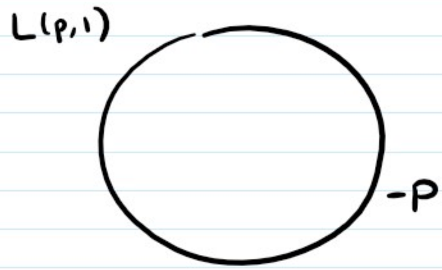
Week 4, 9/12 and 9/14

Tuesday, September 12, 2023 12:34 PM

**Example:**




$$\boxed{E_p = D^4 \cup_h (D^2 \times D^2)} \Rightarrow L(p,1) \text{ null-bordant}$$

**Corollary:**

Any closed 3-mfd $Y$ is null-bordant

**proof:**

By Lickorish-Wallace, $Y$ is integral surgery on a link $L \subset S^3$

so previous theorem implies $Y$ is cobordant to $S^3$

and $S^3$ bounds $B^4$!

hence is null-bordant

///

**Example:**

$p = 0$

$E_0 = S^2 \times D^2$

$\partial E_0 = S^2 \times S^1$

**Example:**

$p = 1$

$E_1 = \overline{\mathbb{C}P^2} - \text{int}(D^4)$

$\partial E_1 = S^3$

$$\mathbb{C}P^2 = \left\{ (z_0, z_1, z_2) \in \mathbb{C}P^3 \setminus \{0\} \right\} / \mathbb{C}^*$$

modulo multiplication

$[z_0 : z_1 : z_2]$

3 charts: $\mathcal{U}_i = \{ z_i \neq 0 \}$

$$h_0 : \mathcal{U}_0 \longrightarrow \mathbb{C}^2 \qquad [z_0 : z_1 : z_2] \longmapsto \left( z_1/z_0 , z_2/z_0 \right)$$

$$h_1 : \mathcal{U} \longrightarrow \mathbb{C}^2 \qquad [z_0 : z_1 : z_2] \longmapsto \left( z_0/z_1 , z_2/z_1 \right)$$

$$h_2 : \mathcal{U} \longrightarrow \mathbb{C}^2 \qquad [z_0 : z_1 : z_2] \longmapsto \left( z_0/z_2 , z_1/z_2 \right)$$

$$\mathcal{U}_0 \cup \mathcal{U}_1 = \mathbb{C}P^2 - [0:0:1]$$

$$h_0(\mathcal{U}_0 \cap \mathcal{U}_1) = \{ (z, w) \in \mathbb{C}^2 : z \neq 0 \} = h_1(\mathcal{U}_0 \cap \mathcal{U}_1)$$

$z \in \mathbb{C}^* \quad w \in \mathbb{C}$

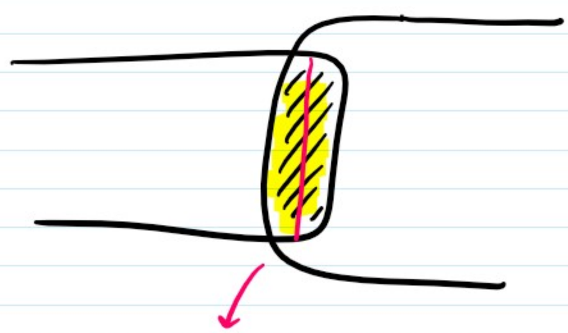
$$h(\mathcal{U}_0 \cap \mathcal{U}_1) \xrightarrow{h_1^{-1}} \mathcal{U}_0 \cap \mathcal{U}_1 \xrightarrow{h_0} h_0(\mathcal{U}_0 \cap \mathcal{U}_1)$$

$$(z, w) \longmapsto (z^{-1}, w z^{-1})$$

$|z| \geq 1$ goes to $|z| \leq 1$

We can truncate $h(\mathcal{U}_0)$ and $h(\mathcal{U}_1)$ by $|z| \leq 1$

gluing happens at $|z| = 1$



can truncate so these are edge to edge

still is the map

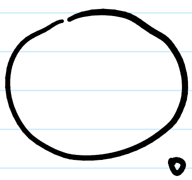
$$(z,w) \longmapsto (z^{-1}, wz^{-1})$$

truncated so that $\mathbb{C}^* \times \mathbb{C}$ is an annulus

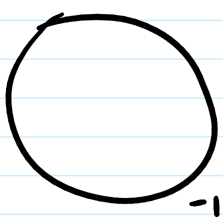
$$(D^2\times D^2) \cup_{hoh^{-1}} (D^2\times D^2)$$

$$hoh_1^{-1}: \partial D^2\times D^2 \longrightarrow \partial D^2 \times D^2$$

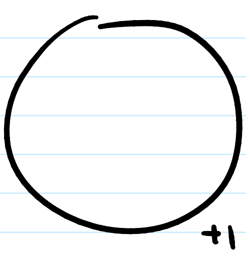
**Examples:**




$= S^2\times D^2$

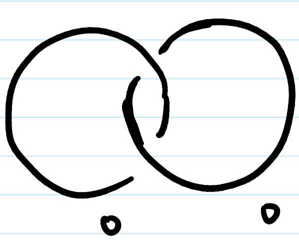



$= \overline{\mathbb{C}P^2} - \mathrm{int}(B^4)$




$= \mathbb{C}P^2 - \mathrm{int}(B^4)$

**Exercise:**




describes 4-mfd $S^2\times S^2 - \mathrm{int}(D^4)$

- the 3-mfd it describes is $S^3$

script $\mathcal{L}$ = framed link

# KIRBY MOVES

**Theorem:**

Integral surgery on framed links $\mathcal{L}, \mathcal{L}'$ are homeomorphic as oriented manifolds

