On the algebraic solutions of the Painlevé-III (D7) equation

Peter D. Miller

Department of Mathematics
University of Michigan
○ ○ ○
joint work with R. Buckingham (Cincinnati)

June 27, 2022
Isomonodromic Deformations, Painlevé Equations, and Integrable Systems
Virtual ICM-2022 Satellite Conference
Columbia University, New York, NY

The six Painlevé equations:

$$\text{PI:} \quad u'' = 6u^2 + x. \qquad \text{PII:} \quad u'' = 2u^3 + xu + \alpha.$$

$$\text{PIII:} \quad u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \gamma u^3 + \frac{\delta}{u}.$$

$$\text{PIV:} \quad u'' = \frac{(u')^2}{2u} + \frac{3u^3}{2} + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u}.$$

$$\text{PV:} \quad u'' = \left(\frac{1}{2u} + \frac{1}{u-1}\right)(u')^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{x} + \frac{\delta u(u+1)}{u-1}.$$

$$\text{PVI:} \quad u'' = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)(u')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)u' + \frac{u(u-1)(u-2)}{x^2(x-1)^2}\left[\alpha + \frac{\beta x}{u^2} + \frac{\gamma(x-1)}{(u-1)^2} + \frac{\delta x(x-1)}{(u-x)^2}\right].$$

All but PI contain one or more free parameters.

Although typical solutions are highly transcendental, PII-PVI admit, for special values of the parameters, solutions expressible explicitly in terms of elementary functions or classical linear special functions (e.g., Airy, Bessel, etc.)

The parameter values for which the special solutions exist are related by a finitely-generated group action.

The group acts on the solutions via *Bäcklund transformations* that preserve the functional character of the solution (rational, algebraic, etc.)

R. Fuchs and Garnier: every Painlevé equation defines an isomonodromic deformation of a certain second-order linear ODE.

Working with first-order systems instead, every Painlevé equation is the compatibility condition for a certain *Lax pair* of linear equations for an auxiliary unknown $\boldsymbol{\Psi}$.

The inverse problem of constructing $\boldsymbol{\Psi}$ from its monodromy data can be formulated as a *Riemann-Hilbert problem*.

The Bäcklund symmetry group acts on $\boldsymbol{\Psi}$ as linear gauge transformations called *Schlesinger transformations*. These leave the monodromy data invariant but change formal exponents at singular points encoding the special parameter values.

Therefore, the Riemann-Hilbert representation of the whole family of special solutions related by a particular group action can be found explicitly once it is known for just one solution in the family — the *seed solution*.

While in general the monodromy data for a given solution cannot be obtained explicitly, for elementary function seed solutions the direct problem for the Lax pair can frequently be solved in terms of classical special functions.

The monodromy data can then be obtained with the use of classical *connection formulæ*.

Then one can use the Riemann-Hilbert representation to deduce useful information about the special solutions, especially in the case of extreme parameter values, in which case asymptotic methods like the *steepest descent method* apply.

This is the *isomonodromy method*. It has been successfully applied to *rational* solutions of

- The PII equation. There is a $\mathbb{Z}$-parametrized family of rational solutions with a single seed. The Jimbo-Miwa Lax pair evaluated on the seed solution is solved explicitly in terms of Airy functions.
- The PIII equation. The generic (D6 type) form admits rational solutions parametrized by $\mathbb{Z} \times \mathbb{C}$, and for the seed (independent of the $\mathbb{C}$ parameter) the Jimbo-Miwa Lax pair is solved using confluent hypergeometric (Whittaker) functions.
- The PIV equation. There are two distinct families of rational solutions, each parametrized by pairs of integers in $\mathbb{Z} \times \mathbb{Z}$:
  - For the seed of the family of *generalized Hermite* rational solutions, the Lax pair is solved in terms of elementary functions.
  - For the seed of the family of *generalized Okamoto* rational solutions, the Lax pair is solved in terms of Airy functions.

The (D7) degeneration of the PIII equation occurs for $\gamma = 0$ and $\alpha\delta \neq 0$:

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \frac{\delta}{u}, \quad \alpha\delta \neq 0.$$

Unlike the (D6) form with $\gamma\delta \neq 0$, it does not have rational solutions. However, for $\alpha > 0$, $\beta = 2n \in 2\mathbb{Z}$, and $\delta = -1$, it has a unique algebraic solution $u = u_n(x)$ that is a rational function of $x^{\frac{1}{3}}$.

Scaling by $(x, u) \mapsto (cx, cu)$ leaves β and δ invariant, but scales α; WLOG we then take $\alpha = 8$ and then the seed solution is $u_0(x) := \frac{1}{2}x^{\frac{1}{3}}$.

More generally, $u_n(x) = \frac{1}{2}x^{\frac{1}{3}}(1 + \mathcal{O}(x^{-\frac{1}{3}}))$ as $x \to \infty$ for each $n \in \mathbb{Z}$.

Since $u_n(x)$ is a rational function of $Z = x^{\frac{1}{3}}$, it can be represented in terms of polynomials. The relevant polynomials are called the *Ohyama polynomials* $P_0(Z), P_1(Z), P_2(Z), \ldots$. They are defined by the recurrence relation

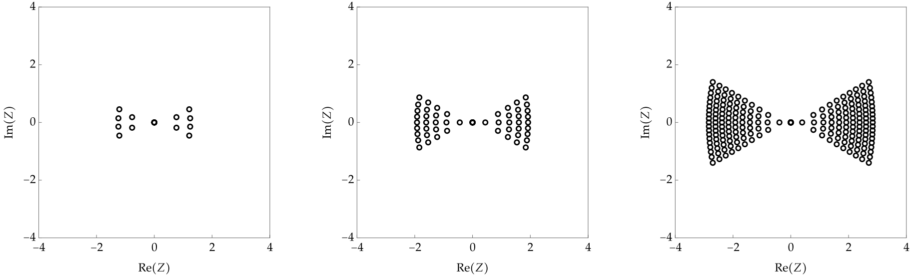
$$2\sqrt{3}ZP_{n+1}(Z)P_{n-1}(Z) = -\frac{1}{3}P_n(Z)P_n''(Z) + \frac{1}{3}P_n'(Z)^2 - \frac{1}{3Z}P_n(Z)P_n'(Z) + 2(3Z^2 - n)P_n(Z)^2$$

with initial conditions $P_0(Z) := 1$ and $P_1(Z) := 3Z^2$. Then

$$u_n(x) = \frac{P_{n+1}(x^{\frac{1}{3}})P_{n-1}(x^{\frac{1}{3}})}{2\sqrt{3}P_n(x^{\frac{1}{3}})^2}, \quad n \in \mathbb{Z}, \quad n > 0.$$

The zeros of the Ohyama polynomials form a “bow-tie” shape in the Z-plane:

Roots of $P_5(Z)$ Roots of $P_{10}(Z)$ Roots of $P_{20}(Z)$



This is a phenomenon that should be explained.... It has something to do with the limit $n \to \infty$.

When n is large in

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{8u^2 + 2n}{x} - \frac{1}{u},$$

a natural balance is achieved by the scaling $u = n^{\frac{1}{2}}U$ and the change of independent variable $x \mapsto z$ given by $x = n^{\frac{3}{2}}(y + n^{-1}z)$ where $y \neq 0$ is a fixed parameter. Then one gets

$$U''(z) = \frac{U'(z)^2}{U(z)} + \frac{8}{y}U(z)^2 + \frac{2}{y} - \frac{1}{U(z)} + \mathcal{O}(n^{-1}).$$

Dropping the formally-small error term gives the *approximating equation* for $U = U(z)$, which is autonomous, but parametrized by y.

