Week 12, 11/7 and 11/9

last time: Rokhlin invariants

we're using 4-mfds to build a 3-mfd invariant

- Y a $\mathbb{Z}HS^3$
- W^4 a smooth, simply connected, Q_W even 4-mfd with $\partial W = Y$

define $\boxed{\mu(Y) := \frac{1}{8}\sigma(W) \pmod 2}$

Last time we discussed why W exists, why $\mu(Y) \in \mathbb{Z}$, and why $\mu(Y)$ does not depend on the choice of W (by Rokhlin's theorem)

- μ is 0, 1 valued ... is it useful?

HOMOLOGY COBORDISM

def: 3-manifolds Y_0 and Y_1 are $\mathbb{Z}$-homology cobordant iff $\exists$ a smooth compact W^4 with $\partial W = -Y_0 \amalg Y_1$ and the inclusions $i_0 : Y_0 \hookrightarrow W$, $i_1 : Y_1 \hookrightarrow W$ induce isomorph's on homology:

$$i_{0*} : H_*(Y_0; \mathbb{Z}) \longrightarrow H_*(W; \mathbb{Z})$$

$$i_{1*} : H_*(Y_1; \mathbb{Z}) \longrightarrow H_*(W; \mathbb{Z})$$

W is called a Homology cobordism from Y_0 to Y_1.

- can also have R-homology cobordisms using $H_*(Y; R)$ instead

Example:

$W = Y \times I$ is a homology cobordism from Y to itself

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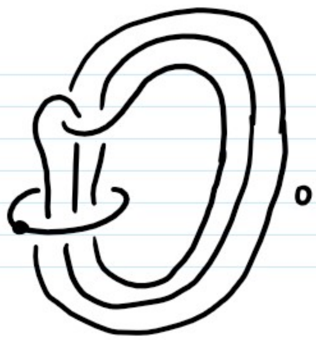
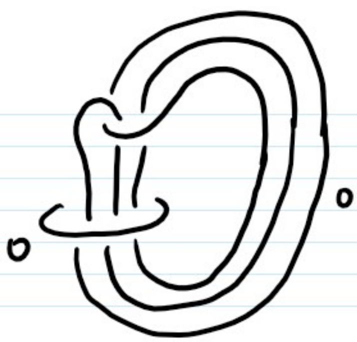
Interesting Example:

Let W be the 4-mfd obtained by removing a 4-ball from this manifold:



Exercise:

Check W is a homology cobordism from S^3 to this 3-mfd:

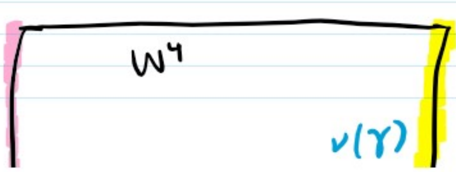


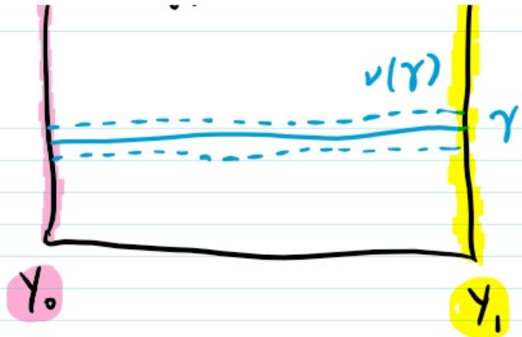
Exercise:

Homology cobordism is an equivalence class

Exercise:

Let Y_i, $i=0,1$, be $\mathbb{Z}HS^3$'s. If W is a homology cobordism from Y_0 to Y_1 and γ is an arc in W connecting Y_0 and Y_1, then $W \setminus \nu(\gamma)$ is a $\mathbb{Z}HB^4$





hint: Mayer-Vietoris

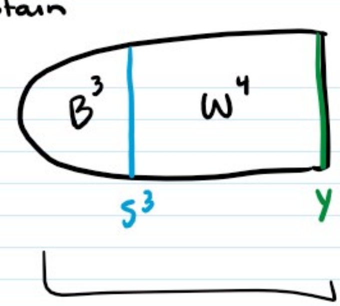
note: $\partial(W \setminus \nu(\gamma)) \cong -Y_0 \# Y_1$

