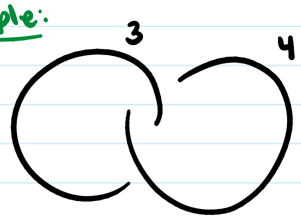
Week 8, 10/12

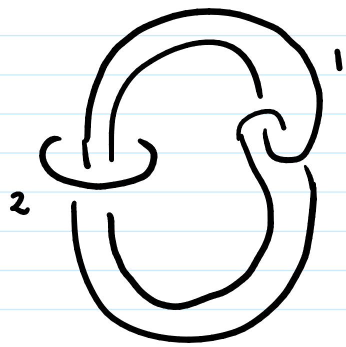
Thursday, October 12, 2023 12:33 PM

Example:



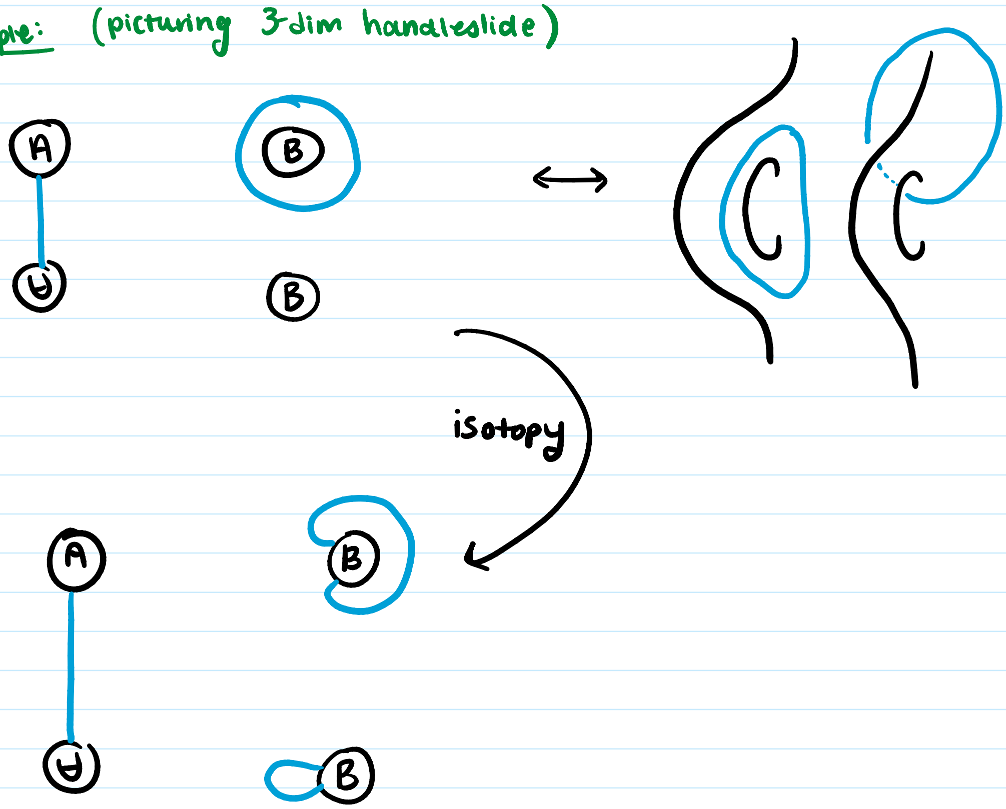
linking matrix: $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$

Ex:

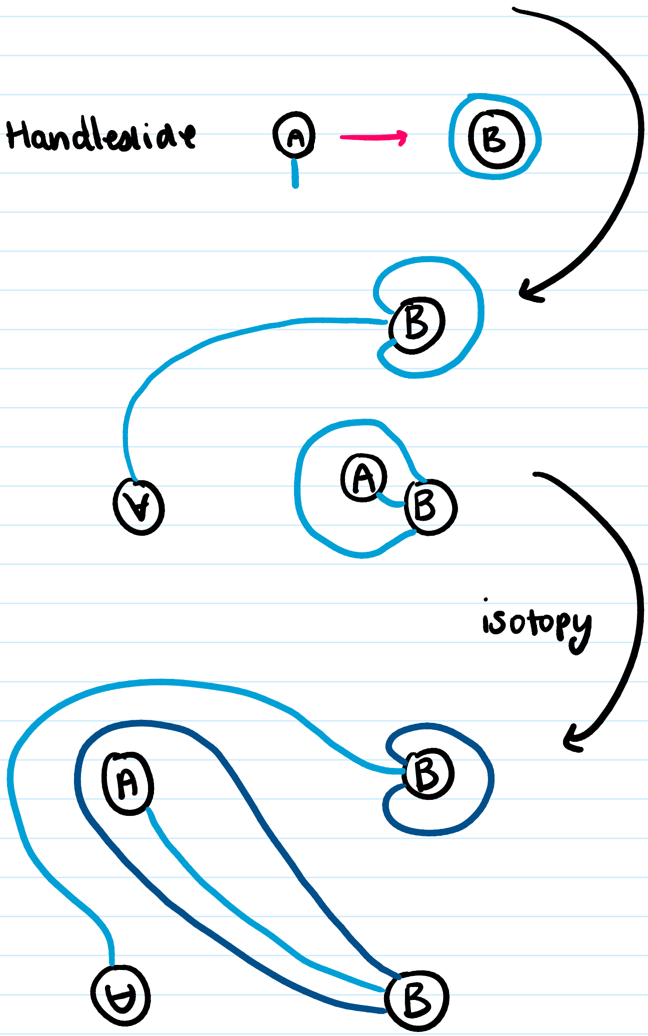


$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

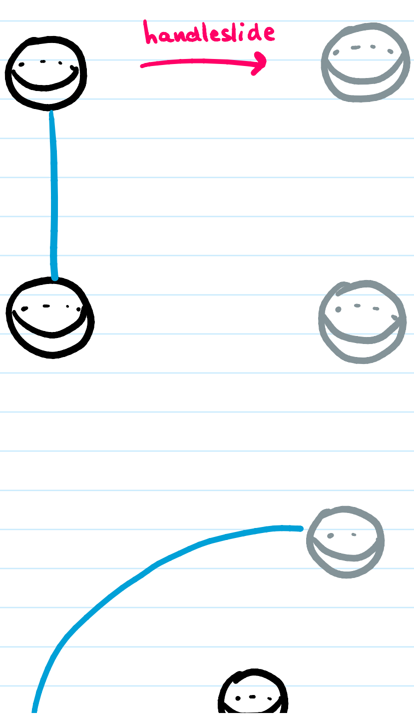
Example: (picturing 3-dim handleslide)

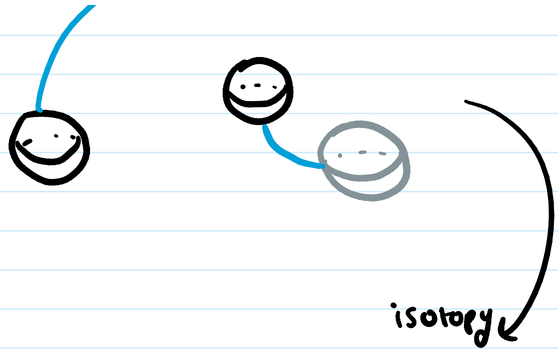


Handleslide

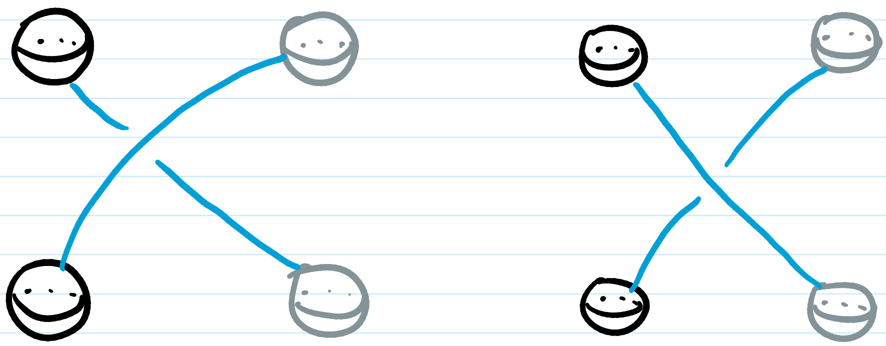


Example: (picturing 4-dim handleslide)