There are two types of solutions of the approximating equation

$$U''(z) = \frac{U'(z)^2}{U(z)} + \frac{8}{y}U(z)^2 + \frac{2}{y} - \frac{1}{U(z)}.$$

Equilibrium solutions (U independent of z). These are roots of the cubic equation $8U^3 + 2U - y = 0$. As $y \to \infty$, $U \sim \frac{1}{2}y^{\frac{1}{3}}$ or rotations by $\frac{2}{3}\pi$. In the original variables we have $u \sim \frac{1}{2}x^{\frac{1}{3}}$ as $x \to \infty$ (matches $u_n(x)$ in that limit for each n);

Non-equilibrium solutions. Multiplying through by $U'(z)/U(z)^2$ and integrating gives

$$U'(z)^2 = \frac{16}{y}U(z)^3 + 2EU(z)^2 - \frac{4}{y}U(z) + 1, \quad E = \text{constant},$$

i.e., the Weierstraß equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ with

$$U(z) = \frac{1}{4}y\wp(z) - \frac{1}{24}yE, \quad g_2 = \frac{16}{y^2} + \frac{E^2}{3}, \quad g_3 = -\frac{16}{y^2} - \frac{8E}{3y^2} - \frac{E^3}{27}.$$

So, it is reasonable to expect that $u_n(x)$ might be modeled by an equilibrium solution for large x and by a modulated Weierstraß function for smaller x. However:

- The formal scaling arguments are not rigorous, so some proofs are required;
- It is not at all clear where/why one type of approximation should give way to the other in the complex x-plane.

To deal with both of these issues, we can

- Apply the isomonodromy method to obtain a Riemann-Hilbert representation of $u_n(x)$.
- Apply the Deift-Zhou steepest descent method to the resulting problem to rigorously address the limit $n \to \infty$.

But first, for context, we briefly describe how the method handles the rational solutions of PIII (D6). WLOG we take $\gamma = -\delta = 4$ and write $\alpha = 4(n+m)$ and $\beta = 4(n-m)$, to get

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{4(n+m)u^2 + 4(n-m)}{x} + 4u^3 - \frac{4}{u}$$

which has a rational solution iff $m \in \mathbb{Z}$ or $n \in \mathbb{Z}$:

- Two rational solutions if $m \in \mathbb{Z}$ and $n \in \mathbb{C} \setminus \mathbb{Z}$ or vice-versa;
- Four rational solutions if both $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

All of them can be obtained by assuming that $n \in \mathbb{Z}$, in which case there is a unique rational solution $u = u_n(x;m)$ with $u_n(\infty;m) = 1$.

When $n = 0$ we have $u_0(x;m) \equiv 1$. This is the seed solution.

The Jimbo-Miwa Lax pair for PIII (D6) involves four potentials $v(x)$, $y(x)$, $s(x)$, and $t(x)$ and a parameter $\Theta_\infty := m - n + 1$:

$$\frac{\partial \mathbf{\Psi}}{\partial \lambda} = \left(\frac{\mathrm{i}x}{2}\sigma_3 + \frac{1}{\lambda}\begin{bmatrix} -\frac{1}{2}\Theta_\infty & y \\ v & \frac{1}{2}\Theta_\infty \end{bmatrix} + \frac{1}{\lambda^2}\begin{bmatrix} \frac{1}{2}\mathrm{i}x - \mathrm{i}st & \mathrm{i}s \\ -\mathrm{i}t(st-x) & -\frac{1}{2}\mathrm{i}x + \mathrm{i}st \end{bmatrix}\right)\mathbf{\Psi};$$

$$\frac{\partial \mathbf{\Psi}}{\partial x} = \left(\frac{\mathrm{i}\lambda}{2}\sigma_3 + \frac{1}{x}\begin{bmatrix} 0 & y \\ v & 0 \end{bmatrix} - \frac{1}{\lambda x}\begin{bmatrix} \frac{1}{2}\mathrm{i}x - \mathrm{i}st & \mathrm{i}s \\ -\mathrm{i}t(st-x) & -\frac{1}{2}\mathrm{i}x + \mathrm{i}st \end{bmatrix}\right)\mathbf{\Psi}.$$

The complicated-looking matrix coefficient is simply the most general parametrization of a matrix having eigenvalues $\pm\frac{1}{2}\mathrm{i}x$. So the Lax pair presented here is diagonalized at $\lambda = \infty$, but could be conjugated into a gauge-equivalent one diagonalized at $\lambda = 0$ instead.

Compatibility yields a first-order system on v, y, s, t that admits a first integral $\Theta_0 = n + m$ and that implies the PIII (D6) equation for $u(x) = -y(x)/s(x)$.

Rational Solutions of PIII (D6)

Lax pair for the seed. M, T. Bothner and Y. Sheng, *Stud. Appl. Math.* **141**, 626–679, 2018.

When $n = 0$ and hence $u_0(x;m) \equiv 1$, the potentials $v(x)$, $y(x)$, $s(x)$, and $t(x)$ are determined up to inessential constants, and they are simple functions of x:

$$v(x) = -\frac{1}{4}(1-2\Theta_\infty)(4x+1+2\Theta_\infty)K^{-1}\mathrm{e}^{-2x}x^{-\Theta_\infty}, \quad y(x) = -\frac{1}{4}K\mathrm{e}^{2x}x^{\Theta_\infty},$$

$$s(x) = \frac{1}{4}K\mathrm{e}^{2x}x^{\Theta_\infty}, \quad t(x) = (1-2\Theta_\infty)K^{-1}\mathrm{e}^{-2x}x^{-\Theta_\infty}.$$

This dramatically simplifies the x-equation in the Lax pair; in fact it reduces to a confluent hypergeometric equation: taking

$$\mathbf{\Psi} = \mathrm{e}^{x\sigma_3}x^{(m+1)\sigma_3/2}x^{-\frac{1}{2}}\mathbf{W}\mathbf{C}(\lambda)$$

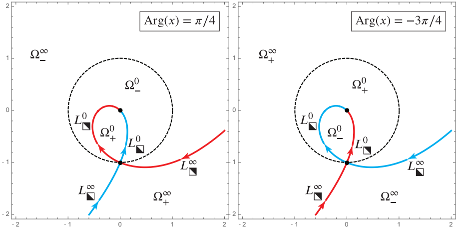
the x-equation implies that with $\mu = \frac{1}{4}$ and $\kappa = \frac{1}{2}m$, the first row $W = W_{1j}$ satisfies

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} + \left[-\frac{1}{4} + \frac{\kappa}{\zeta} + \frac{1-4\mu^2}{4\zeta^2}\right]W = 0, \quad \zeta := \mathrm{i}x(\lambda + 2\mathrm{i} - \lambda^{-1}).$$

Rational Solutions of PIII (D6)

Lax pair for the seed. M, T. Bothner and Y. Sheng, *Stud. Appl. Math.* **141**, 626–679, 2018.

The λ-equation then implies that $\mathbf{C}(\lambda) = (\lambda + \mathrm{i})^{-\frac{1}{2}}\mathbf{C}$ for any constant matrix $\mathbf{C}$. For a fundamental pair of solutions W we choose the *Whittaker functions* $W_{\pm\kappa,\mu}(\pm\zeta)$ which have branch cuts where $\zeta \in \mathbb{R}$:



Red contours: $\zeta < 0$. Blue contours: $\zeta > 0$.

We choose $\mathbf{C}$ for each domain Ω^0_+, Ω^∞_+, Ω^0_-, and Ω^∞_- to obtain suitably normalized solutions $\mathbf{\Psi} = \mathbf{\Psi}(\lambda; x)$ as $\lambda \to 0, \infty$. We find *no jump* across the unit circle, so $\mathbf{\Psi}$ has jumps only across $\zeta \in \mathbb{R}$, explicitly computed (for $n = 0$ but arbitrary $m \in \mathbb{C}$) from known connection formulæ.

Rational Solutions of PIII (D6)

Riemann-Hilbert problem. M, T. Bothner and Y. Sheng, *Stud. Appl. Math.* **141**, 626–679, 2018.