Example:

$$Y \sim_{\substack{\text{homology}\\ \text{cobord.}}} S^3 \iff Y \text{ bounds a } \mathbb{Z}HB^4$$

proof sketch:

$(\Rightarrow)$ attach a B^4 with a diffeo $S^3 \longrightarrow S^3$ to obtain

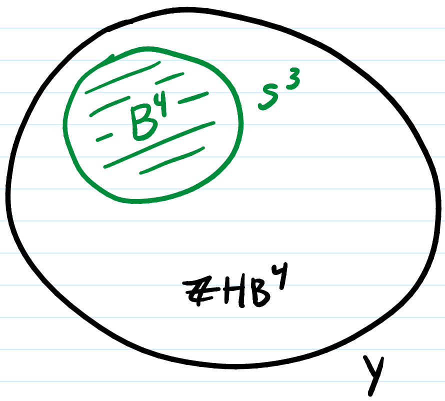


Check this is a $\mathbb{Z}HB^4$ with $\partial = Y$

Also, we could use the previous exercise, since if γ is an arc connecting Y to S^3, $W \setminus \nu(\gamma)$ is a $\mathbb{Z}HB^4$ and $\partial(W \setminus \nu(\gamma))$ is $\cong -Y \# S^3 \cong -Y$

$(\Leftarrow)$ If Y bounds a $\mathbb{Z}HB^4$, remove a B^4 and verify the result is a homol. cobord. $Y \sim S^3$

verify the result is a homol. cobord. $Y \sim S$.



Exercise:

If Y is a $\mathbb{Z}HS^3$, $Y \# -Y$ bounds a $\mathbb{Z}HB^4$

defn: The Homology Cobordism Group

$$\Theta^3_{\mathbb{Z}} := \left(\{\mathbb{Z}HS^3\text{'s}\} \big/ \sim_{\text{hom. cobor.}} \, , \; \# \right)$$

Exercise:

Check this is a well-defined group:

- the operation $\#$ is defined
- closed under $\#$
- Identity? $\quad$ (it's $[S^3]$)
- Inverses? $\quad$ ($-[Y] = [-Y]$)

Question: How does $\Theta^3_{\mathbb{Z}}$ relate to the Rokhlin invariant?

Theorem:

if $Y \sim S^3$ then $\mu(Y) = 0$

Theorem:

$$\boxed{\text{if } Y \underset{\substack{\text{hom.}\\ \text{cobord.}}}{\sim} S^3, \text{ then } \mu(Y) = 0}$$

Exercise:

$\mu : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is a surjective group homorphism

Note: this implies $\Theta^3_{\mathbb{Z}}$ is nontrivial, unlike Ω_3

the bordism group of 3-mfds

Remark:

Manolescu's proof of the triangulation conjecture

($\forall$ $n \geq 4$, $\exists$ a closed n-dim manifold which does not admit a simplicial triangulation)

uses the homology coborelism group!

Remark: using gauge theory and Floer theory, we now have more surjective homomorphisms

$h : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ (Frøyshov)

$d : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ (Ozsváth-Szabó)

$\varphi_i : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ $i \in \mathbb{N}$ (Dai-Hom-Stoffregen-Truong)

The strongest result to date is that $\Theta^3_{\mathbb{Z}}$ admits a $\mathbb{Z}^\infty$ direct summand.

(Dai-Hom-Stoffregen-Truong)

Open Question:

Does $\Theta^3_{\mathbb{Z}}$ have torsion?

Does $\Theta^3_{\mathbb{Z}}$ have torsion?