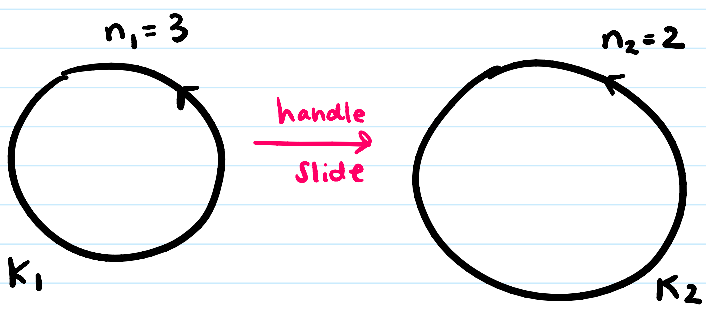
TWO OPTIONS:

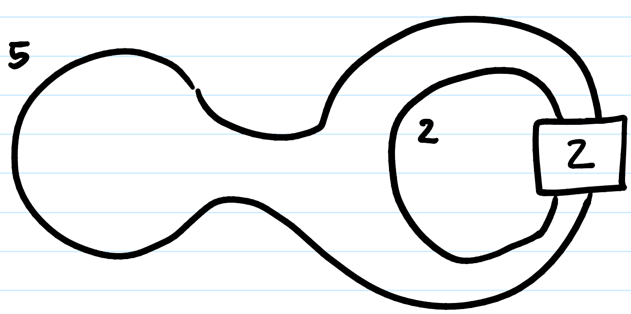


4-dim 2-handle slide:

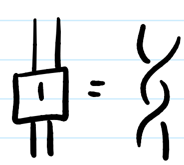
slide K_1 over K_2

Example:

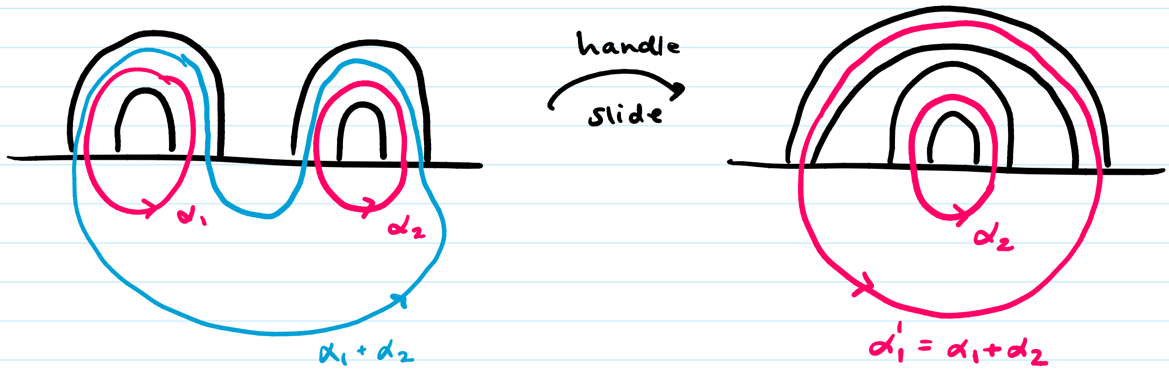




Recall:



change of basis induced by handleslide



α_1, α_2 was canonical basis for $H_2(X)$. New framing on K_1' is

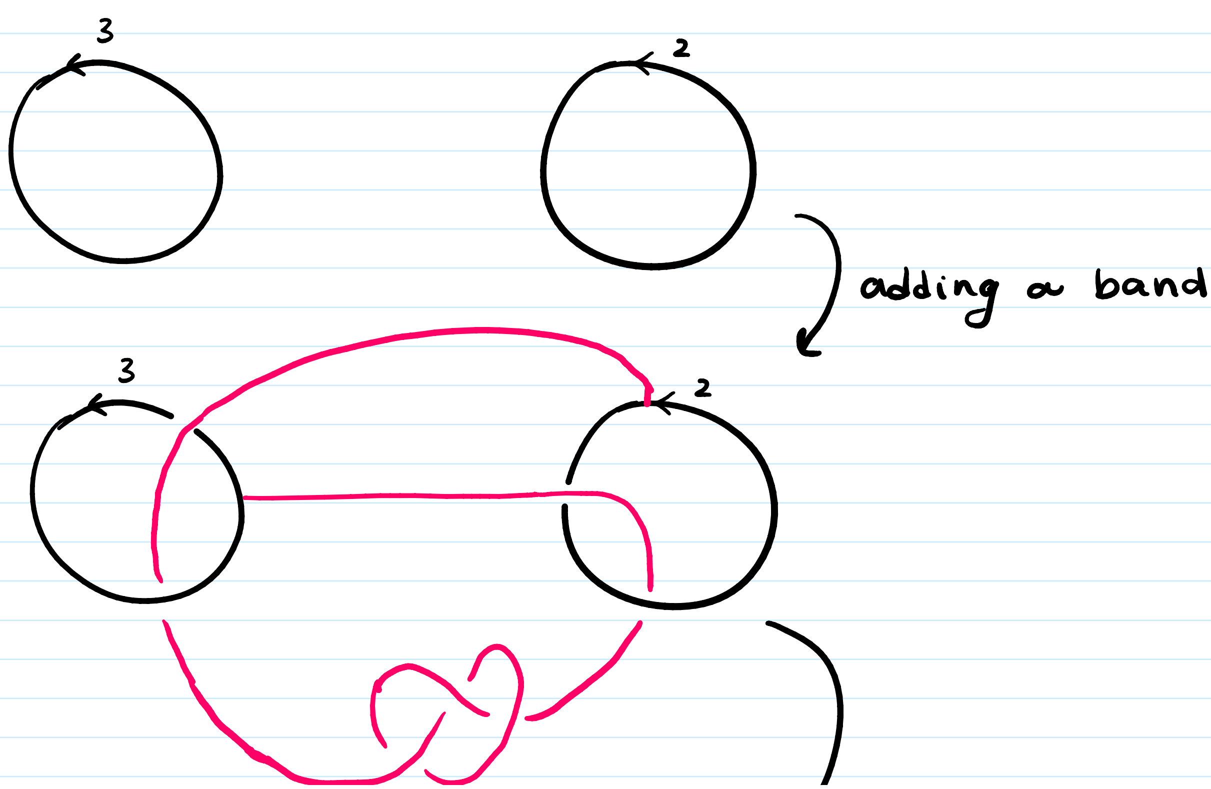
$$(\alpha_1 \pm \alpha_2)^2 = \alpha_1^2 + \alpha_2^2 \pm 2\alpha_1 \cdot \alpha_2$$