$$\Longleftrightarrow$$

$\mathcal{L}'$ can be obtained from $\mathcal{L}$ by a sequence of Kirby moves

**MOVE K1**

$$\mathcal{L} \longmapsto \mathcal{L} \amalg \bigcirc^{\pm 1}$$

add (or delete) disjoint $\pm 1$ framed unknots

**MOVE K2**

Slide one component of $\mathcal{L}$ over another, say $K_1$ over $K_2$

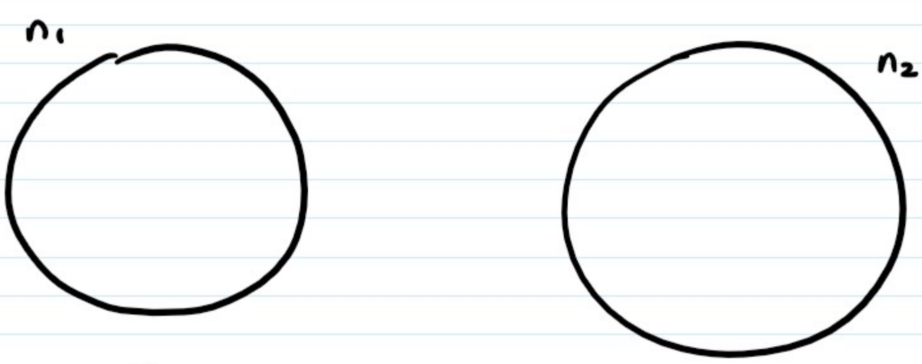
$n_1^{th}$ framed $\quad$ $n_2^{th}$ framed

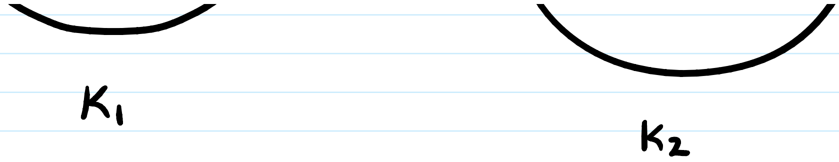
Replace $K_1 \cup K_2$ by $K_{\#} \cup K_2$

where $K_{\#} = K_1 \#_b K_2'$

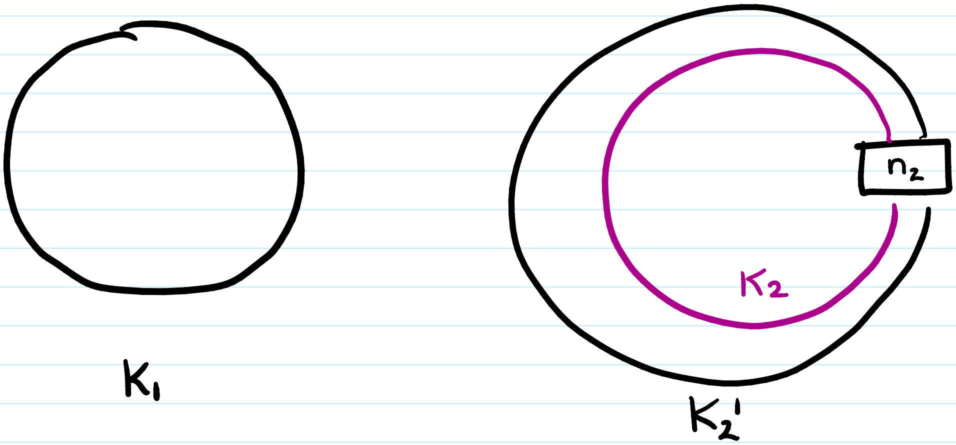
band-summed, $b$ = any band from $K_1$ to $K_2'$ disjoint from other components

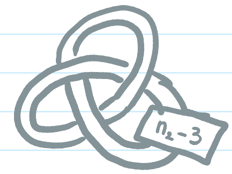
$K_2' = n_2^{th}$ framed pushoff of $K_2$

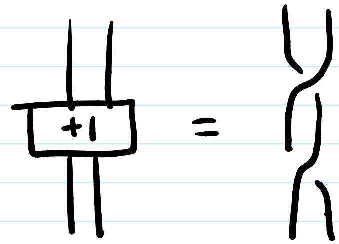


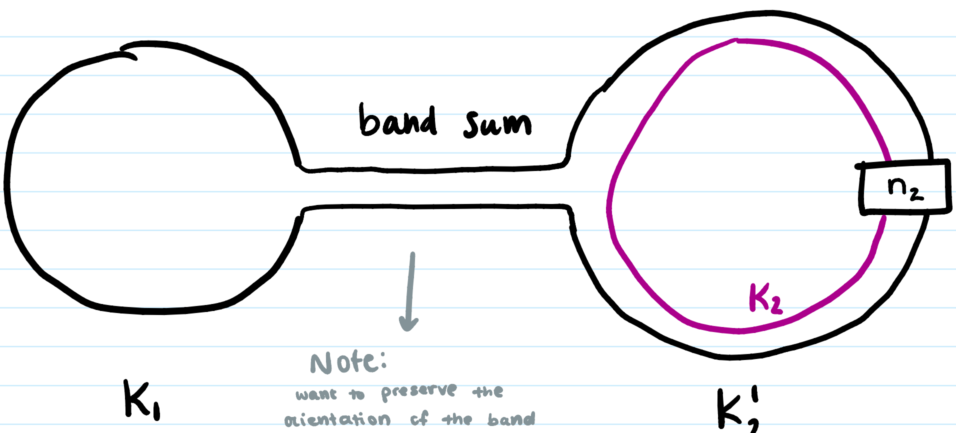



takes care of the writhe of the diagram!



