Stationary measure for the open KPZ equation

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Random growth models $(1+1)d$

Random Deposition

Ballistic Deposition
Random growth models (1+1)d

Random Deposition

Ballistic Deposition
Open KPZ

**KPZ:** \( \partial_T H(T, X) = \frac{1}{2} \partial_X^2 H(T, X) + \frac{1}{2} \left( \partial_X H(T, X) \right)^2 + \xi(T, X) \)
Open KPZ

**KPZ:** \( \partial_T H(T, X) = \frac{1}{2} \partial_X^2 H(T, X) + \frac{1}{2} (\partial_X H(T, X))^2 + \xi(T, X) \)

Open KPZ: for all \( T > 0 \), we impose with \( u, v \in \mathbb{R} \)

\[ \partial_X H(T, X) \big|_{X=0} = u, \quad \partial_X H(T, X) \big|_{X=1} = -v. \]
Stochastic heat equation (SHE)

SHE with inhomogeneous Robin boundary conditions:

\[ \partial_T Z(T, X) = \frac{1}{2} \partial_X^2 Z(T, X) + \xi(T, X)Z(T, X), \]

\[ T \geq 0 \text{ and } X \in [0, 1]; \]

\[ \partial_Z Z(T, X) \Big|_{X=0} = \left( u - \frac{1}{2} \right) Z(T, 0), \quad \partial_Z Z(T, X) \Big|_{X=1} = -\left( v - \frac{1}{2} \right) Z(T, 0), \]

for all \( T > 0 \).

The Hopf-Cole solution to the open KPZ is defined as \( H(T, X) := \log Z(T, X) \), which formally solves the open KPZ equation.
Definition: A stationary measure for the open KPZ equation is the law on a random function $H_0 : [0, 1] \to \mathbb{R}$ with $H_0(0) = 0$ such that the law of $X \mapsto H(T, X) - H(T, 0)$, as a process in $X$, is $T$-independent for all $T \geq 0$ if we start with $H(0, X) = H_0(X)$.

[Funaki–Quastel ’15], [Hairer–Mattingly ’18]
Open ASEP

Open ASEP (with system size $N = 10$) and its height function.
Height function

Height function is defined for $t \geq 0$ and $x \in \{0, \ldots, N\}$ as

$$h_N(t, x) := h_N(t, 0) + \sum_{i=1}^{x} (2\tau_i(t) - 1) \text{ with } \tau_i(t) = 0 \text{ or } 1 \text{ and}$$

$$h_N(t, 0) := -2N_N(t),$$

where the net current $N_N(t)$ equals the number of particles to enter into site 1 minus the number of particles to exit from site 1 up to time $t$. 
Open ASEP is an irreducible continuous time Markov chain with finite state space. We will denote this by $\pi_N(\tau)$ its unique invariant measure. For a function $f$ on the state space denote its expectation under the invariant measure by

$$\langle f \rangle_N := \sum_{\tau \in \{0,1\}^{\{1,\ldots,N\}}} f(\tau) \cdot \pi_N(\tau).$$

Stationary current $J_N := \frac{\langle \alpha(1 - \tau_1) - \gamma \tau_1 \rangle_N}{1 - q}$
Phase diagram

[Derrida–Evans–Hakim–Pasquier ’93], [Sandow ’94], [Uchiyama–Sasamoto–Wadati ’03]
Law of Large Numbers

Theorem:

For \( \tau \) distributed according to the invariant measure \( \pi_N \) the following limits hold for \( x \in [0, 1] \)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{[Nx]} \tau_j = \begin{cases} 
\frac{1}{2}x, & \text{maximal current phase;} \\
\rho_1 x, & \text{low density phase;} \\
\rho_r x, & \text{high density phase.}
\end{cases}
\]
Weakly asymmetric scaling

Let \( u, v \in \mathbb{R} \)

\[
q = \exp \left( -\frac{2}{\sqrt{N}} \right);
\]

\[
\alpha = \frac{1}{1 + q^u}, \quad \beta = \frac{1}{1 + q^v}, \quad \gamma = \frac{q^{u+1}}{1 + q^u}, \quad \delta = \frac{q^{v+1}}{1 + q^v}.
\]

Define \( H_N(X) := N^{-1/2} h_N(NX) \).