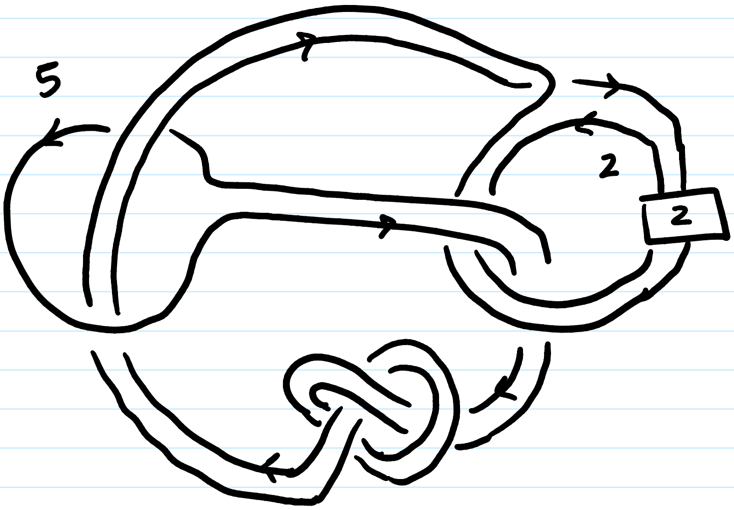
given by framing on K_1

given by framing on K_2

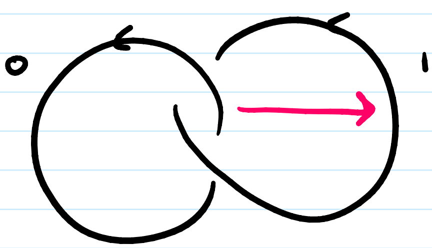
twice the linking number of K_1 & K_2

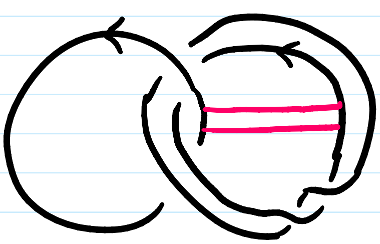
Example:

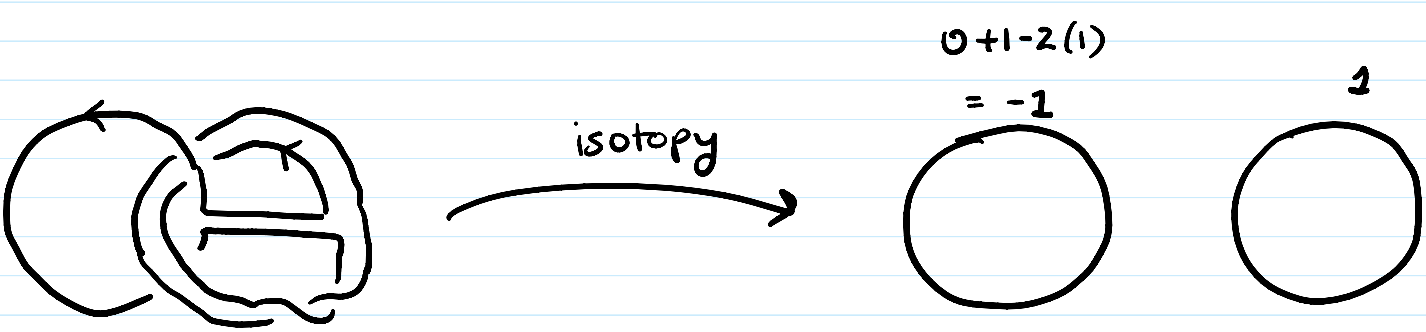




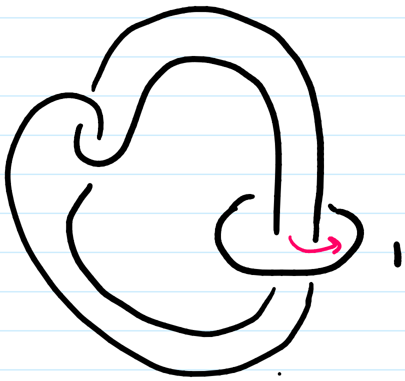
Example:



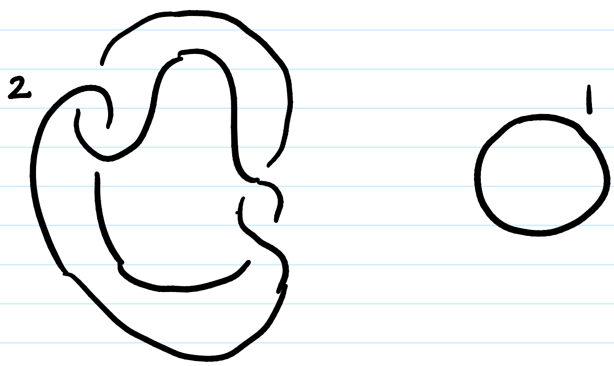




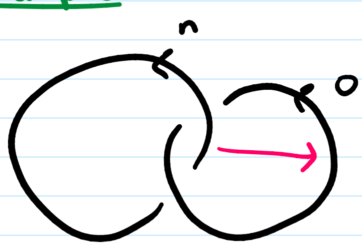
Example: (Exercise - check this 1 slide at a time)