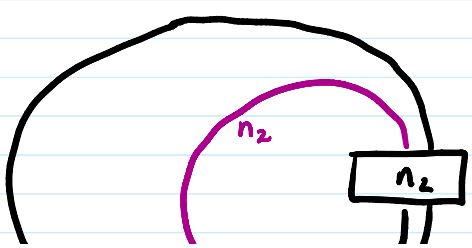



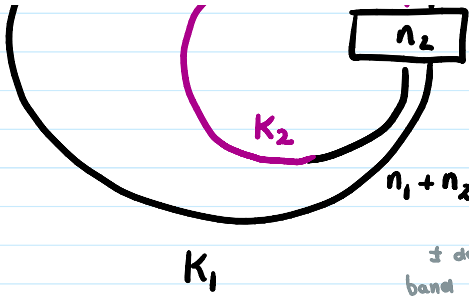




Note: want to preserve the orientation of the band sum for $K_1, K_2$ to be oriented coherently

What's the framing of the new thing?



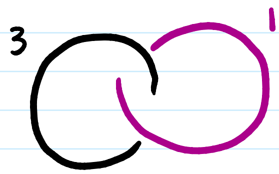


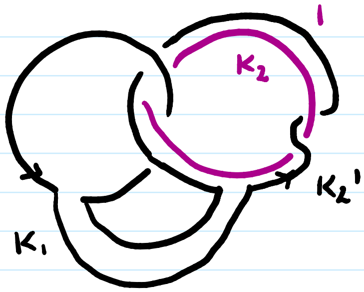

$n_1 + n_2 + 2lk(K_1, K_2)$

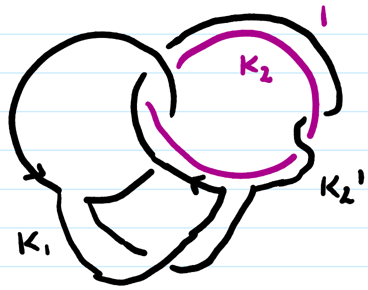
$\downarrow$

$\pm$ depends on choice of band sum on oriented links

## Example:




$3 + 1 + 2(1) = 6$ ← new framing

$3 + 1 + 2(-1) = 2$ ← new framing

## proof of theorem:

$\Leftarrow$

K1 is just $\# S^3$

K2 WLOG $\mathcal{L} = K_1 \cup K_2$

let $Y = S^3_{n_2}(K_2)$

$K_1$ and $K_\#$ are isotopic in $Y$ by pushing $K_1$ over the meridinal disk of surgery solid torus

$$(K_\# = K_1 \#_b K_2)$$

$\Rightarrow$ much harder.

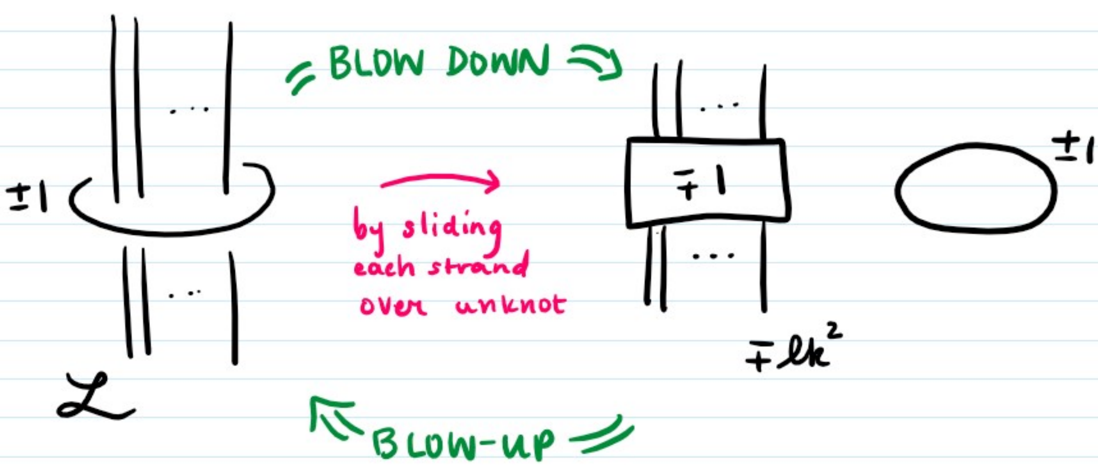
What do the moves do to the cobordism $W$ associated to $\mathcal{L}$?

