Week 16, 12/5

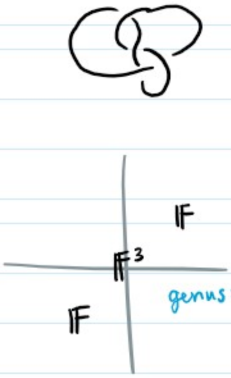
Tuesday, December 5, 2023 12:30 PM

Ex:

$\widehat{HFK}$(unknot)

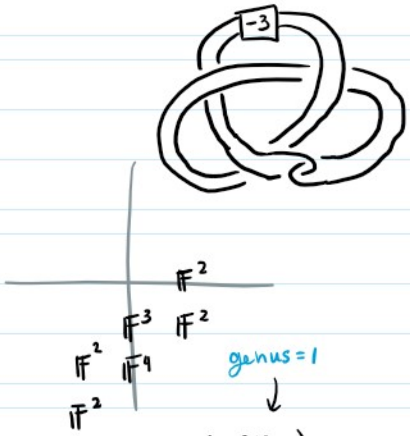
0 everywhere else!

$\widehat{HFK}(4_1)$



- each column take Euler char w/ grading associated to column you're in
- gives Alexander polynomial

$\widehat{HFK}(Wh^+(RHT))$ (Hedden)



(see Theorem below)

Recall:

$$\underset{\substack{\uparrow \\ \text{symmetrized}}}{\deg \Delta_K(t)} \leq g(K)$$

Theorem (Ozsváth-Szabó)

$\widehat{HFK}$ detects genus

$$g(K) = \max\{i \mid \widehat{HFK}(K,i) \neq 0\}$$

Recall:

If K is fibered, then $\Delta_K(t)$ is monic

Theorem (Ghiggini, Ni)

$\widehat{HFK}$ detects fiberedness

$$K \text{ fibered} \iff \underset{\substack{\uparrow \\ \text{Rightmost column of } \widehat{HFK}}}{\widehat{HFK}(K,g)} = 1$$

Proposition (O-S)

- $\widehat{HFK}(K_1 \# K_2) = \widehat{HFK}(K_1) \otimes \widehat{HFK}(K_2)$
- $CFK(K_1 \# K_2) \simeq CFK(K_1) \otimes_{\mathbb{F}[u,v]} CFK(K_2)$

chain homotopy equiv.

dual

- $CFK(-K) \simeq CFK(K)^* := \mathrm{Hom}(CFK(K), \mathbb{F}[u,v])$
- $$(u,v)^{-1} H_*(CFK(K)) \cong \mathbb{F}[u,v,u^{-1},v^{-1}]$$
  $$\shortparallel$$
  $$H_*(CFK)) \otimes_{\mathbb{F}[u,v]} \mathbb{F}[u,v,u^{-1},v^{-1}]$$
- $HFK^-(K) = H_*(CFK(K)/_{v=0})$

$$\cong \bigoplus_{i=1}^{N} \mathbb{F}[u] \oplus \bigoplus_j \mathbb{F}[u]/u$$

$\mathbb{F}[u]$-module

$$u^{-1}(HFK^-) \cong \mathbb{F}[u]$$

Theorem (Zemke)

A concordance from K_0 to K_1 induces a chain map

$$F: \ CFK(K_0) \longrightarrow CFK(K_1)$$

such that

$$F_*: (u,v)^{-1} H_*(CFK(K_0)) \longrightarrow (u,v)^{-1} H_*(CFK(K_1))$$

Remark:

By turning the concordance around, we also get a map

$$G: \ CFK(K_1) \longrightarrow CFK(K_0)$$

s.t. G is an isomorphism on $(u,v)^{-1} H_*$

defn:

two knot Floer complexes C_0 and C_1 are locally equivalent if there exists maps

$$\begin{tikzcd} C_0 \arrow[r, bend left, "F"] & C_1 \arrow[l, bend left, "G"] \end{tikzcd}$$

such that F_* and G_* induce isomorphisms on $(U,V)^{-1} H_*$

Corollary

If K_0 and K_1 are concordant, then $CFK(K_0)$ and $CFK(K_1)$ are locally equivalent