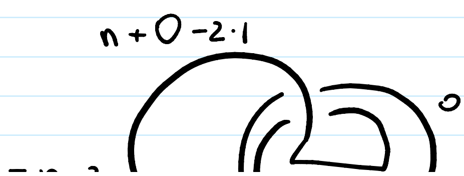
, handleslide both strands over :

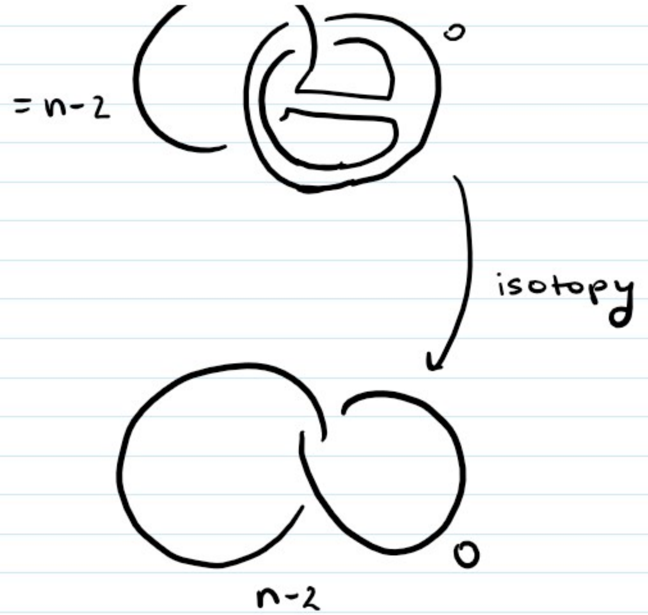


Example:



handleslide





Can always adjust by ± 2

Note: Nothing else goes through K_2 geometrically

Proposition

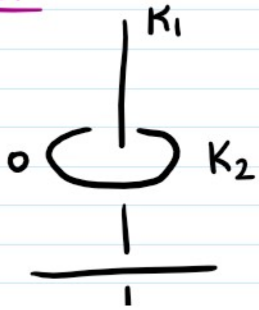
Let X^4 be given by a Kirby diagram. Suppose K_1 and K_2 are attaching circles such that K_1 lies entirely in ∂D^4 and K_2 is a 0-framed meridian of K_1

Then $X = X' \# S$

where X' is obtained from X by erasing K_1 and K_2 and $S = S^2 \times S^2$ if framing coeff. n_1 of K_1 is even and

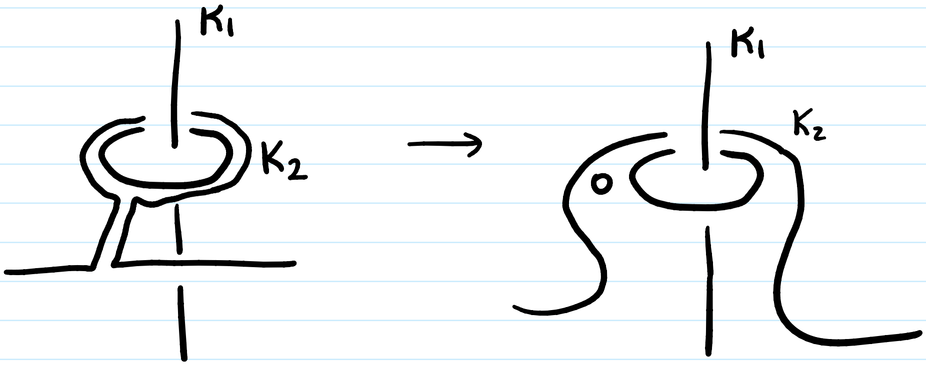
$S = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ otherwise

proof:



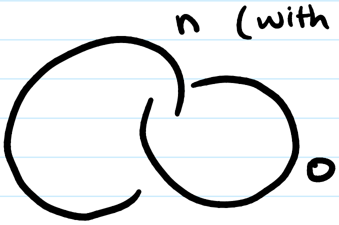
Use 0-framed meridian to bring K_1 and K_2 entirely to the front.