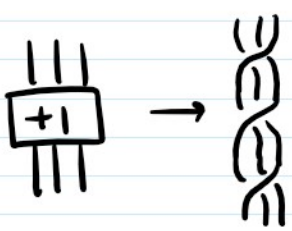
**K1:** $W \rightsquigarrow W \# \mathbb{CP}^2$ or $W \# \overline{\mathbb{CP}^2}$

**K2:** does not change $W$ at all.

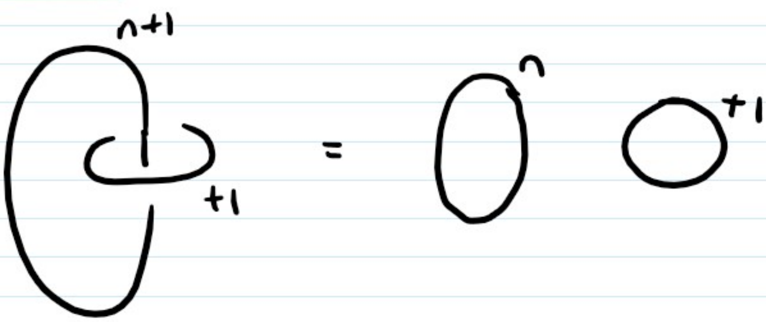
these don't change $\partial W$

**Blow down** and **Blow ups**:

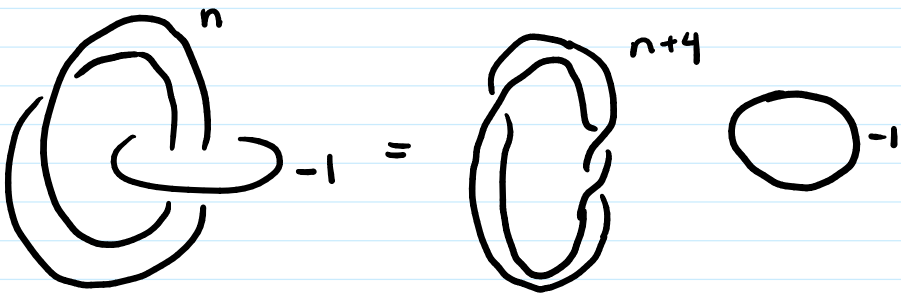



**Example:**

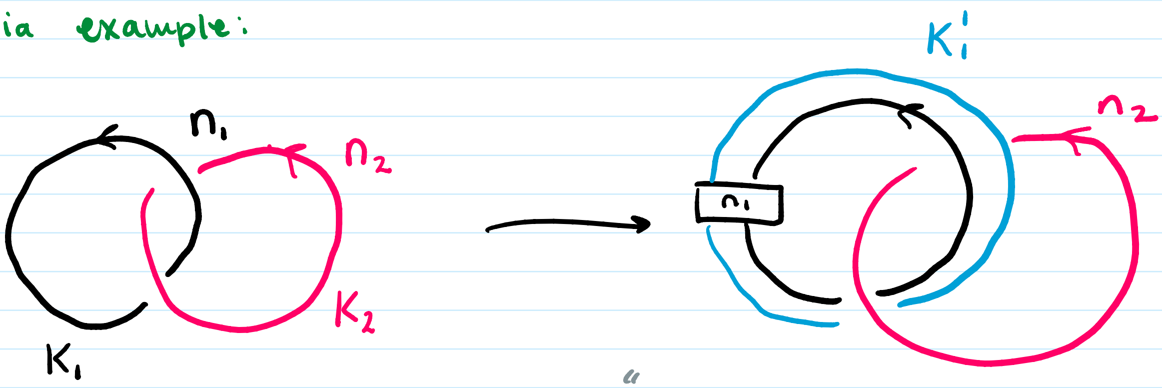


## Example:




Framing change in K2:

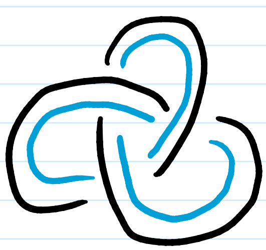
via example:




"double-stranded notation"

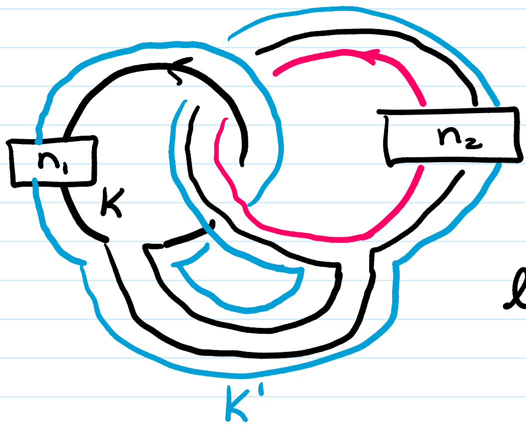
ASIDE:

- may have to consider the writhe



"blackboard framing"

handleslide:

