The complex elliptic Ginibre ensemble at weak non-Hermiticity

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Figure 1: Alex Little.

and based on the forthcoming arXiv:220?..?????
What’s the problem?

Random matrices without any symmetry constraints often appear in natural phenomenological models. For instance, the time evolution of a system of interacting agents \( \mathbf{u} = (u_1, \ldots, u_n) \) may be described by a linear ODE system of the form

\[
\frac{d}{dt} \mathbf{u}(t) = \mathbf{X} \mathbf{u}(t)
\]

where we assume the coefficient matrix \( \mathbf{X} \) to be random (May 1972). Such models have been studied extensively in neuroscience and ecology and they often appear in the form

\[
\frac{d}{dt} \mathbf{u}(t) = (-\mathbf{I} + g\mathbf{X}) \mathbf{u}(t), \tag{1}
\]
where the identity matrix represents an exponential decay at unit rate and the coupling constant $g > 0$ expresses the strength of the random couplings in the model. The main task is to tune $g$ so that the resulting system is stable. However, the maximal growth rate of the solution of (1) is determined by the maximal real part of the spectrum of $-I + gX$, thus we wish to understand accurately the real part of the rightmost eigenvalue of a large non-Hermitian random matrix.

We will achieve this for an interpolating random matrix ensemble.
Consider the Gaussian Unitary Ensemble (GUE), i.e. matrices

\[ X = \frac{1}{2}(Y + Y^\dagger) \in \mathbb{C}^{n \times n} : \quad Y_{jk} \overset{iid}{\sim} \mathcal{N} \left(0, \frac{1}{\sqrt{2}}\right) + i\mathcal{N} \left(0, \frac{1}{\sqrt{2}}\right) \]

as in (Porter 1965). Equivalently think of a log-gas system \( \{x_j\}_{j=1}^n \subset \mathbb{R} \) with joint pdf for the particles’ locations equal to (Mehta 1967)

\[
p_n(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |x_k - x_j|^2 \exp \left(-\sum_{j=1}^{n} x_j^2\right).
\]

Question: How do the particles \( \{x_j\}_{j=1}^n \) behave for large \( n \)?
The particles $\{x_j\}_{j=1}^n$ form a DPP on $\mathbb{R}$ (Dyson 1970),

$$R_k(x_1, \ldots, x_n) := \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(x_1, \ldots, x_n) \prod_{j=k+1}^n dx_j = \det [K_n(x_i, x_j)]_{i,j=1}^k$$

with correlation kernel

$$K_n(x, y) = \frac{e^{-\frac{1}{2}(x^2+y^2)}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{1}{2^k k!} H_k(x) H_k(y), \quad H_n(z) = \frac{n!}{2\pi i} \oint e^{2zt-t^2} \frac{dt}{t^{n+1}}.$$ 

Now analyze $R_k$ asymptotically in different scaling regimes:
(A) The global eigenvalue regime: define the ESD

$$\mu_X(s) = \frac{1}{n} \# \{1 \leq j \leq n, \ x_j \leq s\}, \quad s \in \mathbb{R},$$

then, as $n \to \infty$, the random measure $\mu_X/\sqrt{n}$ converges almost surely to the Wigner semi-circular distribution (Wigner 1955)

$$\rho(x) = \frac{1}{\pi} \sqrt{(2 - x^2)_+} \, dx$$  \hspace{1cm} (2)
**Figure 2:** Wigner’s law for one (rescaled) $2000 \times 2000$ GUE matrix on the left, plotted is the rescaled histogram of the 2000 eigenvalues and the semicircular density $\rho(x)$. On the right we compare Wigner’s law to the exact eigenvalue density for $n = 4$ and the associated eigenvalue histogram (sampled 4000 times).
(B) The local eigenvalue regime: We shall zoom in on $x_0 = \sqrt{2n}$ only
(Bowick, Brézin 1991, Forrester 1993, Nagao, Wadati 1993),

$$\frac{1}{\sqrt{2n^{\frac{1}{6}}}} K_n \left( \sqrt{2n + \frac{x}{\sqrt{2n^{\frac{1}{6}}}}}, \sqrt{2n + \frac{y}{\sqrt{2n^{\frac{1}{6}}}}} \right) \to K_{Ai}(x, y), \quad (3)$$