$K \rightsquigarrow CFK(K)$

$-K \rightsquigarrow CFK(K)^*$

$K_1 \# K_2 \rightsquigarrow CFK(K_1) \otimes CFK(K_2)$

$$\underset{\text{concordant}}{K_0 \sim K_1} \implies CFK(K_0) \underset{\substack{\text{locally} \\ \text{equivalent}}}{\sim} CFK(K_1)$$

$$\left(\{\text{knot Floer complexes}\} \big/ \text{local equivalence} \; , \; \otimes \right) =: \mathcal{C}FK$$

forms a group

and a group homomorphism

$$\mathcal{C} \longrightarrow \left(\left\{ \begin{matrix} \text{knot Floer} \\ \text{complexes} \end{matrix} \right\} \Big/ \begin{matrix} \text{local} \\ \text{equivalence} \end{matrix} \; , \; \otimes \right)$$

$$[K] \longmapsto [CFK(K)]$$

Open Question:

Give a complete description of the group structure of $\mathcal{C}FK$

But we do know some things:

Dai-Hom-Stoffregen-Truong

$$\begin{tikzcd} \mathcal{C}FK \arrow[r, "\bigoplus_{i=1}^{\infty} \varphi_i"] & \mathbb{Z}^\infty \arrow[l, bend right, dashed] \end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{CFK} \arrow[r, "\bigoplus_{i=1}^{\infty} \varphi_i"] & \mathbb{Z}^{\infty} \arrow[l, dashed, bend right]
\end{tikzcd}$$

Remark:

In our definition of concordance, we required the concordance annulus to be smooth

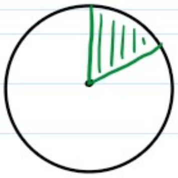
Can replace "smooth" with "topologically locally flat" to obtain the topological concordance group $\mathcal{C}_{TOP}$

"locally flat" $\parallel$ every point has a local product neighborhood

Remark:

Every knot K in S^3 bounds a non-locally flat topological disk in B^4 since

$$\text{cone}\,(S^3, K) = (B^4, D^2)$$



let's exclude this case with condition locally flat

$$\begin{tikzcd}
\mathcal{C} \arrow[r, two heads] & \mathcal{C}_{top} \arrow[r] & \mathcal{A}
\end{tikzcd}$$

$\mathcal{A}$: Levine's algebraic concordance group

while

$$\mathcal{C} \longrightarrow \mathcal{CFK}$$ does not factor through $\mathcal{C}_{TOP}$

Theorem (Freedman)

$$\boxed{\text{If } \Delta_K(t) = 1 \text{ then } K \text{ is topologically slice}}$$

topologically slice: i.e. trivial in $\mathcal{C}_{TOP}$

Example:

$K = Wh^+(J)$

$\Delta_K(t)=1 \underset{\text{Freedman}}{\Longrightarrow}$ K is topologically slice

Work of Donaldson $\Longrightarrow$ $Wh^+(RHT)$ is $\underset{\sim}{\text{not}}$ smoothly slice

Similar construction for homology cobordism:

$$\Theta^3_{\mathbb{Z}} = \left(\{\mathbb{Z}HS^3\text{'s}\}\Big/_{\substack{\text{homology}\\ \text{cobordism}}} \,,\ \# \right)$$