Want to slide over K_2:



Then, unknot K_1

Self-crossing of $K_1 \rightsquigarrow$ framing changes by 2



n (with same parity as original framing of K_1)

$n \equiv n \pmod 2$

n even : this is $S^2 \times S^2$

n odd : this is $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

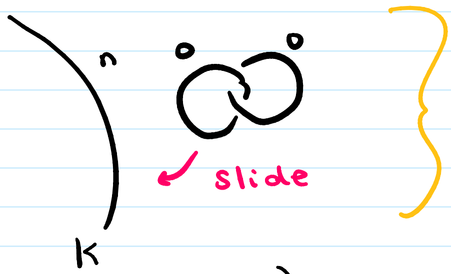
Corollary

Let X^4 be given by a Kirby diagram without 1-handles and with odd intersection form. Then $X \# S^2 \times S^2$ and $X \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are diffeomorphic

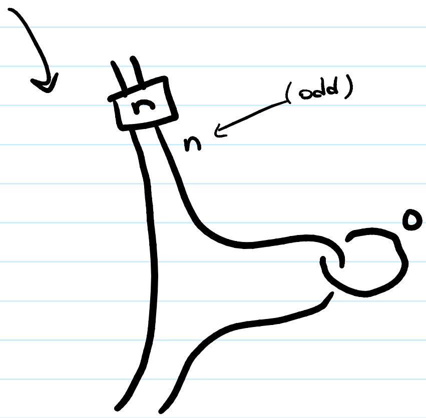
proof:

Odd intersection form $\Rightarrow$ Kirby diagram has a component

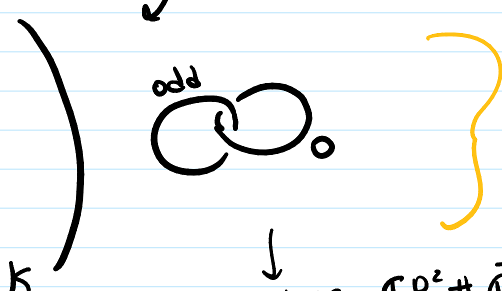
K with odd framing



$X \# S^2 \times S^2$



apply Proposition



$X \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

gives $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

Corollary

Let X^4 consist of a 0-h and m 2-handles. Then the double

$$DX \underset{\text{diffeo}}{\cong} \begin{cases} \#_m S^2 \times S^2 & \text{if } Q_X \text{ even} \\ \#_m \mathbb{C}P^2 \#_m \overline{\mathbb{C}P^2} & \text{if } Q_X \text{ odd} \end{cases}$$

In particular, if X is a closed 4-mfd without 1- or 3-handles, then $X \# \overline{X}$ admits such a connected sum splitting.

Open Question:

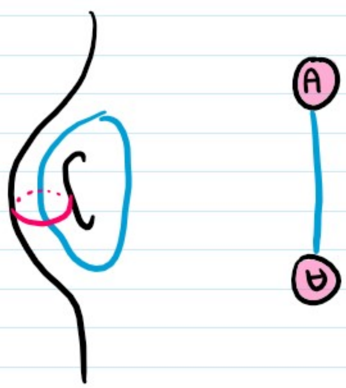
Does every simply-connected closed 4-mfd admit a handle decomposition without 1- or 3- handles?

Weaker: " " without 1- handles?

Handle cancellation:

$(k-1)$ handle h_{k-1} and a k-handle h_k can cancel if attaching sphere of h_k intersects belt sphere of h_{k-1} in a single point (regardless of framings)

Example: 3-dim 1/2 cancelling pair



attaching sphere of 1-handle

belt sphere of 2-handle

Example: 4-dim 1/2 cancelling pair