as $n \to \infty$ uniformly in $x, y \in \mathbb{R}$ chosen from compact subsets, with

$$K_{Ai}(x, y) = \int_0^\infty \text{Ai}(x + z)\text{Ai}(z + y) \, dz,$$

which yields a trace class operator on $L^2(t, \infty)$.
In turn, the largest eigenvalue in the GUE obeys

$$\max_{i=1,\ldots,n} \lambda_i(X) \Rightarrow \sqrt{2n} + \frac{1}{\sqrt{2n^6}} F_2, \quad n \to \infty,$$

where the cdf of $F_2$ equals (Forrester 1993)

$$\text{Prob}(F_2 \leq t) = \det(I - K_{\text{Ai}} \mathcal{I}_{L^2(t,\infty)}),$$

which famously connects to Painlevé special function theory (Tracy, Widom 1994).

**Universality**

Wigner’s law (2) is a universal limiting law (Arnold 1967, ...) and so is the soft edge law (3) (Soshnikov 1999). Both laws holds true for centered and scaled Hermitian Wigner matrices $X = (X_{jk})_{j,k=1}^n$ with $\mathbb{E}|X_{jk}|^2 < \infty$ where $X_{jk}, j < k$ are iid complex variables and $X_{jj}$ iid real variables independent of the upper triangular ones ($\oplus$ decay).
The other side of the coin

Consider the Complex Ginibre ensemble (GinUE), i.e. matrices

\[ X = Y \in \mathbb{C}^{n \times n} : \quad Y_{jk} \overset{\text{iid}}{\sim} \mathcal{N} \left( 0, \frac{1}{\sqrt{2}} \right) + i\mathcal{N} \left( 0, \frac{1}{\sqrt{2}} \right) \]

as in (Ginibre 1965). Equivalently think of a log-gas system \( \{z_j\}_{j=1}^n \subset \mathbb{C} \) with joint pdf for the particles’ locations equal to (Ginibre 1965)

\[
p_n(z_1, \ldots, z_n) = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \exp \left( - \sum_{j=1}^n |z_j|^2 \right).
\]

**Question:** How do the particles \( \{z_j\}_{j=1}^n \) behave for large \( n \)?
The particles $\{z_j\}_{j=1}^n$ form a $DPP$ on $\mathbb{C} \simeq \mathbb{R}^2$ (Mehta 1967),

$$R_k(z_1, \ldots, z_n) := \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} p_n(z_1, \ldots, z_n) \prod_{j=k+1}^n d^2z_j = \det [K_n(z_i, z_j)]_{i,j=1}^k$$

with correlation kernel

$$K_n(z, w) = \frac{e^{-\frac{1}{2}(|z|^2+|w|^2)}}{\pi} \sum_{k=0}^{n-1} \frac{1}{k!} (zw^*)^k.$$

Now analyze $R_k$ asymptotically in different scaling regimes:
(A) The global eigenvalue regime: define the ESD

\[ \mu_X(s, t) = \frac{1}{n} \# \{ 1 \leq j \leq n, \ \Re z_j \leq s, \ \Im z_j \leq t \}, \quad s, t \in \mathbb{R} \]

then, as \( n \to \infty \), the random measure \( \mu_X / \sqrt{n} \) converges almost surely to the uniform distribution on the unit disk (Ginibre 1965)

\[ \rho(z) = \frac{1}{\pi} \chi_{|z|<1}(z)d^2z \quad (4) \]
Figure 3: The circular law for 1000 complex (rescaled) Ginibre matrices of varying dimensions $n \times n$ in comparison with the unit circle boundary. We plot $n = 4, 8, 16$ from left to right.
Figure 4: Rescaled eigenvalue density for $\mathbf{X} \in \text{GinUE}$ with $n = 5, 50, 250$ from left to right. The larger $n$, the better its approach to the uniform density on $x^2 + y^2 \leq 1$. 
(B) The local eigenvalue regime: We shall zoom in on $|z_0| = \sqrt{n}$ only (Ginibre 1965, Mehta 1967)