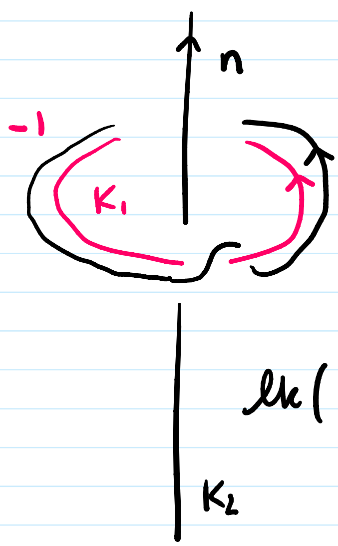



**exercise:**

Check that

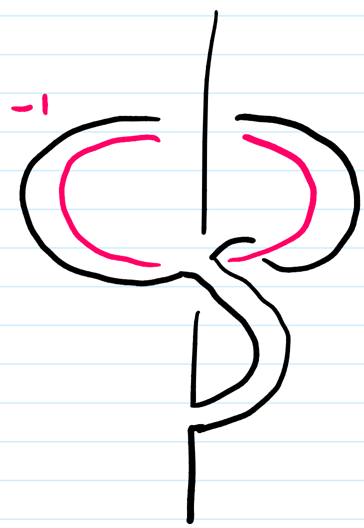
$$lk(K, K') = n_1 + n_2 + 2\,lk(K_1, K_2)$$

## Framings in blow-down:

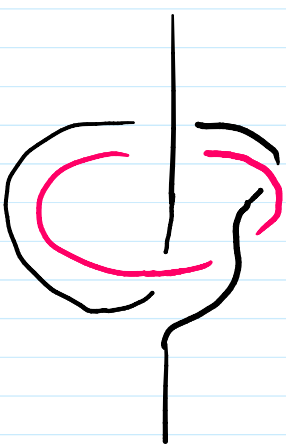



$lk(K_1, K_2) = +1$

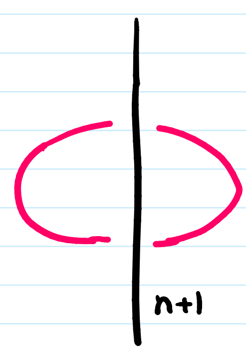
handleslide




isotopy



isotopy



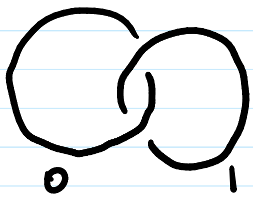

$lk = n - 1 + 2(1)$

$= n+1$

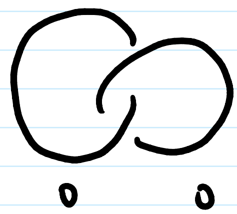
**exercise:** generalize this

See also Rolfsen Ch. 9.H

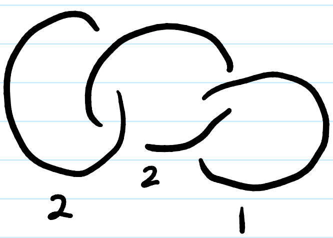
**Examples:**




$= S^3$




$= S^3$

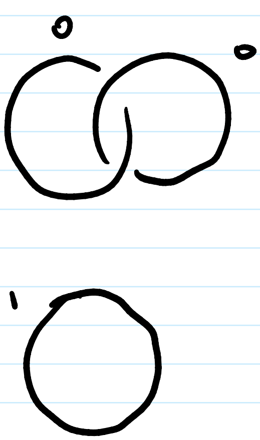



$= S^3$


$= S^3$

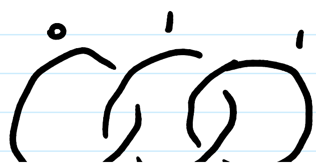
all describe some 4-mfds with $\partial = S^3$

**Example:**

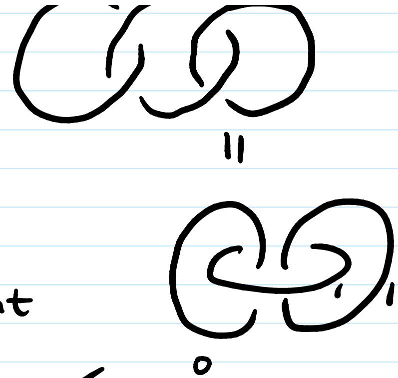



$= (S^2 \times S^2) \# \mathbb{C}P^2$

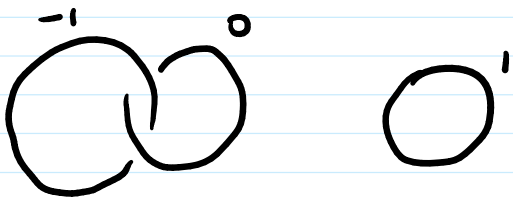
Claim: this is the same as



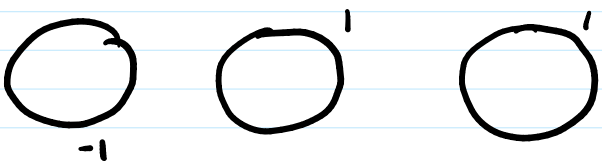
Claim: this is the same as after a blow-up




blow-down middle component



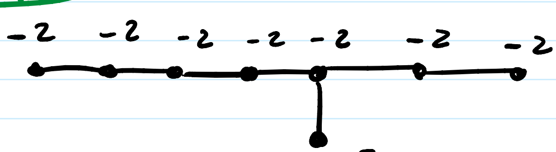

$\| $ blow-down




$= \mathbb{C}P^2 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

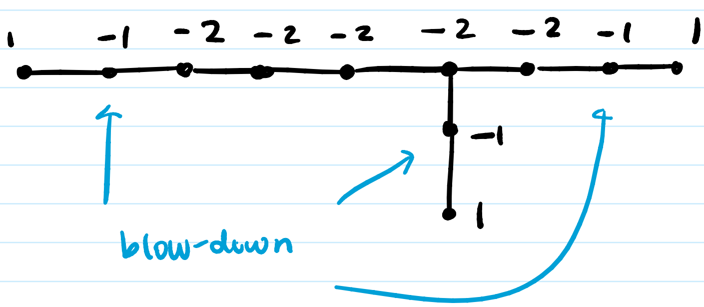
unexpectedly: $(S^2 \times S^2) \# \mathbb{C}P^2 \cong_{\text{diffeo}} \mathbb{C}P^2 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

