Columbia University in the City of New York | *New York, N.Y. 10027*

DEPARTMENT OF MATHEMATICS 508 Mathematics Building
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Fall Semester 2007
Professor Ioannis Karatzas

G4151-G6105: ANALYSIS AND PROBABILITY I

TENTATIVE COURSE SYLLABUS

I. Measure Theory

- *i.* Construction of the integral, limits and integration
- ii. L^p - spaces of functions
- iii. Construction of measures, Lebesgue-Stieltjes and product measures
- iv. Examples: ergodicity, Liouville measure, Hausdorff measure

II. Elements of Probability

- i. The coin-tossing or random walk model
- ii. Independent events and independent random variables
- iii. The Khintchin weak law and the Kolmogorov strong law of large numbers
- iv. Notions of convergence of random variables
- v. The Central Limit, Cramer and Iterated Logarithm Theorems

III. Elements of Fourier Analysis

- i. Fourier transforms of measures, Fourier-Lévy Inversion Formula
- ii. Convergence of distributions and characteristic functions
- iii. Proof of the Central Limit Theorem
- iv. Fourier transforms on Euclidean spaces
- v. Fourier series, the Poisson summation formula
- vi. Spectral decompositions of the Laplacian
- vii. The Heat equation and heat kernel, the Wave equation and D'Alembert's formula

IV. Brownian Motion

- i. Brownian motion as a Gaussian process
- ii. Brownian motion as scaling limit of random walks
- iii. Brownian motion as random Fourier series
- iv. Brownian motion and the heat equation
- v. Elementary properties of Brownian paths

Recommended Texts: G.B. FOLLAND: *"Real Analysis: Modern Techniques And Applications"* Wiley-Interscience, 2nd Edition (1999).

K.L. CHUNG: *"A First Course in Probability"*. 3rd Edition, Academic Press (2001).

Recommended Text for a review of undergraduate Probability:
Sh. ROSS: *"A First Course in Probability".* 5th Edition, Prentice-Hall (1998).

Also recommended: P. Billingsley: *Probability and Measure*, 3rd Edition (Wiley)
C.D. Aliprantis & O. Burkinshaw: *Problems in Real Analysis - A Workbook with Solutions*, 2nd Edition (Academic Press).

Copies of the Lecture Notes for this class will be made available.

Prerequisites: A solid, working knowledge of Advanced Calculus, Linear Algebra, Principles of Mathematical Analysis (at the level of Rudin's and Browder's books), and of Probability at the undergraduate level.

COURSE SYLLABUS

. *Lecture #1: Tuesday, 4 September.*
Definition and properties of measure. Examples. Sigma-algebras of sets. Measurable functions and their properties. Integration of simple functions, properties.

. *Lecture #2: Thursday, 6 September.*
Integration of measurable functions. Monotone Convergence Theorem, Fatou's lemma, Dominated Convergence Theorem.

. *Lecture #3: Tuesday, 11 September.*
Approximation of measurable functions by simple functions. Linearity properties of the integral. Outer measure, Karatheodory's theorem.

Assignment # 1: Elementary properties of measures.
Assignment # 3: Measurable and integrable functions.

. *Lecture #4: Thursday, 13 September*
The Hahn extension theorem. Complete measures, notion of completion of a measure space.

. *Lecture #5: Tuesday, 18 September.*
Proof of the Hahn extension theorem. Construction of Lebesgue and Lebesgue-Stieltjes measures. Borel- and Lebesgue-measurable sets.

Assignment # 2: Outer measure, completions, properties of Lebesgue-Stieltjes measures.

. *Lecture #6: Thursday, 20 September.*
L^p - spaces and their norms. Holder, Minkowski and Chebyshev inequalities.
Convex functions, the Jensen inequality.

Assignment # 4: Elementary properties of integration.

. Lecture #7: Tuesday, 25 September.
Product measure and Tonelli-Fubini theorems. Applications:
Young's inequality. Definition and properties of convolution.

. Lecture #8: Thursday, 27 September.
Modes of convergence for measurable functions:
almost-everywhere, in measure, in L^p. Comparisons. Egorov's theorem.

Assignment # 5: Convergence in measure and in other modes.