When $n = 0$, the matrix $\mathbf{Y} = \mathbf{\Psi}\lambda^{\frac{1}{2}(m-n+1)\sigma_3}\mathrm{e}^{-\frac{1}{2}\mathrm{i}x(\lambda-\lambda^{-1})\sigma_3}$ satisfies the Riemann-Hilbert problem:

- **Analyticity:** $\mathbf{Y}$ is analytic in $\mathbb{C} \setminus (\zeta \in \mathbb{R})$ with continuous boundary values.
- **Jump conditions:** $\mathbf{Y}_+ = \mathbf{Y}_-\mathbf{V}$ for $\zeta \in \mathbb{R}$ where

$$\mathbf{V}_{\text{red}}^{\infty} = \begin{bmatrix} 1 & \dfrac{\sqrt{2\pi}\lambda^{-(m+1)}}{\Gamma(\frac{1}{2}-m)}\lambda^{n}\mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})} \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_{\text{red}}^{0} = \begin{bmatrix} 1 & -\dfrac{\sqrt{2\pi}\lambda^{-(m+1)}}{\Gamma(\frac{1}{2}-m)}\lambda^{n}\mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})} \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{V}_{\text{blue}}^{\infty} = \begin{bmatrix} 1 & 0 \\ \dfrac{\sqrt{2\pi}\langle\lambda^{m+1}\rangle}{\Gamma(\frac{1}{2}+m)}\lambda^{-n}\mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})} & 1 \end{bmatrix}, \quad \mathbf{V}_{\text{blue}}^{0} = \begin{bmatrix} -\mathrm{e}^{2\pi\mathrm{i}m} & 0 \\ \dfrac{\sqrt{2\pi}\langle\lambda^{m+1}\rangle}{\Gamma(\frac{1}{2}+m)}\lambda^{-n}\mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})} & -\mathrm{e}^{-2\pi\mathrm{i}m} \end{bmatrix}.$$

- **Normalization:** $\mathbf{Y} \to \mathbb{I}$ as $\lambda \to \infty$; $\mathbf{Y}\lambda^{-(m+\frac{1}{2})\sigma_3}$ bounded as $\lambda \to 0$.

Care must be taken in the choice of branch for λ^p, cut on the blue contours. Notice what happens when $m \in \mathbb{Z} + \frac{1}{2}$!

Rational Solutions of PIII (D6)

Schlesinger transformations. M, T. Bothner and Y. Sheng, *Stud. Appl. Math.* **141**, 626–679, 2018.

We work out how to generalize from $n = 0$ by applying Schlesinger transformations. If (as is true for $n = 0$ when $\Theta_\infty = m + 1$ and $\Theta_0 = m$)

$$\boldsymbol{\Psi}(\lambda;x)\lambda^{\frac{1}{2}\Theta_\infty\sigma_3}\mathrm{e}^{-\frac{1}{2}\mathrm{i}x\lambda\sigma_3} = \mathbb{I} + \boldsymbol{\Psi}_1^\infty(x)\lambda^{-1} + \cdots, \quad \lambda \to \infty,$$

$$\boldsymbol{\Psi}(\lambda,x)\lambda^{-\frac{1}{2}\Theta_0\sigma_3}\mathrm{e}^{\frac{1}{2}\mathrm{i}x\lambda^{-1}\sigma_3} = \boldsymbol{\Psi}_0^0(x) + \boldsymbol{\Psi}_1^0(x)\lambda + \cdots, \quad \lambda \to 0,$$

then, for instance, one can simultaneously increment Θ_0 and decrement Θ_∞ by 1 (same as replacing n with $n+1$) without (essentially) changing the jump matrices by a (Schlesinger) gauge transformation:

$$\widehat{\boldsymbol{\Psi}}(\lambda;x) := \left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\lambda^{\frac{1}{2}} + \widehat{\mathbf{B}}(x)\lambda^{-\frac{1}{2}}\right)\boldsymbol{\Psi}(\lambda;x), \quad \text{where}$$

$$\widehat{\mathbf{B}}(x) := \begin{bmatrix}\Psi^0_{0,21}(x)\Psi^\infty_{1,12}(x)/\Psi^0_{0,11}(x) & -\Psi^\infty_{1,12}(x)\\ -\Psi^0_{0,21}(x)/\Psi^0_{0,11}(x) & 1\end{bmatrix}.$$

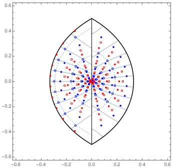
There is a similar (in fact, inverse) Schlesinger transformation that decrements Θ_0 and increments Θ_∞ by 1.

Rational Solutions of PIII (D6)

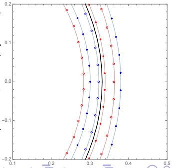
Large-n asymptotics. M, T. Bothner, *Constr. Approx.* **51**, 123–224, 2020.

The Riemann-Hilbert representation of $u_n(x;m)$ can be analyzed by the Deift-Zhou steepest descent method. Some results:

- When $n \in \mathbb{Z}$ is large and $m \in \mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$ is fixed, there is a specific eye-shaped domain E such that:
  - When $x \in nE$, $u_n(x;m)$ is approximated by a specific elliptic function of x with phase shift and elliptic parameter that depend (nonanalytically) on the slow variable x/n.
  - When $x \in \mathbb{C} \setminus nE$, $u_n(x;m)$ is approximated by an analytic function of the slow variable x/n.

  The approximations are non-equilibrium and equilibrium solutions, respectively, of an approximating equation.
- When $n \in \mathbb{Z}$ is large and $m \in \mathbb{Z} + \frac{1}{2}$ is fixed, the equilibrium approximation extends to the complement of one or the other "eyebrows". Near the eyebrow, $u_n(x;m)$ is approximated by a trigonometric function that slowly varies along the eyebrow.



Algebraic solutions of Painlevé-III (D7)

Basic definitions

Recall that starting from the nondegenerate (D6) version of PIII

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \gamma u^3 + \frac{\delta}{u}, \quad \gamma\delta \neq 0,$$

the (D7) degeneration arises by taking $\gamma = 0$:

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \frac{\delta}{u}, \quad \alpha\delta \neq 0.$$

We take $\alpha = 8$ and $\delta = -1$. Then there is a unique algebraic solution

$$u = u_n(x) = \tfrac{1}{2} x^{\frac{1}{3}} (1 + \mathcal{O}(x^{-\frac{1}{3}})), \quad x \to \infty$$

if and only if $\beta = 2n \in 2\mathbb{Z}$. It is a rational function of $Z = x^{\frac{1}{3}}$ expressible in terms of Ohyama polynomials.

We follow the steps of the isomonodromy method to characterize these solutions and study them for large n.

In Jimbo-Miwa (1981 — Part II) a Lax pair for PIII (D6) is given with $\gamma = -\delta = 4$. Arbitrary values of γ, δ with $\gamma\delta \neq 0$ can be restored by scaling, but *one cannot just set* $\gamma = 0$ to get a Lax pair for PIII (D7).

That is because the required modification to obtain PIII (D7) is to admit a coefficient matrix of the most singular terms at $\lambda = 0$ that is *not diagonalizable*. The following Lax pair was given by Kitaev-Vartanian (2004):

$$\frac{\partial \boldsymbol{\Psi}}{\partial \lambda} = \left(-\mathrm{i}x\sigma_3 + \frac{1}{\lambda}\begin{bmatrix} -\frac{1}{2}\mathrm{i}a & \frac{x^2 p}{4u} \\ \frac{x^2 q}{4u} & \frac{1}{2}\mathrm{i}a \end{bmatrix} + \frac{1}{\lambda^2}\begin{bmatrix} -\frac{1}{2}\mathrm{i}u & \frac{1}{2}\mathrm{i}u e^{\mathrm{i}\varphi} \\ -\frac{1}{2}\mathrm{i}u e^{-\mathrm{i}\varphi} & \frac{1}{2}\mathrm{i}u \end{bmatrix} \right) \boldsymbol{\Psi};$$

$$\frac{\partial \boldsymbol{\Psi}}{\partial x} = \left(-\mathrm{i}\lambda\sigma_3 + \frac{1}{x}\begin{bmatrix} \frac{1}{2}\mathrm{i}a & \frac{x^2 p}{4u} \\ \frac{x^2 q}{4u} & -\frac{1}{2}\mathrm{i}a \end{bmatrix} - \frac{1}{\lambda x}\begin{bmatrix} -\frac{1}{2}\mathrm{i}u & \frac{1}{2}\mathrm{i}u e^{\mathrm{i}\varphi} \\ -\frac{1}{2}\mathrm{i}u e^{-\mathrm{i}\varphi} & \frac{1}{2}\mathrm{i}u \end{bmatrix} \right) \boldsymbol{\Psi}.$$

It involves a constant parameter a and unknown functions $p(x)$, $q(x)$, $\varphi(x)$, and $u(x)$.