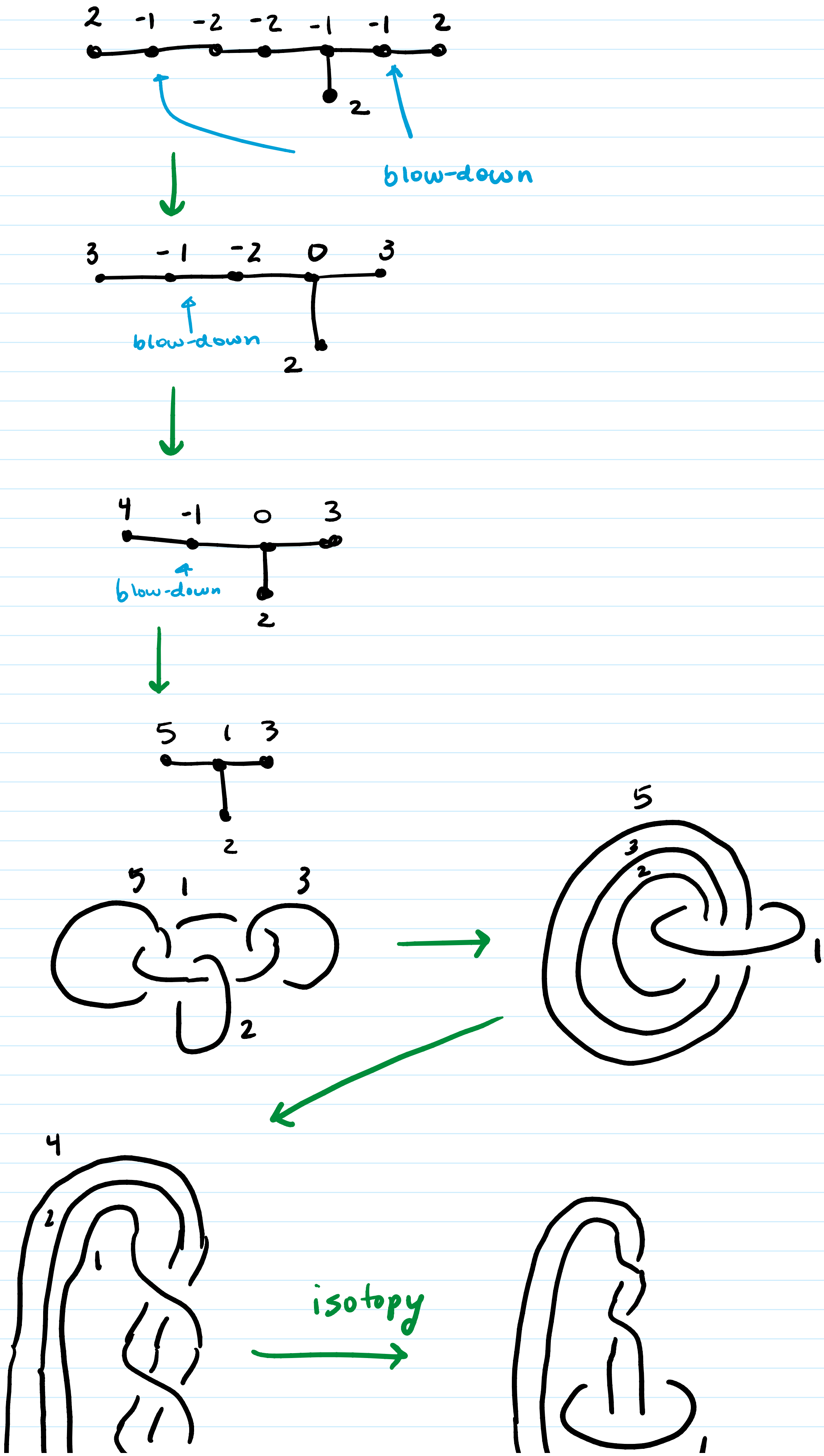
**Example:**





blow-up the ends (as 3-mfds)