[Bertini–Giacomin ’97], [Corwin–Shen ’16], [Parekh ’19]
Corwin-K. '21:

- **WASEP-stationary measures:** For $u, v \in \mathbb{R}$ we construct stationary measures $H_{u,v}$ for the open KPZ equation as the limits of $H_{u,v}^{(N)}(\cdot)$;

- **Characterization:** For $u, v \in \mathbb{R}$ with $u + v > 0$ we characterize $H_{u,v}$ through an explicit multi-point Laplace transform.
Theorem (Corwin–K. ’21):

- **WASEP-stationary measures**: For $u, v \in \mathbb{R}$, the sequence of laws of $H_{u,v}^{(N)}(\cdot)$ are tight in the space of measures on $C[0,1]$. All subsequential limits $H_{u,v}$ are stationary measures for the open KPZ equation and are Hölder $1/2$– almost surely.

- **Coupling**: Let $M \in \mathbb{Z}_{\geq 2}$, $u_1 \leq \cdots \leq u_M$ and $v_1 \geq \cdots \geq v_M$. For the corresponding collection \( \{H_{u_i,v_i}\}_{i=1}^{M} \) of WASEP-stationary measures there exists a coupling such that for $0 \leq X \leq X' \leq 1$

\[
H_{u_i,v_i}(X') - H_{u_i,v_i}(X) \leq H_{u_j,v_j}(X') - H_{u_j,v_j}(X).
\]

- **Characterization**: For $u, v \in \mathbb{R}$ with $u + v > 0$, there is a unique WASEP-stationary measure $H_{u,v}$ for the open KPZ equation whose law is determined by its explicit multi-point Laplace transform.
A simple case for $u, v > 0$ and $c \in (0, 2u)$:

$$
\mathbb{E} \left[ e^{-cH_{u,v}(1)} \right] = e^{c^2/4} \cdot \frac{\int_{0}^{\infty} e^{-r^2} \cdot \left| \frac{\Gamma\left(\frac{c}{2} + u + ir\right)\Gamma\left(-\frac{c}{2} + v + ir\right)}{\Gamma(2ir)} \right|^2 dr}{\int_{0}^{\infty} e^{-r^2} \cdot \left| \frac{\Gamma(u + ir)\Gamma(v + ir)}{\Gamma(2ir)} \right|^2 dr}.
$$

$H_{u,v}(1)$ records the net height change across the interval $[0, 1]$. 

Laplace transform
Theorem (Bryc-Wesolowski ’17): In the fan region for $0 < t_1 \leq t_2 \leq \cdots \leq t_n$ the joint generating function of the stationary distribution of the ASEP

$$
\left\langle \prod_{j=1}^{N} t_j^{T_j} \right\rangle_N = \frac{\mathbb{E} \left[ \prod_{j=1}^{N} \left( 1 + t_j + 2\sqrt{t_j} Y_j \right) \right]}{2^N \mathbb{E} \left[ (1 + Y_1)^N \right]},
$$

where $\{Y_t\}_{t \geq 0}$ is the Askey-Wilson process with parameters determined by the model.

Connection to Askey-Wilson polynomials [Uchiyama-Sasamoto-Wadati ’03]
The Askey-Wilson process:

It is a continuous Markov process on \([-1,1]\) for

\[
t \in \left[q^2; \min(q^{-2}, 1/q^2)\right]
\]

with

\[
\pi_t(y) = \text{AW}_y \left( q^u \sqrt{t}, -q \sqrt{t}, \frac{q^v}{\sqrt{t}}, -\frac{q}{\sqrt{t}} \right), \text{ and for } s < t,
\]

\[
p_{s,t}(x, y) = \text{AW}_y \left( q^u \sqrt{t}, -q \sqrt{t}, \frac{\sqrt{s}}{\sqrt{t}} e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{t}} e^{-i\theta_x} \right),
\]

where \(x = \cos \theta_x\) and

\[
\text{AW}_x(a, b, c, d) = \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi(abcd; q)_\infty \sqrt{1-x^2}} \left| \frac{(e^{2i\theta})_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta})_\infty} \right|^2,
\]

\[
q-Pochhammer symbol: (z; q)_\infty = (1-z)(1-qz)(1-q^2z)\ldots
\]
Key asymptotic result

Proposition (Corwin–K '21):

Let $A^+[\kappa, z] = -\frac{\pi^2}{6\kappa} - \left( z - \frac{1}{2} \right) \log \kappa - \log \left[ \frac{\Gamma(z)}{\sqrt{2\pi}} \right]$ and

$A^-[\kappa, z] = \frac{\pi^2}{12\kappa} - \left( z - \frac{1}{2} \right) \log 2, \quad z \in \mathbb{C}, \kappa > 0.$

Then for $q = e^{-\kappa}$ we have

$$\log(q^z; q)_\infty = A^+[\kappa, z] + \text{Error}^+[\kappa, z];$$

$$\log(-q^z; q)_\infty = A^-[\kappa, z] + \text{Error}^- [\kappa, z],$$

with good control over the error term in $z$ and $\kappa$.