The conditions for compatibility of the Kitaev-Vartanian Lax pair are:

$$p(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u(x)\mathrm{e}^{\mathrm{i}\varphi(x)}}{x}\right), \quad q(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u(x)\mathrm{e}^{-\mathrm{i}\varphi(x)}}{x}\right),$$

$$8u(x)^3 - u(x)u'(x) + xu'(x)^2 - 2au(x)^2\varphi'(x) + xu(x)^2\varphi'(x)^2 - xu(x)u''(x) = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)\varphi'(x) - \frac{2au(x)}{x}\right) = 0.$$

Integrating the last equation with a concrete choice of integration constant gives

$$u(x)\varphi'(x) = \frac{2au(x)}{x} + \mathrm{i}.$$

Explicitly eliminating $\varphi'(x)$ then gives PIII (D7) for $u(x)$ in the form

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{8u^2+\beta}{x} - \frac{1}{u}, \quad \beta = 2\mathrm{i}a.$$

Algebraic solutions of Painlevé-III (D7)

Lax pair for the seed. Buckingham and M., `arXiv:2202.04217`

For the seed, we take $\beta = 2n = 0$ and $u(x) = u_0(x) = \frac{1}{2}x^{\frac{1}{3}}$. Using the nonlinear differential equations from compatibility, we first get $\varphi(x) = 3\mathrm{i}x^{\frac{2}{3}} + \varphi_0$ (take $\varphi_0 = 0$ WLOG) so $\mathrm{e}^{\pm\mathrm{i}\varphi(x)} = \mathrm{e}^{\mp 3x^{2/3}}$, and then

$$p(x) = -\left(\frac{1}{x} + \frac{1}{3}x^{-\frac{5}{3}}\right)\mathrm{e}^{-3x^{2/3}} \qquad q(x) = -\left(-\frac{1}{x} + \frac{1}{3}x^{-\frac{5}{3}}\right)\mathrm{e}^{3x^{2/3}}.$$

The exponential factors $\mathrm{e}^{\pm 3x^{2/3}}$ can be easily eliminated from the coefficients by the gauge transformation $\mathbf{\Psi} = \mathrm{e}^{-\frac{3}{2}x^{2/3}\sigma_3}\mathbf{\Phi}$, after which the Lax system reads

$$\frac{\partial \mathbf{\Phi}}{\partial \lambda} = \left(-\mathrm{i}x\sigma_3 - \frac{x}{\lambda}\begin{bmatrix} 0 & \frac{1}{6}x^{-1} + \frac{1}{2}x^{-\frac{1}{3}} \\ \frac{1}{6}x^{-1} - \frac{1}{2}x^{-\frac{1}{3}} & 0 \end{bmatrix} + \frac{\mathrm{i}x^{\frac{1}{3}}}{4\lambda^2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)\mathbf{\Phi};$$

$$\frac{\partial \mathbf{\Phi}}{\partial x} = \left(-\mathrm{i}\lambda\sigma_3 - \begin{bmatrix} -x^{-\frac{1}{3}} & \frac{1}{6}x^{-1} + \frac{1}{2}x^{-\frac{1}{3}} \\ \frac{1}{6}x^{-1} - \frac{1}{2}x^{-\frac{1}{3}} & x^{-\frac{1}{3}} \end{bmatrix} - \frac{\mathrm{i}x^{-\frac{2}{3}}}{4\lambda}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)\mathbf{\Phi}.$$

We want to solve the x-equation explicitly. It is useful to first simplify it by scaling the spectral parameter: $\lambda = \Lambda/X$ and $x = X$ so that

$$\frac{\partial}{\partial \Lambda} = \frac{1}{X}\frac{\partial}{\partial \lambda} \quad \text{and} \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial x} - \frac{\Lambda}{X^2}\frac{\partial}{\partial \lambda}.$$

In the new variables we then have

$$\frac{\partial \mathbf{\Phi}}{\partial \Lambda} = \left(-\mathrm{i}\sigma_3 - \frac{X}{\Lambda}\begin{bmatrix} 0 & \frac{1}{6}X^{-1} + \frac{1}{2}X^{-\frac{1}{3}} \\ \frac{1}{6}X^{-1} - \frac{1}{2}X^{-\frac{1}{3}} & 0 \end{bmatrix} + \frac{\mathrm{i}X^{\frac{4}{3}}}{4\Lambda^2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)\mathbf{\Phi};$$

$$\frac{\partial \mathbf{\Phi}}{\partial X} = \left(X^{-\frac{1}{3}}\sigma_3 - \frac{\mathrm{i}X^{\frac{1}{3}}}{2\Lambda}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)\mathbf{\Phi}.$$

Now make the substitution $Z = X^{\frac{1}{3}} = x^{\frac{1}{3}}$ and the constant gauge transformation $\mathbf{\Omega} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{\Phi}$ in the X-equation:

Algebraic solutions of Painlevé-III (D7)

Lax pair for the seed. Buckingham and M., arXiv:2202.04217

We get the off-diagonal system

$$\frac{\partial \mathbf{\Omega}}{\partial Z} = \begin{bmatrix} 0 & 3Z \\ 3Z + 3\mathrm{i}\Lambda^{-1}Z^3 & 0 \end{bmatrix} \mathbf{\Omega}.$$

A few more manipulations reduce this system to the *Airy equation*. A fundamental solution matrix is $\mathbf{\Omega} = \mathbf{\Omega}_0(Z, \Lambda) := \mathbf{\Delta}(\Lambda)\mathbf{F}(\xi)$, where

$$\mathbf{\Delta}(\Lambda) := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -\left(\frac{2}{3}\right)^{\frac{1}{3}}(\mathrm{i}\Lambda)^{-\frac{1}{3}} \end{bmatrix}, \quad \mathbf{F}(\xi) := \begin{bmatrix} f_1(\xi) & f_2(\xi) \\ f_1'(\xi) & f_2'(\xi) \end{bmatrix},$$

where $f_j(\xi)$ are a fundamental pair for $f''(\xi) - \xi f(\xi) = 0$ and

$$\xi := \left(\frac{3}{2}\right)^{\frac{2}{3}} (\mathrm{i}\Lambda)^{\frac{2}{3}} \left(1 - \frac{Z^2}{\mathrm{i}\Lambda}\right).$$

The general solution is therefore $\mathbf{\Omega}(Z, \Lambda) = \mathbf{\Omega}_0(Z, \Lambda)\mathbf{H}(\Lambda)$.

Algebraic solutions of Painlevé-III (D7)

Now use the Λ-equation to find $\mathbf{H}(\Lambda)$ using compatibility. Get $\mathbf{H}(\Lambda) = \Lambda^{\frac{1}{6}}\mathbf{H}_0$ where $\mathbf{H}_0$ is an arbitrary absolute constant matrix, which can be absorbed in the choice of basis $(f_1(\xi), f_2(\xi))$.

