric spaces, open and closed sets, completeness and compactness, perfect sets. Sequences and series of real numbers, especially power series; the number *e*. Continuous maps.

Functions of real variable: continuity and differentiability, the chain and L'Hopital rules. The Riemann integral: characterizations, mean-value theorems, the fundamental theorem of calculus. Uniform convergence; its relevance in continuity, integration and differentiation. Sequences and series of functions; double series.

Approximations: the Stone-Weierstrass theorem, Bernstein polynomials. Euler/Mac Laurin, De Moivre, Wallis and Stirling. Taylor approximations, Newton's method. The DeMoivre/Laplace and Poisson approximations to the bimomial distribution; examples.

Monotone functions, functions of finite variation. Infinitely-differentiable functions. Continuous functions which are nowhere differentiable. Convex sets, their separation properties. Convex functions, their differentiability and their relevance.

Prerequisites: Calculus IV (Math V1202) and Linear Algebra (Math V2010).

Required Text: W. RUDIN: *Principles of Mathematical Analysis*. Third Edition, 1976. McGraw-Hill Publishing Co. New York.

Detailed *Lecture Notes*, generously made available by Professor P.X. Gallager, will be distributed regularly.

Homework will be assigned and discussed regularly, in recitation sections by TA's.

There will be a *Mid-Term* and a *Final* Examination.

<u>COURSE SYLLABUS</u> (Tentative)

. *Lecture #1: Wednesday, 7 September.*
Algebra of subsets.

. *Lecture #2: Monday, 12 September.*
Algebra of maps.

<u>Assignment #1</u>: To be turned in Monday, 19 September.

. *Lecture #3: Wednesday, 14 September.*
Partitions. Equivalence relations. Cardinal numbers.

. *Lecture #4: Monday, 19 September*
Countable and uncountable sets.

. *Lecture #5: Wednesday, 21 September*
Properties of the rational number system. Notions of total ordering, field, totally ordered field, upper bound, supremum, the least-upper-bound property. *Cuts*, as subsets of the rationals. The Dedekind construction of the reals.

Reading: Chapter 1 of Rudin, including the Appendix.

. *Lecture #6: Monday, 26 September*
Metric and Topological Spaces. Open and closed sets; interior and closure of a set; properties.

. *Lecture #7: Wednesday, 28 September*
Notions of limit points of sets; perfect sets. Equivalent characterization of closed sets, in terms of their limit points. Continuous functions – global and local notions, relationship.

Reading: Chapter 2 of Rudin, pp.24-36.
<u>Assignment #2</u>: To be turned in Wednesday, 5 October.

. *Lecture #8: Monday, 3 October*

Notions of sequence, subsequence, and convergence in a metric space. Characterizations of closure and of completeness, in terms of convergence of sequences. Notion of a Cauchy sequence. Definition of compactness.

. *Lecture #9: Wednesday, 5 October*

Properties and characterizations of compactness. Notion of "completeness" of a metric space (convergence of every Cauchy sequence).

Reading: Chapter 2 of Rudin, pp. 36-40. Chapter 3 of Rudin, pp. 47-58.
Assignment # 3: Rudin, Chapter 2: # 10, 11, 12, 14, 16.
Rudin, Chapter 3: # 3, 16, 20, 23.

. *Lecture #10: Monday, 10 October*

Compactness, Bolzano-Weierstrass and Heine-Borel theorems.

. *Lecture #11: Wednesday, 12 October*

Maxima of continuous functions over compact sets. Uniform continuity and its properties.

Reading: Chapter 4 of Rudin, pp. 83-98.
Assignment # 4: Problems # 1, 2, 3, 6, 17, 20, 22, 23, 24 in Chapter 4.

. *Lecture #12: Monday, 17 October*

Uniform convergence of functions; relations with continuity and integration.

Reading: Chapter 7 of Rudin, pp. 147-152. Chapter 3, pp. 58-71.

. *Lecture #13: Wednesday, 19 October*

The comparison, Weierstrass and integral tests for series.
Radius of Convergence. Absolute convergence.

Assignment # 5: Read Chapter 3, pp. 72-78.
Chapter 3 of Rudin, Problems # 6 (a,b), 7, 8, 9, 11.

. *Lecture #14: Monday, 24 October*

Mid-Term Examination.

. *Lecture #15: Wednesday, 26 October*

Divergence of the series of the reciprocals of prime numbers.
Leibnitz test. Rearrangements. Double Series.

. *Lecture #16: Monday, 30 October*
The definition and properties of the Riemann integral.

. *Lecture #17: Wednesday, 2 November*
Definition and properties of the derivative.
Intermediate and mean-value theorems.
The fundamental theorem of calculus.

Reading: Chapter 5 of Rudin, pp. 103-111.
Chapter 6 of Rudin, pp. 123-134.

Assignment # 6: To be handed in on Wedn. 9 November.
Problems # 1, 2, 4, 6, 7, 10, 11 of Chapter 5 in Rudin.
Problems # 2, 4 of Chapter 6 in Rudin.

. *Lecture #18: Monday, 7 November*
University Holiday

. *Lecture #19: Wednesday, 9 November*
Mean Value Theorems for Integrals.
Uniform convergence and integration;
uniform convergence and differentiation.
The Holder and triangle inequalities for the integral.

Assignment # 7: Not to be handed in.
Read pp. 147-154 in Rudin.
Exercises 1-7 on p. 19.6 of Prof. Gallager's notes.
Problems # 22, 24, 26, 27 of Chapter 5 in Rudin.
Problem # 15 of Chapter 6 in Rudin.

. *Lecture #20: Monday, 14 November*
Differentiation under the integral sign.
Double integrals; improper integrals.
The Gamma function.

Assignment # 8: Due Monday, 21 November.
Exercises 1-3 on p. 22.6 of Prof. Gallager's notes.
Problems # 7, 8, 9, 16 (pp. 138-141) of Chapter 6 in Rudin.
Problem # 4, 7, 12 (pp. 165-167) of Chapter 7 in Rudin.

. *Lecture #21: Wednesday, 16 November*
Properties of the exponential and Gamma functions.
Transcendence of *e*.

Assignment # 9: Not to be handed in.
Problems # 1, 4, 5 (a,b), 6, 9 (pp. 196-197) of Chapter 8 in Rudin.
Exercises 1, 2, 3 on p. 24.5 of Prof. Gallager's notes.

. *Lecture #22: Monday, 21 November*
Euler-MacLaurin summation formula .
Formula of Wallis. The Stirling approximation.

. *Lecture #23: Wednesday, 23 November*
Taylor approximation. Newton's method.

Assignment # 10: Not to be handed in.
Problems # 15, 16, 17, 18, 19, 25 (pp. 115-118) of Chapter 5 in Rudin.

. *Lecture #24: Monday, 28 November*
The Binomial theorem, the Binomial distribution.
Computation of moments. The weak law of large numbers.

. *Lecture #25: Wednesday, 30 November*
Newton's Binomial Series. Bernstein's proof of the Weierstrass approximation theorem. The Gauss-Laplace function. Statement and significance of the DeMoivre-Laplace limit theorem.

Assignment # 11: Not to be handed in.

. *Lecture #26: Monday, 5 December*
Proof of the DeMoivre-Laplace limit theorem.
The Poisson approximation to the binomial probabilities.

. *Lecture #27: Wednesday, 7 December*
Irrationality – and computation – of *pi*. Computation of *e*.

. *Lecture #28: Monday, 12 December*
The Laplace asymptotic formula.

. *Lecture #29: Wednesday, 14 December*
Problem-solving session.