. Lecture #9: Tuesday, 2 October.
Completeness of L^p – spaces. First Borel-Cantelli Lemma, characterizations of a.e.-convergence.
Monote Class Theorem, proof of the Product Measure and Fubini-Tonelli theorems.

. Lecture #10: Thursday, 4 October.
Probability spaces, events, random variables. Example: Bernoulli, binomial, Poisson
and normal (Gaussian) variables. Poisson and normal approximations to the binomial.
Independence of events. Second Borel-Cantelli Lemma.

. Lecture #11: Tuesday, 9 October.
Independence of random variables; additivity of the variance. Elementary versions
of the Weak and Strong (Rajchman, Markov) Laws of Large Numbers.

Assignment # 6: Probability theory, modes of convergence, laws of large numbers.

. Lecture #12: Thursday, 11 October.
Khincin and Kolmogorov Laws of Large Numbers (with proofs). Statement and
significance of the Central Limit Theorem; statement of the Berry-Esseen Theorem.

. Lecture #13: Tuesday, 16 October.
Fourier transforms of measures, and their elementary properties. Examples.
The determination of a distribution from its Fourier transform, Fourier-Levy
inversion theorem. The simple Fourier inversion theorems (for functions).

. Lecture #14: Thursday, 18 October.
Convergence of probability measures, connections to other modes of convergence.
The Skorohod representation.

Assignment # 8: Modes of Convergence for Random Variables.

. Lecture #15: Tuesday, 23 October. MID-TERM EXAMINATION.

. Lecture #16: Thursday, 25 October.

The Helly-Bray lemma, tightness, the basic convergence result. Convergence results for characteristic functions, proof of the Central Limit Theorem.

. *Lecture #17: Tuesday, 30 October.*
Properties of characteristic functions. The Plancherel and Parseval identities. Applications of Fourier analysis to ordinary differential equations.

Assignment # 9: On Fourier analysis.

. *Lecture #18: Thursday, 1 November.*
Applications of Fourier analysis to the Wave Equation and the Heat Equation. The D'Alembert and Laplace formulae.

. *Lecture #17: Tuesday, 6 November.* UNIVERSITY HOLIDAY

. *Lecture #18: Thursday, 8 November.*
The Heat equation: fundamental solutions, existence, uniqueness, the method of images, initial and boundary-value problems.

Assignment # 10: Applications of Fourier analysis to differential equations.

. *Lecture #19: Tuesday, 13 November.*
Overview of Hilbert space theory: basic properties, the parallelogram law, the Pythagorean theorem. Orthonormal systems, the Bessel inequality and the Parseval identity.

. *Lecture #20: Thursday, 15 November.*
The projection theorem. The representation theorem for linear mappings on Hilbert space. Absolute continuity and singularity of measures. Weak convergence and duality for L^p – spaces.

Assignment # 7: Uniform Integrability, properties, the Dunford-Pettis and Komlos theorems.

. *Lecture #21: Tuesday, 20 November.*
The Lebesgue Decomposition and Radon-Nikodym Theorems.
The Daniell-Kolmogorov Theorem.

. *Lecture #22: Thursday, 22 November.* THANKSGIVING HOLIDAY

. *Lecture #23: Tuesday, 27 November.*
Brownian Motion: definition, visualization as a limit of random walks, elementary properties, Law of Large Numbers, Quadratic Variation. Interpretation of the solution of the Heat Equation in terms of Brownian Motion.

. Lecture #24: Thursday, 29 November.
Representation of the solution of Initial and Boundary Value problems for the heat equation, in terms of reflected and absorbed Brownian motion. Computation of the distribution of the first-hitting time. The Daniell- Kolmogorov consistency theorem. The Centsov-Kolmogorov theorem.

. Lecture #25: Tuesday, 4 December.
The Wiener-Ciesielski construction of Brownian motion, using Haar functions.
Levy modulus of continuity, Law of the Iterated Logarithm, and the (strong) Markov property for Brownian motion.

Assignment # 11: On Hilbert spaces.

. Lecture #26: Thursday, 6 December.
The Cantor ternary set and function. The zero-set of the Brownian motion and its properties. Notion and elementary properties of Brownian Local Time.

. Lecture #27: Tuesday, 11 December.
Notion and elementary properties of relative entropy. The Neyman-Pearson lemma.

. Lecture #28: Thursday, 13 December.
Review session; problem—solving.