Restoring the original variables, the general simultaneous solution of the PIII (D7) Lax pair for the seed algebraic solution $u = u_0(x) = \frac{1}{2}x^{\frac{1}{3}}$ with $\beta = 2n$ and $n = 0$ is

$$\mathbf{\Psi}(\lambda, x) = \mathrm{e}^{-\frac{3}{2}x^{2/3}\sigma_3} \frac{1}{2} \begin{bmatrix} 1 & -\left(\frac{2}{3}\right)^{\frac{1}{3}} \\ -1 & -\left(\frac{2}{3}\right)^{\frac{1}{3}} \end{bmatrix} (\mathrm{i}x\lambda)^{\frac{1}{6}\sigma_3}\mathbf{F}(\xi),$$

where with $f_1(\xi)$ and $f_2(\xi)$ a fundamental pair for $f''(\xi) - \xi f(\xi) = 0$,

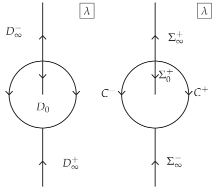
$$\mathbf{F}(\xi) = \begin{bmatrix} f_1(\xi) & f_2(\xi) \\ f_1'(\xi) & f_2'(\xi) \end{bmatrix}, \quad \xi = \left(\frac{3}{2}\right)^{\frac{2}{3}} (\mathrm{i}x\lambda)^{\frac{2}{3}} \left(1 - \frac{x^{\frac{2}{3}}}{\mathrm{i}x\lambda}\right).$$

Algebraic solutions of Painlevé-III (D7)

Normalized solutions, Stokes phenomenon, and asymptotics. Buckingham and M., arXiv:2202.04217

The general solution is branched at $\lambda = 0, \infty$ and also exhibits Stokes phenomenon in both limits because $\mathbf{F}(\xi)$ does as $\xi \to \infty$, which happens both for $\lambda \to 0$ and $\lambda \to \infty$.

It is easiest to specify solutions with simple asymptotics as $\lambda \to 0, \infty$ by assuming first that $x > 0$. Then we can define three domains with oriented boundary curves as shown:

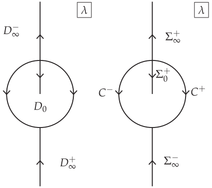


Then for $\lambda \in D_\infty^\pm$ we use the basis $f_1(\xi) := c_1^\pm \mathrm{Ai}(\xi)$, $f_2(\xi) := c_2^\pm \mathrm{Ai}(\mathrm{e}^{\mp\frac{2}{3}\pi\mathrm{i}}\xi)$ with $c_1^\pm := 2\sqrt{\pi}(\frac{3}{2})^{\frac{1}{6}}$, $c_2^\pm := -\mathrm{e}^{\mp\frac{1}{6}\mathrm{i}\pi}2\sqrt{\pi}(\frac{3}{2})^{\frac{1}{6}}$.

Similarly, for $\lambda \in D_0$ we use the basis $f_1(\xi) := c^+ \mathrm{Ai}(\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\xi)$, $f_2(\xi) := c^- \mathrm{Ai}(\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\xi)$ with $c^+ := 2\mathrm{i}\sqrt{\pi}(\frac{3}{2})^{\frac{1}{6}}$, $c^- := 2\sqrt{\pi}(\frac{3}{2})^{\frac{1}{6}}$.

Algebraic solutions of Painlevé-III (D7)

Normalized solutions, Stokes phenomenon, and asymptotics. Buckingham and M., `arXiv:2202.04217`



These choices yield three simultaneous solutions of the Lax pair for the seed, denoted $\mathbf{\Psi}_\infty^\pm(\lambda, x)$ for $\lambda \in D_\infty^\pm$ and $\mathbf{\Psi}_0(\lambda, x)$ for $\lambda \in D_0$. They are analytic functions of λ in their domains of definition.

The solutions are normalized in the sense that as $\lambda \to \infty$ from within $D_\infty^\pm$,

$$\mathbf{\Psi}_\infty^\pm(\lambda, x)\mathrm{e}^{\mathrm{i}x\lambda\sigma_3} \sim \mathbb{I} + \sum_{p=1}^{\infty}\left(\frac{\mathrm{i}\lambda}{x}\right)^{-p}\mathbf{A}_p(x).$$

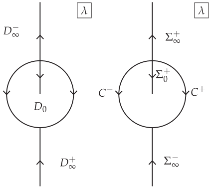
It holds also that in the limit $\lambda \to 0$,

$$\mathbf{\Psi}_0(\lambda, x)\mathrm{e}^{-\mathrm{i}x(\mathrm{i}x\lambda)^{-1/2}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mathrm{e}^{\frac{1}{3}\mathrm{i}\pi} \\ \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi} & 1 \end{bmatrix}\left(\frac{\mathrm{i}\lambda}{x}\right)^{-\frac{1}{4}\sigma_3} \sim \sum_{p=0}^{\infty}\left(\frac{\mathrm{i}\lambda}{x}\right)^{p}\mathbf{B}_p(x).$$

Algebraic solutions of Painlevé-III (D7)

Normalized solutions, Stokes phenomenon, and asymptotics. Buckingham and M., `arXiv:2202.04217`

Because $\mathrm{Ai}(\xi) + \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}\mathrm{Ai}(\mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}\xi) + \mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi}\mathrm{Ai}(\mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi}\xi) = 0$, we can relate the three solutions to each other, i.e., compute jumps across the contours in the diagram. We also have to take into account that $\lambda \mapsto \xi$ has a branch cut on the positive imaginary axis.



We set $\mathbf{\Psi}(\lambda, x) := \mathbf{\Psi}_\infty^\pm(\lambda, x)$ for $\lambda \in D_\infty^\pm$ and $\mathbf{\Psi}(\lambda, x) := \mathbf{\Psi}_0(\lambda, x)$ for $\lambda \in D_0$. Then, $\mathbf{\Psi}_+(\lambda, x) = \mathbf{\Psi}_-(\lambda, x)\mathbf{V}$, where $\mathbf{V}$ is the piecewise-constant matrix

$$\begin{bmatrix} 1 & -\mathrm{i} \\ 0 & 1 \end{bmatrix} \text{ for } \lambda \in \Sigma_\infty^-; \quad \begin{bmatrix} 1 & \mathrm{e}^{-\frac{1}{6}\mathrm{i}\pi} \\ \mathrm{e}^{-\frac{5}{6}\mathrm{i}\pi} & 0 \end{bmatrix} \text{ for } \lambda \in \Sigma_0^+; \quad \begin{bmatrix} 1 & 0 \\ \mathrm{i} & 1 \end{bmatrix} \text{ for } \lambda \in \Sigma_\infty^+;$$

$$\begin{bmatrix} 0 & \mathrm{e}^{-\frac{1}{3}\mathrm{i}\pi} \\ \mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi} & \mathrm{e}^{-\frac{5}{6}\mathrm{i}\pi} \end{bmatrix} \text{ for } \lambda \in C^-; \quad \begin{bmatrix} \mathrm{e}^{-\frac{5}{6}\mathrm{i}\pi} & 0 \\ \mathrm{e}^{\frac{1}{3}\mathrm{i}\pi} & \mathrm{e}^{\frac{5}{6}\mathrm{i}\pi} \end{bmatrix} \text{ for } \lambda \in C^+.$$

Algebraic solutions of Painlevé-III (D7)

The matrix $\mathbf{\Psi}(\lambda, x)$ therefore satisfies the conditions of a Riemann-Hilbert problem. We can replace the index $n = 0$ by an arbitrary integer in the conditions by working out the effect of iterated Schlesinger transformations. *This effect is somewhat unusual for the PIII (D7) problem.*

The relevant gauge transformations are described in Kitaev-Vartanian. For instance, a transformation to map $n = 0$ to $n = 1$ is to replace $\mathbf{\Psi}^{(0)}(\lambda, x) := \mathbf{\Psi}(\lambda, x)$ with $\mathbf{\Psi}^{(1)}(\lambda, x) := \mathbf{G}^{(0)\uparrow}(\lambda, x)\mathbf{\Psi}^{(0)}(\lambda, x)$ where

$$\mathbf{G}^{(0)\uparrow}(\lambda, x) :=$$
$$\left(\begin{bmatrix} 1 & 0 \\ -A_{1,21}(x) & 1 \end{bmatrix} + \frac{x}{\mathrm{i}\lambda}\begin{bmatrix} 0 & -B_{0,12}(x)/B_{0,22}(x) \\ 0 & A_{1,21}(x)B_{0,12}(x)/B_{0,22}(x) \end{bmatrix}\right)\left(\frac{\mathrm{i}\lambda}{x}\right)^{-\frac{1}{2}\sigma_3}.$$

This preserves analyticity in the three domains $D_\infty^\pm$ and D_0, and it leaves $\mathbf{V}$ invariant except on $\Sigma_0^+ \cup \Sigma_\infty^+$ where it induces $\mathbf{V} \mapsto -\mathbf{V}$.