[Moak '84], [Olde Daalhuis '94], [Mcintosh '99], [Zhang '14]
Continuous dual Hahn Process

Let \( C_{uv} := \begin{cases} 2 & \text{if } u \leq 0 \text{ or } u \geq 1, \\ 2u & \text{if } u \in (0, 1). \end{cases} \)

For \( u, v > 0 \) and \( s \in [0, C_{uv}) \) define a measure \( p_s \) with density given by

\[
p_s(r) := \frac{(v + u)(v + u + 1)}{8\pi} \cdot \frac{\left| \Gamma\left(\frac{s}{2} + v + i \frac{\sqrt{r}}{2}\right) \cdot \Gamma\left(-\frac{s}{2} + u + i \frac{\sqrt{r}}{2}\right) \right|^2}{\sqrt{r} \cdot \left| \Gamma(i \sqrt{r}) \right|^2} 1_{r \geq 0}.
\]
Continuous dual Hahn Process

For $a \in \mathbb{R}$ and $b = \bar{c} \in \mathbb{C} \setminus \mathbb{R}$ with $\text{Re}(b) = \text{Re}(c) > 0$ let

$$\text{CDH}(x; a, b, c) :=$$

$$= \frac{1}{8\pi} \cdot \frac{\left| \Gamma\left( a + i \frac{\sqrt{x}}{2} \right) \cdot \Gamma\left( b + i \frac{\sqrt{x}}{2} \right) \cdot \Gamma\left( c + i \frac{\sqrt{x}}{2} \right) \right|^2}{\Gamma(a + b) \cdot \Gamma(a + c) \cdot \Gamma(b + c) \cdot \sqrt{x} \cdot |\Gamma(i\sqrt{x})|^2} 1_{x > 0}.$$ 

For $s, t \in [0, C_{u,v})$ with $s < t$ and $m, r \in (0, \infty)$ define

$$p_{s,t}(m, r) := \text{CDH}\left( r; u - \frac{t}{2}, \frac{t - s}{2} + i \frac{\sqrt{m}}{2}, \frac{t - s}{2} - i \frac{\sqrt{m}}{2} \right).$$
Explicit characterization

\( \vec{X} = (X_0, \ldots, X_{d+1}) \) where \( 0 = X_0 < X_1 < \cdots < X_d \leq X_{d+1} = 1 \),
\( \vec{c} = (c_1, \ldots, c_d) \) where \( c_1, \ldots, c_d > 0 \),
\( \vec{s} = (s_1 > \cdots > s_{d+1}) \) where \( s_k = c_k + \cdots + c_d \) and \( s_{d+1} = 0 \).

For any \( d \in \mathbb{Z}_{\geq 1} \) provided that \( s_1 < C_{u,v} \)

\[
\mathbb{E} \left[ e^{-\sum_{k=1}^{d} c_k H_{u,v}(X_k)} \right] = \frac{\mathbb{E} \left[ e^{\frac{1}{4} \sum_{k=1}^{d+1} (s_k^2 - T_{s_k})(X_k - X_{k-1})} \right]}{\mathbb{E} \left[ e^{-\frac{1}{4} T_0} \right]},
\]

where \( T_s \) is the continuous dual Hahn process.
Probabilistic description

\[ H_{u,v} : [0,1] \to \mathbb{R} \text{ is equal in law to } 2^{-1/2}B + X, \text{ where} \]

\( B : [0,1] \to \mathbb{R} \) has the law of a standard Brownian motion;

\( X : [0,1] \to \mathbb{R} \) is independent of \( B \) and is absolutely continuous with respect to that of a Brownian motion with diffusion coefficient 1/2 with an explicit Radon-Nikodym derivative.

[Bryc-Kuznetsov-Wang-Wesolowski '21], [Barraquand-Le Doussal '21]