$$\frac{1}{\sqrt{n}} K_n \left( z_0 + \frac{z}{\sqrt{n}}, z_0 + \frac{w}{\sqrt{n}} \right) \to K_e(z, w) \quad (5)$$

as $n \to \infty$ uniformly in $z, w \in \mathbb{C}$ chosen from compact subsets, with

$$K_e(z, w) = \frac{1}{2\pi} \text{erfc} \left( \sqrt{2} \left( e^{i\theta} w^* + e^{-i\theta} z \right) \right) e^{-\frac{1}{2}(|z|^2 + |w|^2) + zw^*}$$

where $\theta = \text{arg} \, z_0$. 
In turn the rightmost eigenvalue in the GinUE obeys

$$\max_{i=1,\ldots,n} \Re z_i(X) \Rightarrow \sqrt{n} + \sqrt{\frac{\gamma_n}{4}} + \frac{G}{\sqrt{4\gamma_n}}, \quad n \to \infty,$$

where $\gamma_n = \frac{1}{2}(\ln n - 5 \ln \ln n - \ln(2\pi^4))$ and the cdf of $G$ equals (Cipolloni, Erdős, Xu, Schröder 2022)

$$\text{Prob}(G \leq t) = e^{-e^{-t}},$$

so no Painlevé transcendents are floating about.

**Universality**

The circular law (4) is a universal limiting law (Girko 1985, ...) and so is the edge law (5) (Cipolloni, Erdős, Xu, Schröder 2022). Both laws holds true for centered and scaled matrices $X = (X_{jk})_{j,k=1}^n$ with iid complex entries so that $E|X_{jk}|^2 < \infty$ (⊕ decay).
Consider the Complex Elliptic Ginibre ensemble (eGinUE), i.e. matrices

\[ X = \sqrt{\frac{1 + \tau}{2}} X_1 + i \sqrt{\frac{1 - \tau}{2}} X_2 \in \mathbb{C}^{n \times n} : X_1, X_2 \in \text{GUE independent} \]

as in (Girko 1986). Here, 0 ≤ \( \tau \) ≤ 1. Equivalently think of a log-gas system \( \{z_j\}_{j=1}^n \subset \mathbb{C} \) with joint pdf equal to (Ginibre 1965)

\[
p^\tau_n(z_1, \ldots, z_n) = \frac{1}{Z^n \tau} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \exp \left( -\frac{1}{1 - \tau^2} \sum_{j=1}^n (|z_j|^2 - \tau \Re z_j^2) \right).
\]

**Question:** How do the particles \( \{z_j\}_{j=1}^n \) behave for large \( n \)?
The particles \( \{z_j\}_{j=1}^n \) from a DPP on \( \mathbb{C} \simeq \mathbb{R}^2 \) (Di Francesco,... 1994),

\[
R_k^\tau(z_1, \ldots, z_n) := \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} p_n^\tau(z_1, \ldots, z_n) \prod_{j=k+1}^n d^2z_j = \det [K_n^\tau(z_i, z_j)]_{i,j=1}^k
\]

with correlation kernel

\[
K_n^\tau(z, w) = e^{-\frac{1}{2(1-\tau^2)}(|z|^2 - \tau \Re z^2 + |w|^2 - \tau \Re w^2)} \sum_{k=0}^{n-1} \frac{\tau^k}{2^k k!} H_k \left( \frac{z}{\sqrt{2\tau}} \right) H_k \left( \frac{w^*}{\sqrt{2\tau}} \right).
\]

Now analyze \( R_k \) asymptotically in different scaling regimes:
(A) The **global eigenvalue regime**: define the ESD

\[ \mu_X(s, t) = \frac{1}{n} \# \{ 1 \leq j \leq n, \Re z_j \leq s, \Im z_j \leq t \}, \quad s, t \in \mathbb{R} \]

then, as \( n \to \infty \), the random measure \( \mu_X/\sqrt{n} \) converges almost surely to the uniform distribution on the ellipse

\[ E_\tau := \{ z \in \mathbb{C} : (\Re z)^2/(1 + \tau)^2 + (\Im z)^2/(1 - \tau)^2 < 1 \}, \]