2 -1 -2 -2 -1 -1 2
2
blow-down
3 -1 -2 0 3
blow-down
2
4 -1 0 3
blow-down
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5 1 3
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5 1 3
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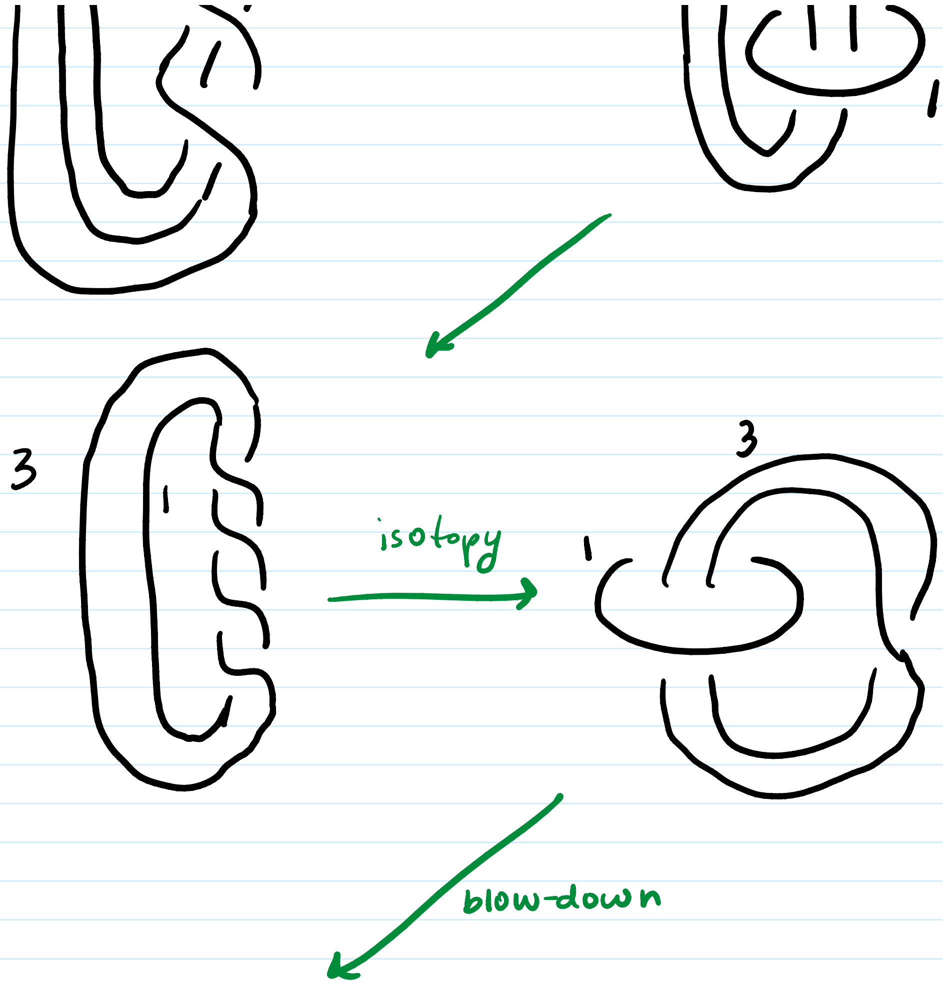
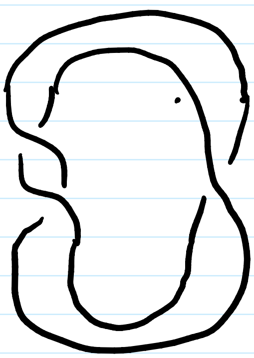
isotopy
1



$3-4=-1$

$= -1$ surgery on LH trefoil

$=$ Poincaré Homology sphere

$= \Sigma(2,3,5)$

Brieskorn Homology sphere

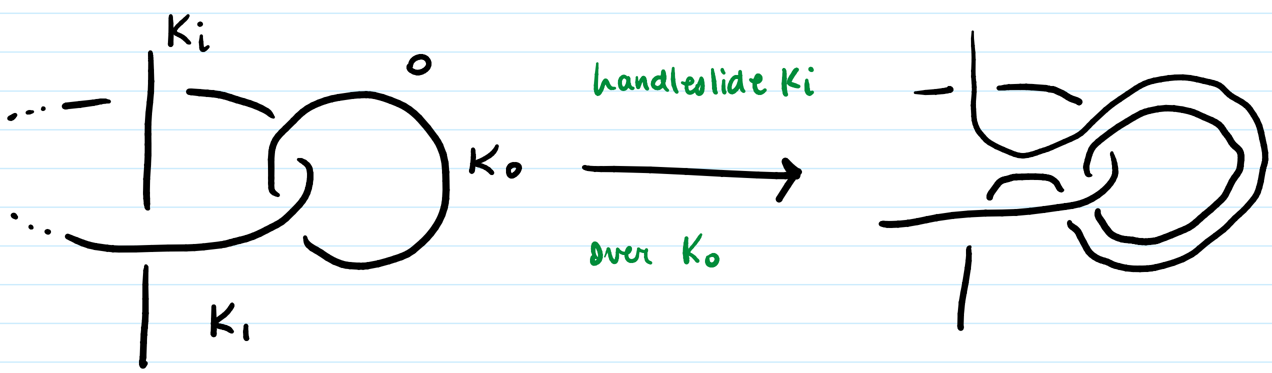
$$\Sigma(p,q,r) = \{\, x^p + y^q + z^r = 0 \,\} \cap S^5_\varepsilon \subset \mathbb{C}^3$$