However, it modifies the asymptotic behavior as $\lambda \to \infty$:

$$\mathbf{\Psi}^{(1)}(\lambda,x)\mathrm{e}^{\mathrm{i}\lambda\sigma_3} \sim \left(\mathbb{I} + \sum_{p=1}^{\infty}\left(\frac{\mathrm{i}\lambda}{x}\right)^{-p}\mathbf{A}_p^{(1)}(x)\right)\left(\frac{\mathrm{i}\lambda}{x}\right)^{\sigma_3}$$

and as $\lambda \to 0$:

$$\mathbf{\Psi}^{(1)}(\lambda,x)\mathrm{e}^{-\mathrm{i}x(\mathrm{i}x\lambda)^{-1/2}\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & \mathrm{e}^{\frac{1}{3}\mathrm{i}\pi} \\ \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi} & 1\end{bmatrix}\left(\frac{\mathrm{i}\lambda}{x}\right)^{-\frac{1}{4}\sigma_3} \sim \left(\sum_{p=0}^{\infty}\left(\frac{\mathrm{i}\lambda}{x}\right)^{p}\mathbf{B}_p^{(1)}(x)\right)\left(\frac{\mathrm{i}\lambda}{x}\right)^{-\frac{1}{2}\sigma_3}$$

So, the power of $(\mathrm{i}\lambda/x)^{\sigma_3}$ has increased by 1 at $\lambda = \infty$ and has decreased by $\frac{1}{2}$ at $\lambda = 0$ (and the coefficients have been modified).

Now we apply the "increment-n" transformation again, getting $\mathbf{\Psi}^{(2)}(\lambda, x) := \mathbf{G}^{(1)\uparrow}(\lambda, x)\mathbf{\Psi}^{(1)}(\lambda, x)$ where

$$\mathbf{G}^{(1)\uparrow}(\lambda, x) := \left(\begin{bmatrix} 1 & 0 \\ -A_{1,21}^{(1)}(x) & 1 \end{bmatrix} + \frac{x}{\mathrm{i}\lambda}\begin{bmatrix} 0 & -B_{0,11}^{(1)}(x)/B_{0,21}^{(1)}(x) \\ 0 & A_{1,21}^{(1)}(x)B_{0,11}^{(1)}(x)/B_{0,21}^{(1)}(x) \end{bmatrix}\right)\left(\frac{\mathrm{i}\lambda}{x}\right)^{-\frac{1}{2}\sigma_3}.$$

To get the necessary potentials out of $\mathbf{\Psi}^{(1)}(\lambda, x)$ instead of out of $\mathbf{\Psi}^{(0)}(\lambda, x)$ it is necessary to replace the ratio of second-column entries of $\mathbf{B}(x) = \mathbf{B}^{(0)}(x)$ with the ratio of first-column entries of $\mathbf{B}^{(1)}(x)$.

Again, analyticity in the three domains $D_\infty^\pm$ and D_0 is preserved, and it leaves $\mathbf{V}$ invariant except on $\Sigma_0^+ \cup \Sigma_\infty^+$ where it induces $\mathbf{V} \mapsto -\mathbf{V}$ (and hence flips the sign back to the original).

Then, one checks that as $\lambda \to \infty$,

$$\mathbf{\Psi}^{(2)}(\lambda, x)\mathrm{e}^{\mathrm{i}\lambda\sigma_3} \sim \left(\mathbb{I} + \sum_{p=1}^{\infty} \left(\frac{\mathrm{i}\lambda}{x}\right)^{-p} \mathbf{A}_p^{(2)}(x)\right) \left(\frac{\mathrm{i}\lambda}{x}\right)^{2\sigma_3},$$

so the power of $(\mathrm{i}\lambda/x)^{\sigma_3}$ at $\lambda = \infty$ has increased again by 1. However, as $\lambda \to 0$,

$$\mathbf{\Psi}^{(2)}(\lambda, x)\mathrm{e}^{-\mathrm{i}x(\mathrm{i}x\lambda)^{-1/2}\sigma_3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{e}^{\frac{1}{3}\mathrm{i}\pi} \\ \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi} & 1 \end{bmatrix} \left(\frac{\mathrm{i}\lambda}{x}\right)^{-\frac{1}{4}\sigma_3} \sim \sum_{p=0}^{\infty} \left(\frac{\mathrm{i}\lambda}{x}\right)^{p} \mathbf{B}_p^{(2)}(x),$$

which means that the power of $(\mathrm{i}\lambda/x)^{\sigma_3}$ at $\lambda = 0$ has *increased* by $\frac{1}{2}$ and hence *reverted back* to that for $n = 0$.

Thus $\mathbf{W}^{(n)}(\lambda, x) := \mathbf{\Psi}^{(n)}(\lambda, x)\mathrm{e}^{\mathrm{i}(x\lambda - x(\mathrm{i}x\lambda)^{-1/2})\sigma_3}$ solves the following.

RH Problem (Algebraic solutions of PIII (D7) for $x > 0$)

Given $x > 0$ and $n \in \mathbb{Z}$, seek a 2×2 matrix function $\lambda \mapsto \mathbf{W}^{(n)}(\lambda, x)$ such that:

- *Analyticity:* $\mathbf{W}^{(n)}(\lambda, x)$ *is analytic for* $\lambda \in \mathbb{C} \setminus (\Sigma_\infty^- \cup \Sigma_\infty^+ \cup \Sigma_0^+ \cup C^+ \cup C^-)$.
- *Jump conditions: with* $\widetilde{\mathbf{V}} = \mathbf{V}$ *on* $\Sigma_\infty^- \cup C^+ \cup C^-$ *and* $\widetilde{\mathbf{V}} := (-1)^n \mathbf{V}$ *on* $\Sigma_0^+ \cup \Sigma_\infty^+$,

$$\mathbf{W}_+^{(n)}(\lambda, x) = \mathbf{W}_-^{(n)}(\lambda, x)\mathrm{e}^{-\mathrm{i}(x\lambda - x(\mathrm{i}x\lambda_-)^{-1/2})\sigma_3}\widetilde{\mathbf{V}}\mathrm{e}^{\mathrm{i}(x\lambda - x(\mathrm{i}x\lambda_+)^{-1/2})\sigma_3}.$$

- *Normalization:* $\mathbf{W}^{(n)}(\lambda, x)\left(\frac{\mathrm{i}\lambda}{x}\right)^{\frac{1}{2}n\sigma_3} \to \mathbb{I}$ *as* $\lambda \to \infty$.
- *Behavior at the origin: the following limit exists:*

$$\mathbf{B}_0^{(n)}(x) := \lim_{\lambda \to 0} \mathbf{W}^{(n)}(\lambda, x)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mathrm{e}^{\frac{1}{3}\mathrm{i}\pi} \\ \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi} & 1 \end{bmatrix}\left(\frac{\mathrm{i}\lambda}{x}\right)^{-\frac{1}{4}(-1)^n\sigma_3}.$$

The Riemann-Hilbert problem therefore takes a different form for even and odd n. A function $u_n(x)$ is extracted from $\mathbf{W}^{(n)}(\lambda, x)$ as follows:

$$u_n(x) = \begin{cases} e^{-\frac{5}{6}i\pi} x B_{0,12}^{(n)} B_{0,22}^{(n)}(x), & n \text{ even}, \\ e^{\frac{5}{6}i\pi} x B_{0,11}^{(n)}(x) B_{0,21}^{(n)}(x), & n \text{ odd}. \end{cases}$$

A dressing argument shows that $u_n(x)$ is a solution of PIII (D7) with $\beta = 2n$.