(Crisanti, Sommers, Sompolinsky, Stein 1988)

\[ \rho(z) = \frac{1}{\pi(1 - \tau^2)} \chi_{E_\tau}(z)d^2z \]
**Figure 5:** The elliptic law for 500 complex (rescaled) elliptic Ginibre matrices of dimension $10 \times 10$ in comparison with the ellipse boundary. We plot $\tau = 0, 0.25, 0.75$ from left to right.
(B) The local eigenvalue regime: One can look at

\[ n \to \infty : \quad 1 - \tau > 0 \quad \text{uniformly in } n \quad \text{strong non-Hermiticity} \]

as done in (Forrester, Jankovici 1996). Or, more interestingly, one can look at

\[ n \to \infty : \quad \tau \uparrow 1 \quad \text{weak non-Hermiticity} \]

as first investigated by (Fyodorov 1997). To this end, set

\[ \sigma_n := n^\alpha \sqrt{1 - \tau_n} > 0, \quad (\tau_n)_{n=1}^\infty \subset [0, 1), \]

which will allow us to interpolate between GUE and GinUE statistics.
We shall zoom in on the rightmost particle of the process \( \{ z_j \}_{j=1}^n \equiv \{(x_j, y_j)\}_{j=1}^n \subset \mathbb{R}^2 \) (Bender 2009). Centering and scaling,

\[
\begin{align*}
x_j &\mapsto \tilde{x}_j = \frac{x_j - c_n}{a_n}, & \quad y_j &\mapsto \tilde{y}_j = \frac{y_j}{b_n}, \quad \alpha = \frac{1}{6},
\end{align*}
\]

accordingly, the eigenvalue process \( P_{n}^{\tau} = \{ (\tilde{x}_j, \tilde{y}_j) \}_{j=1}^n \)

(i) converges weakly to a **Poisson process** on \( \mathbb{R}^2 \) when \( \sigma_n \rightarrow \infty \),

(ii) converges weakly to the **interpolating Airy process** on \( \mathbb{R}^2 \) when \( \sigma_n \rightarrow \sigma \in [0, \infty) \).

The Poisson process is determined by the correlation kernel

\[
K_p(z_1, z_2) = \delta_{z_1z_2} \frac{1}{\sqrt{\pi}} e^{-x_1 - y_1^2}, \quad z_k = (x_k, y_k) \in \mathbb{R}^2
\]
and the interpolating Airy process by the correlation kernel

\[ K_{Ai}^\sigma(z_1, z_2) = \frac{1}{\sigma \sqrt{\pi}} \exp \left[ -\frac{1}{2\sigma^2} (y_1^2 + y_2^2) + \frac{1}{2} \sigma^2 (x_1 + iy_1 + x_2 - iy_2) + \frac{1}{6} \sigma^6 \right] \]

\[ \times \int_0^\infty e^{s\sigma^2} \text{Ai} \left( x_1 + iy_1 + \frac{1}{4} \sigma^4 + s \right) \text{Ai} \left( x_2 - iy_2 + \frac{1}{4} \sigma^4 + s \right) \, ds, \]

where we write \( z_k = (x_k, y_k) \in \mathbb{R}^2 \) for shorthand. In addition

\[ \max_{i=1,...,n} x_j(X) \Rightarrow c_n + a_n B_{\sigma}, \quad \sigma_n \to \sigma \in [0, \infty) \]

where the cdf of \( B_{\sigma} \) equals (Bender 2009)

\[ F(t, \sigma) := \text{Prob}(B_{\sigma} \leq t) = \det(I - K_{Ai}^\sigma \upharpoonright L^2((t, \infty) \times \mathbb{R})). \]
The problem and its solution

Gernot Akemann’s question

What can you say about $F(t, \sigma)$? Any Painlevé transcendent floating around? What about asymptotics?