↓

cobound a 4-mfd which
looks like product on
level of homology

$Y \rightsquigarrow CF^-(Y)$

$-Y \rightsquigarrow CF^-(Y)^*$

$Y_1 \# Y_2 \rightsquigarrow CF^-(Y_1) \otimes CF^-(Y_2)$

$\underset{\text{hom cob}}{Y_0 \sim Y_1} \Longrightarrow CF^-(Y_0) \underset{\substack{\text{local}\\ \text{equivalence}}}{\sim} CF^-(Y_1)$

$$\Theta^3_{\mathbb{Z}} \longrightarrow \left(\left\{\substack{\text{Heegaard Floer}\\ \text{complexes}}\right\}\Big/_{\substack{\text{local}\\ \text{equiv}}} \,,\ \otimes \right)$$

$$\rotatebox{90}{$\cong$}$$

$$\mathbb{Z}$$

$$[Y] \longmapsto [CF^-(Y)]$$

Hendricks-Manolescu endowed Heegaard Floer homology with extra structure, namely a homotopy involution ι

↑

involution — squares to identity \iota

homotopy involution - squares to be homotopic to id

Involutive Heegaard Floer

$Y \rightsquigarrow (CF^-(Y), \iota)$

$Y \rightsquigarrow (CF^-(Y)^*, \iota^*)$

$Y_1 \# Y_2 \rightsquigarrow (CF^-(Y_1) \otimes CF^-(Y_2), \iota_1 \otimes \iota_2)$

$Y_0 \sim Y_1 \Longrightarrow (CF^-(Y_0), \iota_0) \underset{\text{local}}{\sim} (CF^-(Y_1), \iota_1)$

$$\underset{\text{hom. cob}}{Y_0 \sim Y_1} \implies (CF^-(Y_0), \iota_0) \underset{\substack{\text{local}\\ \text{equiv}}}{\sim} (CF^-(Y_1), \iota_1)$$

$$\Theta^3_{\mathbb{Z}} \longrightarrow \Big(\underbrace{\left\{ \substack{\text{involutive Heegaard} \\ \text{Floer complexes}} \right\} \Big/ \text{local equiv.}}_{\substack{\text{Dai-Hom-Stoffregen-Truong :} \\ \text{admits a direct summand} \\ \text{isomorphic to } \mathbb{Z}^\infty}} , \otimes \Big)$$

Theorem (DHST)

$\Theta^3_{\mathbb{Z}}$ has a direct summand isomorphic to $\mathbb{Z}^\infty$, generated by $\Sigma(2i+1, 4i+1, 4i+3)$ $i \in \mathbb{Z}_{>0}$

Q: Which types of manifolds can represent $[Y] \in \Theta^3_{\mathbb{Z}}$?

Example:

Every 3-mfd is homology cobordant to...

... an irreducible 3-mfd (Livingston, 1981)

... a hyperbolic mfd (Myers, 1983)

On the other hand, $\exists$ homology spheres that are not homology cobordant to a Seifert fibered space

(Frøyshov, F. Lin, Stoffregen)

e.g. $\Sigma(2,3,7) \# \Sigma(2,3,7)$

not homology cobordant to any Seifert fibred space

Q: Do Seifert fibered spaces generate $\Theta^3_{\mathbb{Z}}$?

Theorem: (Hendricks - Hom - Stoffregen - Zemke)

Seifert fibered spaces do not generate $\Theta^3_{\mathbb{Z}}$

Let Θ_{SF} be the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by Seifert fibered spaces. The quotient

$\Theta^3_{\mathbb{Z}} / \Theta$

by Seifert fibered spaces. The quotient

$$\Theta^3_{\mathbb{Z}} / \Theta_{SF}$$

is infinitely generated. The family

$$Y_n = S^3_{+1}\left(T_{2,3} \# -2T_{2n,2n+1} \# T_{2n,4n+1}\right) \quad n \geq 3, \text{odd}$$

is linearly independent in $\Theta^3_{\mathbb{Z}} / \Theta_{SF}$

$$\Theta^3_{\mathbb{Z}} \longrightarrow \left(\left\{ \substack{\text{involutive HF} \\ \text{complexes}} \right\} \Big/ \substack{\text{local} \\ \text{equiv}} \ , \ \otimes \right)$$