The Bäcklund transformation that can be derived directly from the Schlesinger gauge matrix $\mathbf{G}^{(n)\uparrow}(\lambda, x)$ preserves the algebraic character of u_n under $n \mapsto n + 1$.

Hence $u_n(x)$ is the unique solution of PIII (D7) that is a rational function of $Z = x^{\frac{1}{3}}$.

Algebraic solutions of Painlevé-III (D7)

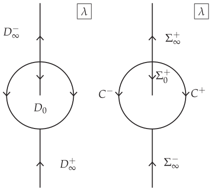
Rotating the branch cut. Buckingham and M., `arXiv:2202.04217`

An explicitly-related matrix $\mathbf{Y}^{(n)}(\lambda, x)$ is easier to work with. In terms of $\widetilde{\mathbf{Y}}^{(n)}(\lambda, x) := \mathbf{Y}^{(n)}(\lambda, x)\mathrm{e}^{-(\mathrm{i}x\lambda - x(-\mathrm{i}x\lambda)^{-1/2})\sigma_3}$ (note that the exponential has a downward branch cut instead of upward), its jump conditions are

$$\widetilde{\mathbf{Y}}_+^{(n)}(\lambda, x) = \widetilde{\mathbf{Y}}_-^{(n)}(\lambda, x)\begin{bmatrix} 1 & 0 \\ \mathrm{i} & 1 \end{bmatrix}, \quad \lambda \in \Sigma_\infty^+ \cup \Sigma_0^+,$$

$$\widetilde{\mathbf{Y}}_+^{(n)}(\lambda, x) = \widetilde{\mathbf{Y}}_-^{(n)}(\lambda, x)(-1)^n\begin{bmatrix} 1 & -\mathrm{i} \\ 0 & 1 \end{bmatrix}, \quad \lambda \in \Sigma_\infty^-,$$

$$\widetilde{\mathbf{Y}}_+^{(n)}(\lambda, x) = \widetilde{\mathbf{Y}}_-^{(n)}(\lambda, x)\begin{bmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{bmatrix}, \quad \lambda \in C^+ \cup C^-,$$



and a new jump appears on Σ_0^-, the oriented segment joining $\lambda = 0$ to $\lambda = -\mathrm{i}$:

$$\widetilde{\mathbf{Y}}_+^{(n)}(\lambda, x) = \widetilde{\mathbf{Y}}_-^{(n)}(\lambda, x)\mathrm{e}^{-\mathrm{i}x\lambda\sigma_3}(-1)^n\mathrm{i}\sigma_1\mathrm{e}^{\mathrm{i}x\lambda\sigma_3}, \quad \lambda \in \Sigma_0^-.$$

Its asymptotic behavior as $\lambda \to \infty, 0$ is implied by its relation to $\mathbf{W}^{(n)}(\lambda, x)$.

Recalling the basic scaling $x = n^{\frac{3}{2}} y$, we also scale the spectral parameter by $\lambda = n^{-\frac{1}{2}} y^{-1} \eta$ so the fundamental exponent becomes

$$\mathrm{i}x\lambda - x(-\mathrm{i}x\lambda)^{-\frac{1}{2}} = n\Phi(\eta, y), \quad \Phi(\eta, y) := \mathrm{i}\eta - y(-\mathrm{i}\eta)^{-\frac{1}{2}}.$$

Then we set $\mathbf{Z}^{(n)}(\eta, y) := (ny)^{-n\sigma_3} \mathbf{Y}^{(n)}(x, \lambda)$. We also rescale the jump contour by $n^{-\frac{1}{2}} y^{-1}$ so it becomes fixed in the η-plane.

The key to analyze $\mathbf{Z}^{(n)}(\eta, y)$ for large $n > 0$ is to introduce a scalar function $\eta \mapsto g(\eta, y)$ independent of n and analytic in the complement of the jump contour such that:

$$\eta \mapsto F(\eta, y) := \left(\frac{\partial g}{\partial \eta}(\eta, y) - \frac{\partial \Phi}{\partial \eta}(\eta, y) \right)^2$$

is continuous except at $\eta = 0$, and there is some $g_0(y)$ so that as $\eta \to \infty$, $g(\eta, y) = -\frac{1}{2} \log(-\mathrm{i}\eta) + g_0(y) + o(1)$ as $\eta \to \infty$.

We use such a $g(\eta, y)$ to obtain a new unknown by $\mathbf{M}^{(n)}(\eta, y) := \mathrm{e}^{ng_0(y)\sigma_3}\mathbf{Z}^{(n)}(\eta, y)\mathrm{e}^{-ng(\eta,y)\sigma_3}$. The normalization condition at $\eta = \infty$ is simplified so that $\mathbf{M}^{(n)}(\eta, y) \to \mathbb{I}$ as $\eta \to \infty$. There are also constant unit-determinant matrices $\widetilde{\mathbf{E}}^{\pm}$ such that

$$\lim_{\substack{\eta \to 0 \\ \pm\mathrm{Re}(\eta)>0}} \mathbf{M}^{(n)}(\eta, y)\widetilde{\mathbf{E}}^{\pm}(-\mathrm{i}\eta)^{-\frac{1}{4}(-1)^n\sigma_3}$$

exists unambiguously.

To determine g, we consider the function $F(\eta, y)$ that is analytic except at $\eta = 0$ and that necessarily satisfies

$$F(\eta, y) = \begin{cases} -1 + \mathrm{i}\eta^{-1} + \mathcal{O}(\eta^{-2}), & \eta \to \infty \\ -\frac{1}{4}y^2(-\mathrm{i}\eta)^{-3} + \mathcal{O}(\eta^{-2}), & \eta \to 0. \end{cases}$$

Therefore, $F(\eta, y) = P(-\mathrm{i}\eta, y)(-\mathrm{i}\eta)^{-3}$ where $P(\mu, y)$ is a polynomial of the form

$$P(\mu, y) := -\mu^3 + \mu^2 + c\mu - \tfrac{1}{4}y^2, \quad c = \mathrm{const}.$$

The polynomial $P(\mu, y)$ is related to the formal approximating equation

$$U'(z)^2 = \frac{16}{y}U(z)^3 + 2EU(z)^2 - \frac{4}{y}U(z) + 1, \quad E = \text{constant}.$$

Indeed, if we match the integration constant E to c by $y^2E = -8c$, then

$$-\frac{64U^3}{y^3}P\left(\frac{y}{2U}; y\right) = \frac{16}{y}U^3 + 2EU^2 - \frac{4}{y}U + 1.$$

The correct assumption to make for large y is that c is chosen such that $P(\mu, y)$ has a double root $\mu = d$ and a simple root $\mu = s$. This implies:

$$s(s-1)^2 = -y^2 \quad \text{(cubic equation for } s\text{)}.$$

For $y > 0$ large enough, we select the unique negative real root which has asymptotic behavior $s = -y^{\frac{2}{3}}(1 + o(1))$ as $y \to +\infty$. Then, we have found $g(\eta, y)$ explicitly.