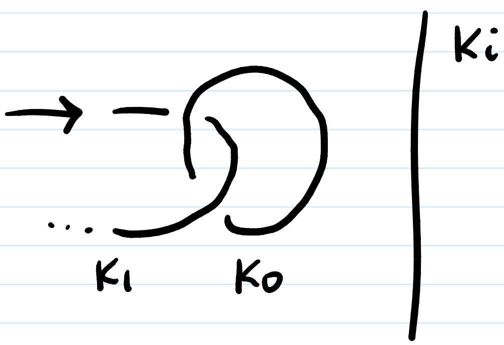
when $p,q,r$ are relatively prime

these are all Seifert fibered spaces

**Proposition:**

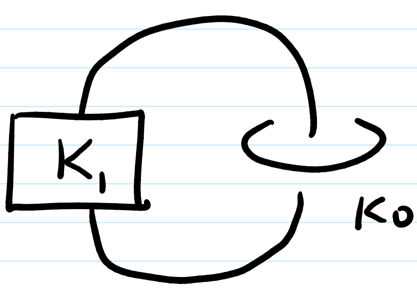
If a framed link $\mathcal{L}$ has a zero-framed unknotted component $K_0$ that links only one other component $K_1$ geometrically once, then $K_0 \cup K_1$ can be moved away from $\mathcal{L}$ without changing framings and cancelled.



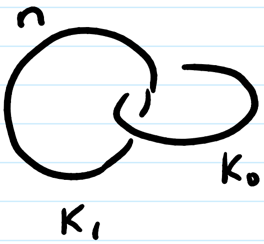

Can use $K_0$ to remove any crossings between $K_1$ and $K_i$

This moves $K_1 \cup K_0$ away from $\mathcal{L}$.



The same move changes crossings of $K_1$ to unknot $K_1$

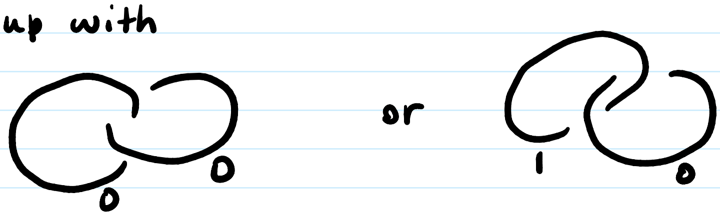
(handleslide over $K_1$)




Sliding left component over right component

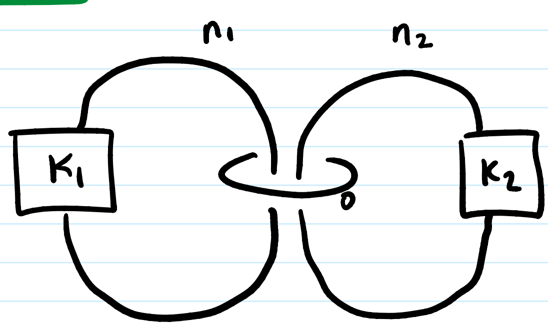
can change framing by $\pm 2$.

End up with

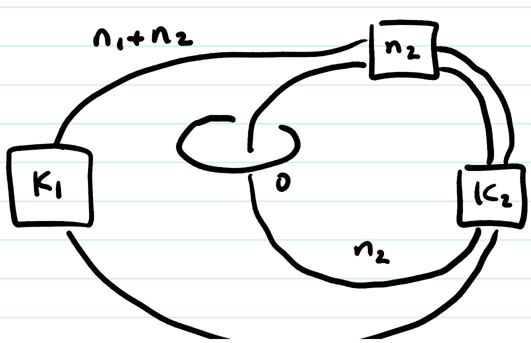


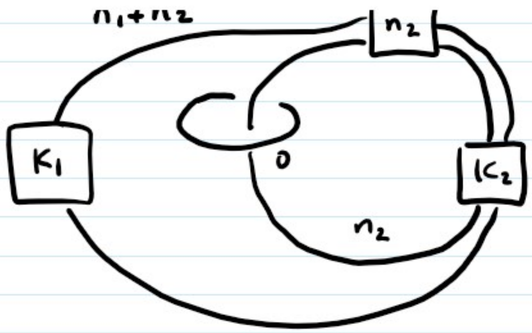

which both describe $S^3$.

Example:



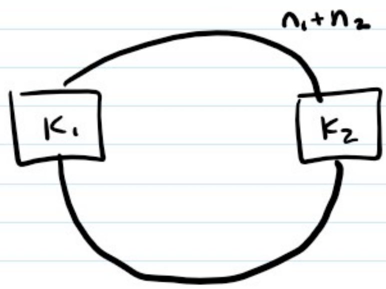

Slide $K_1$ over $K_2$






use previous proposition:

$\downarrow$




useful because it relates surgeries on connected sums

## Linking matrix

$\mathcal{L} = K_1 \cup \ldots \cup K_n$ oriented framed link

$K_i$ is $n_i$-framed

$n \times n$ matrix $A = (a_{ij})$

$$a_{ij} = \begin{cases} n_i & i=j \\ lk(K_i, K_j) & i \neq j \end{cases}$$

symmetric since $lk(K_i, K_j) = lk(K_j, K_i)$

**Exercise:**

suppose $\mathcal{L}$ is a surgery description for $Y$. The

Exercise:

suppose $\mathcal{L}$ is a surgery description for $Y$. The linking matrix is a presentation matrix for $H_1(Y)$

generators = rows + columns

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n$$

$$H_1(Y) = \mathbb{Z}^n / \operatorname{Im} A$$

rowspace gives you the kernel

hint: the generators for $H_1$ are meridians $\mu_i$ of $K_i$

relations are rows and columns of the matrix

(combo of generators that bound a surface)

(think of relations as surfaces)