and our answer

B-Little 2022

For all $(t, \sigma) \in \mathbb{R} \times [0, \infty)$,

$$F(t, \sigma) = \exp \left[ - \int_{t}^{\infty} (s - t) \left\{ \int_{-\infty}^{\infty} q_{\sigma}^{2}(s, \lambda)d\nu_{\sigma}(\lambda) \right\} ds \right], \quad \frac{d\nu_{\sigma}}{d\lambda} = \frac{1}{\sigma \sqrt{\pi}} e^{-\lambda^{2}/\sigma^{2}}$$

where $q_{\sigma}(t, \lambda)$ solve the integro-differential Painlevé-II equation

$$\frac{\partial^{2}}{\partial t^{2}} q_{\sigma}(t, y) = \left[ t + y + 2 \int_{-\infty}^{\infty} q_{\sigma}^{2}(t, \lambda)d\nu_{\sigma}(\lambda) \right] q_{\sigma}(t, y), \quad q_{\sigma}(t, y) \sim \text{Ai}(t + y), \ t \to +\infty.$$
The above shows in particular that

\[ F(t, \sigma) = \det(I - K_{\text{Ai}}^{\sigma} \upharpoonright L^2((t, \infty) \times \mathbb{R})) = \det(I - L_{\sigma} \upharpoonright L^2(t, \infty)), \]

where \( L_{\sigma} \) is trace class on \( L^2(t, \infty) \) with kernel

\[ L_{\sigma}(x, y) = \int_{-\infty}^{\infty} \Phi \left( \frac{z}{\sigma} \right) \text{Ai}(x + z)\text{Ai}(z + y) \, dz, \tag{6} \]

with \( \Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2} \, dy = 1 - \frac{1}{2} \text{erfc}(x) \). Note that (6) is an example of a so-called finite-temperature Airy kernel.
Some details

Why a Painlevé connection? Put $J_t := (t, \infty) \times \mathbb{R} \subset \mathbb{R}^2$.

Trace identities

We have for all $n \in \mathbb{Z}_{\geq 0}$ and $(t, \sigma) \in \mathbb{R} \times [0, \infty)$,

$$\text{tr}_{L^2(J_t)} (K_{\sigma}^n) = \text{tr}_{L^2(J_t)} K^n_{\sigma}$$

where $K_{\sigma}$ is trace class on $L^2(J_t)$ with kernel

$$K_{\sigma}(z_1, z_2) := \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}y_1^2} K_{\text{Ai}}(x_1 + \sigma y_1, x_2 + \sigma y_2) e^{-\frac{1}{2}y_2^2}.$$  (7)

The point is, (7) is an additive Hankel composition kernel in the horizontal variable!
Indeed, \(K_\sigma(z_1, z_2)\) is of the type

\[
K_\sigma(z_1, z_2) = \int_0^\infty \phi_\sigma(x_1 + s, y_1)\phi_\sigma(s + x_2, y_2) \, ds
\]

where

\[
\phi_\sigma(x, y) := \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}y^2} \text{Ai}(x + \sigma y).
\]

Thus the methods of (Krajenbrink 2021) and (Bothner 2022) are readily available in the analysis of \(F(t, \sigma)\) and the integro-differential Painlevé-II equation appears quite naturally.
How about (tail) asymptotics of $F(t, \sigma)$?

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For any $\epsilon \in (0, 1)$, there exists $t_0 = t_0(\epsilon)$ such that

$$F(t, \sigma) = 1 - A(t, \sigma)e^{-B(t, \sigma)}(1 + o(1)), \quad (8)$$

for $t \geq t_0$ and $0 \leq \sigma \leq t^\epsilon$. Here,

$$A(t, \sigma) = \frac{1}{2\pi t^{\frac{3}{2}}} \left(\sqrt{4 + \sigma^4 t^{-1}} - \sigma^2 t^{-\frac{1}{2}}\right)^{-\frac{5}{2}} \left(4 + \sigma^4 t^{-1}\right)^{-\frac{1}{4}},$$

$$B(t, \sigma) = \frac{4}{3} t^{\frac{3}{2}} \left(1 + \frac{\sigma^4}{4t}\right)^{\frac{3}{2}} - t\sigma^2 - \frac{1}{6} \sigma^6.$$