We get, for $y > 0$ sufficiently large:

$$g(\eta, y) = (-\mathrm{i}\eta - s + 1)(-\mathrm{i}\eta - s)^{\frac{1}{2}}(-\mathrm{i}\eta)^{-\frac{1}{2}} + \frac{1}{2}\log\left(\frac{(-\mathrm{i}\eta - s)^{\frac{1}{2}} - (-\mathrm{i}\eta)^{\frac{1}{2}}}{(-\mathrm{i}\eta - s)^{\frac{1}{2}} + (-\mathrm{i}\eta)^{\frac{1}{2}}}\right) + \mathrm{i}\eta - y(-\mathrm{i}\eta)^{-\frac{1}{2}}.$$

This has the desired asymptotic behavior as $\eta \to \infty$ with constant term

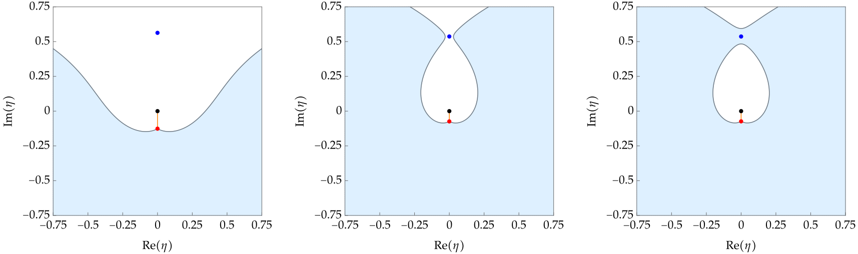
$$g_0(y) := 1 - \frac{3}{2}s + \frac{1}{2}\ln\left(-\frac{1}{4}s\right), \quad s = s(y) < 0.$$

The effect on the jump conditions is that the exponent $\Phi(\eta, y)$ is replaced with $-h(\eta, y)$, where $h(\eta, y) := g(\eta, y) - \Phi(\eta, y)$. One can easily plot sign charts of $\mathrm{Re}(h(\eta, y))$ in the η-plane to assess where jump matrices will decay to the identity as $n \to \infty$.

Algebraic solutions of Painlevé-III (D7)

Application: the limit $n \to \infty$. Buckingham and M., `arXiv:2202.04217`

Here are some sign charts for $\mathrm{Re}(h(\eta, y))$ for positive real y:



$y = 0.4$ $\qquad$ $y = 0.292$ $\qquad$ $y = -0.291$

We can lay the jump contour over the landscapes shown in the left two figures so that all jump matrices decay rapidly to $\mathbb{I}$ except on Σ_0^- (the orange segment). Here, the jump reads, exactly

$$\mathbf{M}_+^{(n)}(\eta, y) = \mathbf{M}_-^{(n)}(\eta, y) \begin{bmatrix} 0 & (-1)^n \mathrm{i} \\ (-1)^n \mathrm{i} & 0 \end{bmatrix}, \quad \eta \in \Sigma_0^- .$$

Algebraic solutions of Painlevé-III (D7)

Application: the limit $n \to \infty$. Buckingham and M., `arXiv:2202.04217`

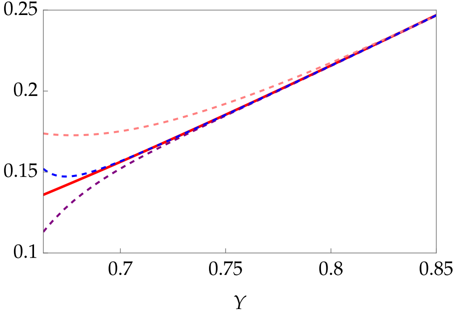
This jump condition is solved exactly by the *outer parametrix* $\breve{\mathbf{M}}^{(n),\mathrm{out}}(\eta, y)$, which also tends to $\mathbb{I}$ as $\eta \to \infty$:

$$\breve{\mathbf{M}}^{(n),\mathrm{out}}(\eta, y) := \mathrm{e}^{(\frac{1}{4}+\frac{1}{2}n)\pi\mathrm{i}\sigma_3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \left(\frac{\eta - \mathrm{i}s}{\eta} \right)^{\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix} \mathrm{e}^{-(\frac{1}{4}+\frac{1}{2}n)\pi\mathrm{i}\sigma_3}.$$

A local parametrix built from Airy functions is needed to resolve nonuniformity of the jump matrices near the point $\eta = \mathrm{i}s$. Then comparing the parametrix to $\mathbf{M}^{(n)}(\eta, y)$ leads to the conditions of a small-norm Riemann-Hilbert problem, and hence to an accurate approximation for $u_n(x)$.

Theorem

There exists $y_c > 0$ such that $u_n(n^{\frac{3}{2}}y) = n^{\frac{1}{2}}U + \mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$ for $y > y_c$, where $U = U(y)$ is the positive real solution of the equilibrium cubic $8U^3 + 2U - y = 0$. The error estimate is valid uniformly for $y \geq y_c + \delta$ for any $\delta > 0$.



The graph of $U(Y^3)$ (red) and the graphs of $n^{-\frac{1}{2}}u_n(n^{\frac{3}{2}}Y^3)$ for $n = 2, 5, 10$ (pink, purple, and blue dotted lines) for $Y > y_c^{\frac{1}{3}}$.

The analysis can also be carried out for complex y with $|y|$ large enough. The lower bound for $|y|$ depends on $\arg(y)$:

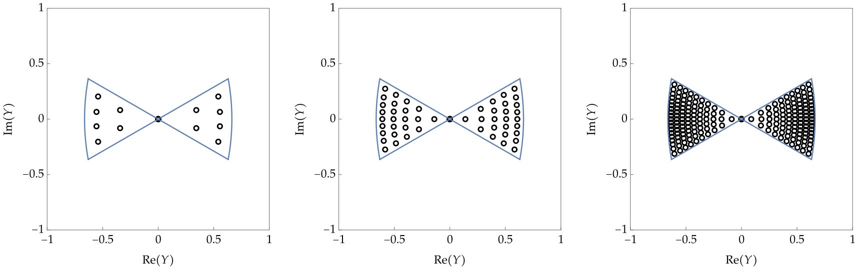
Theorem

$u_n(n^{\frac{3}{2}}Y^3) = n^{\frac{1}{2}}U + \mathcal{O}(n^{-\frac{1}{2}})$ holds uniformly for Y in compact subsets of a certain unbounded domain $\mathcal{E}$, where U satisfies $8U^3 + 2U - Y^3 = 0$ and is analytic for $Y \in \mathcal{E}$ with $U = \frac{1}{2}Y$ as $Y \to \infty$. In particular, $u_n(n^{\frac{3}{2}}Y^3)$ is pole- and zero-free on $\mathcal{E}$ for n large. The complement $\mathbb{C} \setminus \mathcal{E}$ has a "bow-tie" shape with boundary consisting of two line segments joining the pairs $\pm 2^{\frac{1}{3}}3^{-\frac{1}{2}}\mathrm{e}^{\frac{1}{6}\mathrm{i}\pi}$ and their conjugates, and two curved arcs described by $\mathrm{Re}(h(\mathrm{id}, Y^3)) = 0$.

The boundary curves can be easily plotted.

Algebraic solutions of Painlevé-III (D7)

Here is the boundary curve in the Y-plane shown with the zeros of the Ohyama polynomials $P_n(n^{\frac{1}{2}}Y)$.



$n = 5$ $n = 10$ $n = 20$

Dropping the assumption that $P(\mu, y)$ have a double root, the cubic now has simple roots and can be mapped onto the Weierstraß cubic. The integration constant E now would have to be determined as a function of $y = Y^3$ by a suitable *Boutroux condition* to allow the analysis to proceed. The details are for the future. . .

Thank you!

Why straight lines?

The condition $\mathrm{Re}(h(\mathrm{i}d, \Upsilon^3)) = 0$ can be written explicitly in terms of the simple root s satisfying the cubic equation

$$s(s-1)^2 = -\Upsilon^6$$

and $-\Upsilon^6 > 0$ holds on the imaginary Υ-axis and the rays through the "bow-tie" corner points.