Discrete flows of Goncharov-Kenyon integrable systems.

June 30, 2022

GK integrable systems.

Defined by a convex polygon Δ in the plane with integral vertices.

Phase space — pairs $(C, \mathcal{L})$.
$C \subset (\mathbb{C}^\times)^2$ a planar curve of genus g with Δ as a Newton polygon.
$\mathcal{L} \in \mathrm{Pic}^{g-1}(\bar{C})$ a line bundle of degree $g-1$ on the compactification $\bar{C}$ of C.
Hamiltonians and Casimirs: forgetting $\mathcal{L}$.

Polygons and curves:

$$C = \{(\lambda, \mu) | P(\lambda, \mu) = \sum_{ij \in \Delta} c_{ij} \lambda^i \mu^j = 0\}$$

Points inside $\Delta \leftrightarrow$ basis of holomorphic forms:

$$ij \leftrightarrow \text{Res} \frac{\lambda^i \mu^j d\lambda d\mu}{\lambda \mu P(\lambda, \mu)}$$

(Side (a, b) of the polygon) $\leftrightarrow m = \gcd(a, b)$ points of $\bar{C} \setminus C$.

$$(a/m, b/m) \leftrightarrow (\text{ord}\, \mu, -\text{ord}\, \lambda)$$

$P(\lambda, \mu)$ and $\alpha P(\beta\lambda, \gamma\mu)$ as well as $\lambda^e \mu^f P(\lambda^a \mu^b, \lambda^c \mu^d)$ with $ad - bc = 1$ define the same curve.
g=(genus of C)=number of points inside Δ
n=(number of points of $\bar{C} \setminus C$)=(number of boundary points of Δ).

For any side of Δ one can change coordinates and make it connect (0,0) and (k,0) and thus the polynomial becomes

$$P(\lambda, \mu) = P_0(\lambda) + \mu P_1(\lambda) + \cdots$$

Thus roots of P_0 are points at infinity.

Group $\mathcal{G}$ of discrete flows.

Div_0 — group of divisors of degree 0 supported on $\bar{C} \setminus C$.
$\mathrm{div} \subset \mathrm{Div}_0$ — subgroup generated by (λ) and (μ).

$$\mathcal{G} = \frac{\mathrm{Div}_0}{\mathrm{div}}.$$

Obviously acts on the phase space by $(C, \mathcal{L}) \mapsto (C, \mathcal{L} + d)$.
The group $\mathcal{G}$ does not depend on the curve C if the sides of Δ are of length 1 or if the Casimirs are fixed.
Let A — $n \times 2$ matrix given by the sides of the Δ.

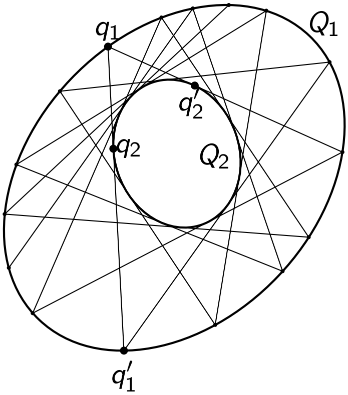
$$\mathcal{G} = \mathbb{Z}^{n-1}/\mathrm{Im}\ A.$$

$$\mathcal{G} = \frac{\mathbb{Z}\text{-valued function on the vertices of } \Delta}{\text{functions} \quad ai + bj + c}$$

Example: for $A(\lambda, \lambda^{-1}) \in \widehat{GL}(N)/H$.
$f : A^{-}(\lambda^{-1})A^{+}(\lambda) \mapsto A^{+}(\lambda)A^{-}(\lambda(\lambda^{-1}))$.

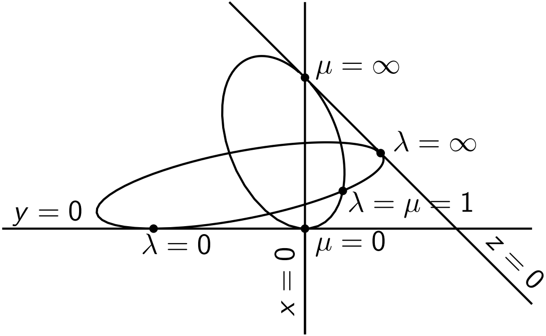
Example: Poncelet porism. (Exactly 200 years old!)

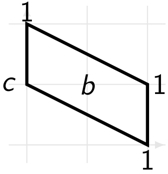
Example: Poncelet



C — Pairs of points $(q_1, q_2) \in (Q_1, Q_2)$ such that the segment (q_1, q_2) is tangent to Q_2.
$\sigma_1(q_1, q_2) \mapsto (q_1', q_2),\ \sigma_2(q_1, q_2) \mapsto (q_1, q_2'),$

Coordinates:

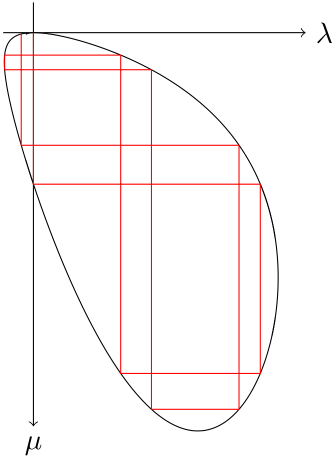




$$P(\lambda,\mu) = \lambda^2 + \mu^2 - \lambda^2\mu + b\lambda\mu + c\mu$$

$\sigma_1 : (\lambda,\mu) \mapsto (\lambda, \lambda^2\mu^{-1})$ and $\sigma_2 : (\lambda,\mu) \mapsto (\mu\lambda^{-1}\dfrac{\mu + c}{1+\mu}, \mu)$

Trajectory in the (λ, μ)-plane



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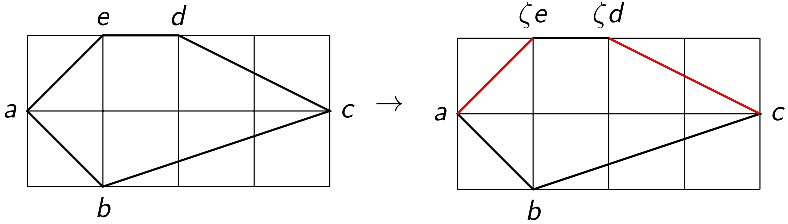
Higher genus generalization:

Aim: Define involutions on the phase space of GK integrable system.
Fix the value of the Casimirs (coefficients in the corners of Δ).
(Phase space)=$\{$Curve,divisor of degree $g\}$=(collections of points $(\lambda_1, \mu_1), \ldots, (\lambda_g, \mu_g)$)



Claim: The curves containing the divisor and corresponding to the two sets of Casimirs intersect in $2g$ points.
Proof: Generic curves with the same Newton polygon Δ intersect in $2S = 2g + n - 2$ points, where S is the area of Δ. But $n - 2$ points of these curves intersect at infinity. □