And beyond $0 \leq \sigma \leq t^\epsilon, \epsilon \in (0, 1)$?
There exist $t_0, \sigma_0 > 0$ such that

$$F(t, \sigma) = \exp \left[ \sigma \frac{3}{2} C \left( \frac{t}{\sigma} \right) + \frac{1}{4} \int_{t/\sigma}^{\infty} \left\{ \frac{d}{du} D(u) \right\}^2 du \right] (1 + o(1)), \quad (9)$$

for $t \geq t_0$ and $\sigma \geq \sigma_0$. Here,

$$C(y) = \frac{1}{\pi} \int_{0}^{\infty} \sqrt{x} \ln \Phi(x + y) \, dx, \quad D(y) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\sqrt{x}} \ln \Phi(x + y) \, dx.$$

Note that (8) and (9) capture the full ($t \to +\infty$) crossover between

$$F_2(t) = 1 - \frac{1}{16\pi t^{3/2}} \exp \left[ -\frac{4}{3} t^{3/2} \right] (1 + o(1)); \quad e^{-e^{-t}} = 1 - e^{-t} (1 + o(1))$$

The left tail (uniformly for all $\sigma \in (0, \infty)$) is work in progress.
Bulk excursions

Zooming in on bulk particles of the process \( \{ z_j \}_{j=1}^n \equiv \{ (x_j, y_j) \}_{j=1}^n \subset \mathbb{R}^2 \), gaps between consecutive \( x_j \), in the weak non-Hermiticity limit, are governed by an interpolating sine process on \( \mathbb{R}^2 \) with kernel

\[
K_{\sin}^\sigma(z_1, z_2) = \frac{1}{\sigma \pi^{\frac{3}{2}}} \exp \left[ -\frac{1}{2\sigma^2} (y_1^2 + y_2^2) \right] \int_0^1 e^{-\left( s\sigma \right)^2} \cos \left( (z_1 - z_2^*) s \right) ds,
\]

where we write \( z_k = x_k + iy_k \) for shorthand. In addition the limiting gap function equals

\[
H(t, \sigma) := \det(I - K_{\sin}^\sigma \upharpoonright L^2((-t,t) \times \mathbb{R})), \quad t > 0, \quad \sigma > 0,
\]

thus generalizing the sine kernel determinant.
Gernot Akemann’s question

What can you say about $H(t, \sigma)$? Any Painlevé transcendents floating around? What about asymptotics?

Trace identities

We have for all $n \in \mathbb{Z}_{\geq 0}$ and $(t, \sigma) \in (0, \infty) \times (0, \infty)$,

$$\text{tr}_{L^2(I_t)} (K^\sigma_{\text{sin}})^n = \text{tr}_{L^2(I_t)} S^n_{\sigma}, \quad I_t := (-t, t) \times \mathbb{R},$$

where $S_{\sigma}$ is trace class on $L^2(I_t)$ with kernel

$$S_{\sigma}(z_1, z_2) := \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2} y_1^2} K_{\text{sin}}(x_1 + \sigma y_1, x_2 + \sigma y_2) e^{-\frac{1}{2} y_2^2}. \quad (10)$$
Based on the above trace identities one then proves

\[ H(t, \sigma) = \det(I - K_\sin^\sigma \upharpoonright L^2((-t,t) \times \mathbb{R})) = \det(I - M_{\sigma} \upharpoonright L^2(-t,t)) \]

where \( M_{\sigma} \) is trace class on \( L^2(-t, t) \) with kernel

\[
M_{\sigma}(x, y) = \frac{t}{2\pi} \int_{-\infty}^{\infty} \Psi_{t/\sigma}(z) \cos(z(x - y)t) \, dz \tag{11}
\]

with \( \Psi_\alpha(x) = \pi(\Phi(\alpha(x + 1)) - \Phi(\alpha(x - 1))) \). Note that (11) is an example of a so-called finite-temperature sine kernel. In turn \( H(t, \sigma) \) relates to an integro-differential Painlevé-V transcendent, (Bothner 2021, unpublished).
Thank you very much for your attention!!!