If it has 2-torsion, there's a $\mathbb{Z}HS^3$ Y with $Y \underset{\text{hom cob}}{\sim} -Y$

but $Y \underset{\text{hom cob}}{\nsim} S^3$ ...

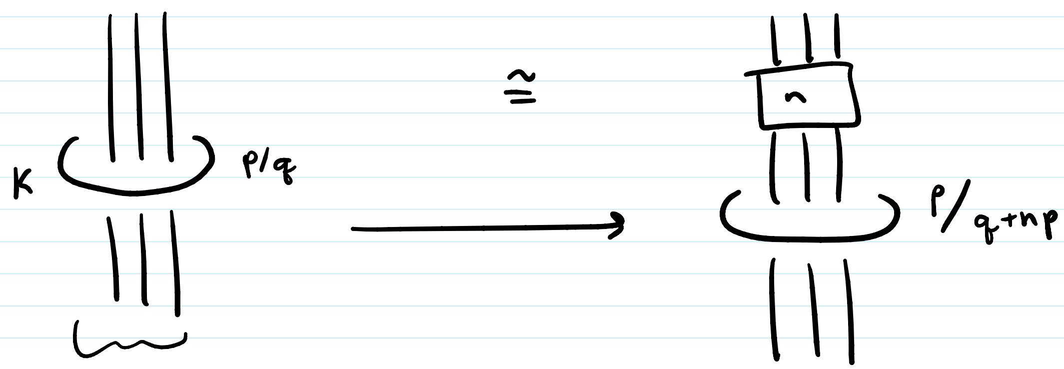
Two Moves for Rational Surgery

We largely focused our discussion of 3-mfds on integral surgery

if we allow rational surgeries, there are two more moves:

1. Rolfsen Twist

If $K \subset S^3$ unknot and $n \in \mathbb{Z}$,



strands of knots K_i w/ surgery coeffs r_i

new coefficients $r_i' = r_i + n\,lk(K, K_i)^2$

- $S^3 \setminus \nu(K)$ is a solid torus, we can perform Dehn twists

Exercise:

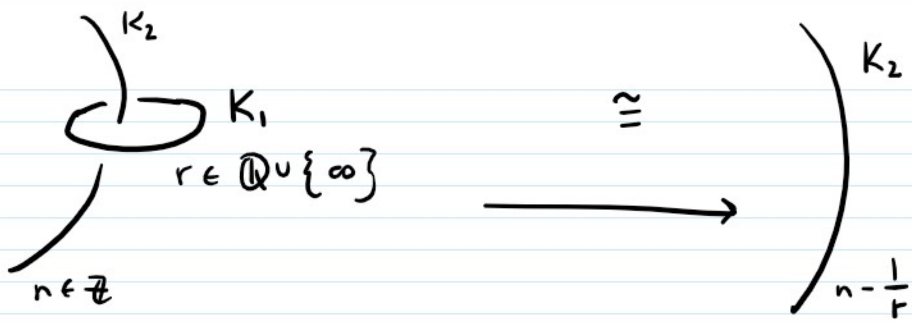
Verify the new surgery coefficients

hint: double strand notation

2. Slam dunk

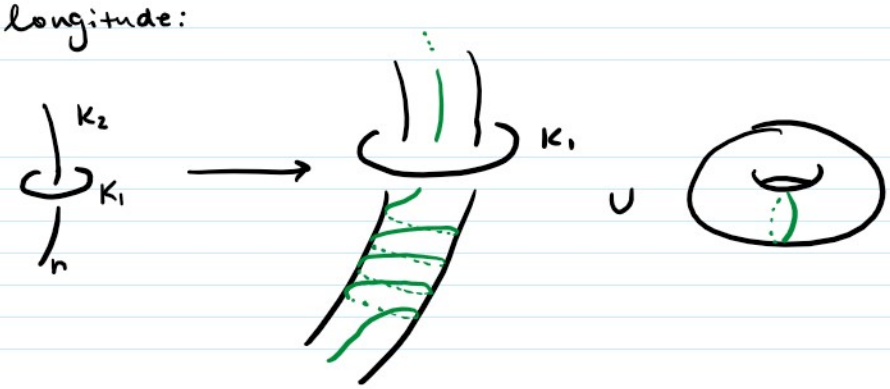
K_2 integrally framed

K_1 a rationally framed meridian of K_2



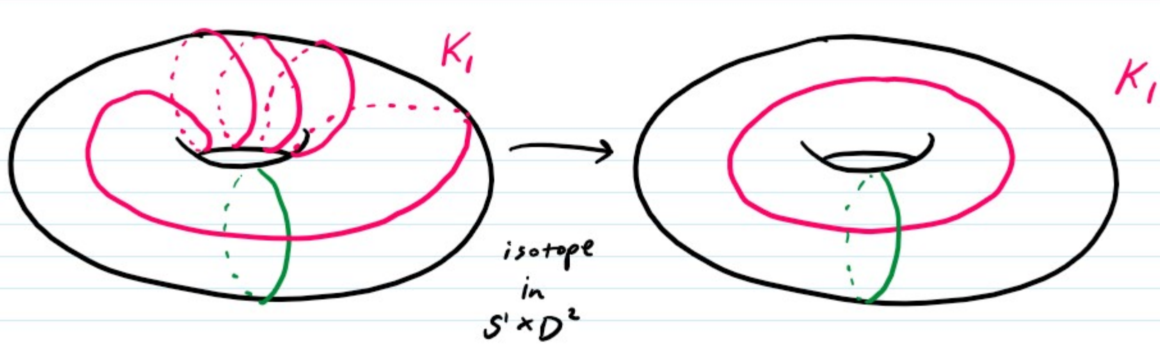
Key idea: it's integral on K_2

This surgery removes $\nu(K_2)$ and glues in a solid torus sending a meridian to n meridians + 1 longitude:



$S^3 \setminus \nu(K_2) \cup_\varphi S^1 \times D^2$ (with the indicated φ)

We can push K_1 into the boundary of the surgery torus to get a curve that intersects the meridian once:



and isotope K_1 to be the core of the surgery torus.

So, we can describe the composition of surgeries on a single knot.

Exercise:

For $\frac{1}{2}$ surgery on K_2 figure out what the core of the surgery solid torus looks like

Exercise:

Verify the slam-dunk surgery coefficient is $n - \frac{1}{r}$

Invariants of Knots and Links

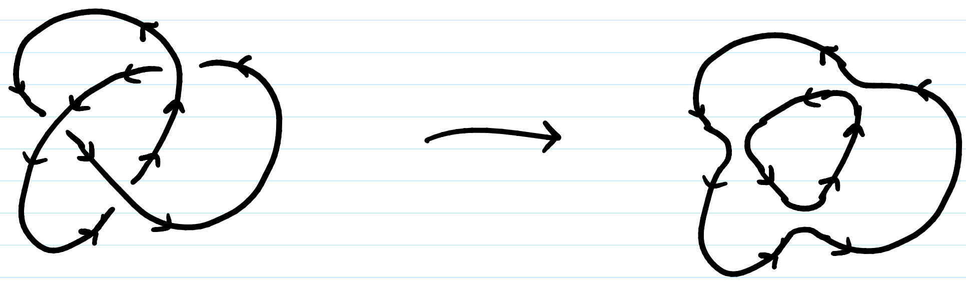
- Recall a Seifert surface of a link L is an oriented connected surface whose boundary is L

- Seifert's algorithm: given an oriented link L, to find a Seifert surface,

  1. Resolve all crossings preserving orientation to get Seifert circles

  2. add bands corresponding to the crossings

ex:



Exercise:

Seifert's algorithm produces a surface of genus g where
$2g = 2 - s - n + c$

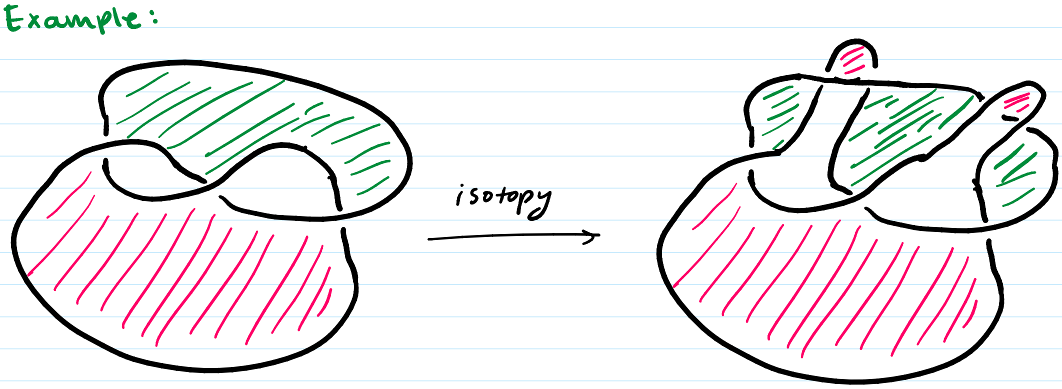
n = # components of link

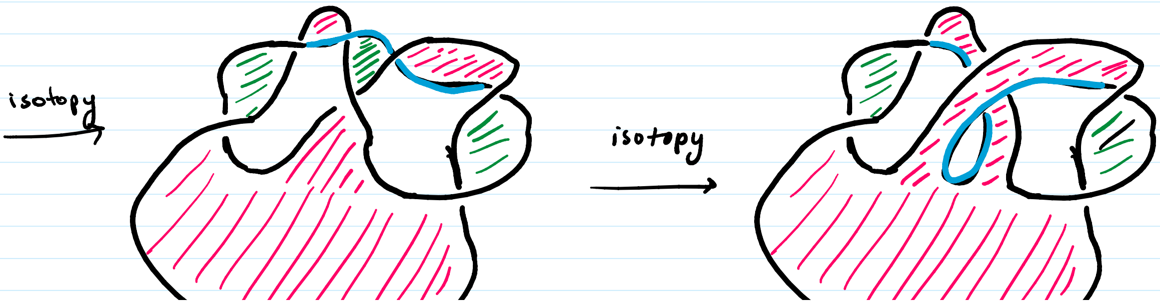
s = # Seifert circles

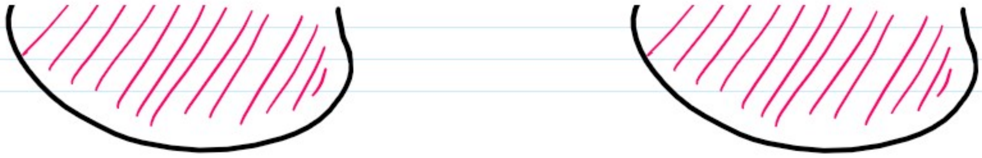
c = # of crossings of link

So above, $n = 1$, $s = 2$, $c = 3$ $\Rightarrow$ $2g = 2-2-1+3 = 2$ $\Rightarrow$ $g = 1$

Example:







isotopy $\longrightarrow$

Note: there are many Seifert surfaces

Can connect sum on a torus, etc

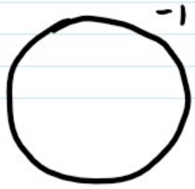
defn: the genus of a knot

$$g(K) := \min\{\text{genus of } \Sigma : \Sigma \text{ a seifert surface for } K\}$$

Example:

If K = trefoil, we know one Seifert surface with genus 1, so $g(K) \leq 1$. If $g(K) = 0$, then K bounds a disk so K is the unknot U. We'll show $K \neq U$ and therefore $g(K) = 1$.

If $K \cong U$ are ambiently isotopic, then the 3-manifolds obtained by surgeries



and

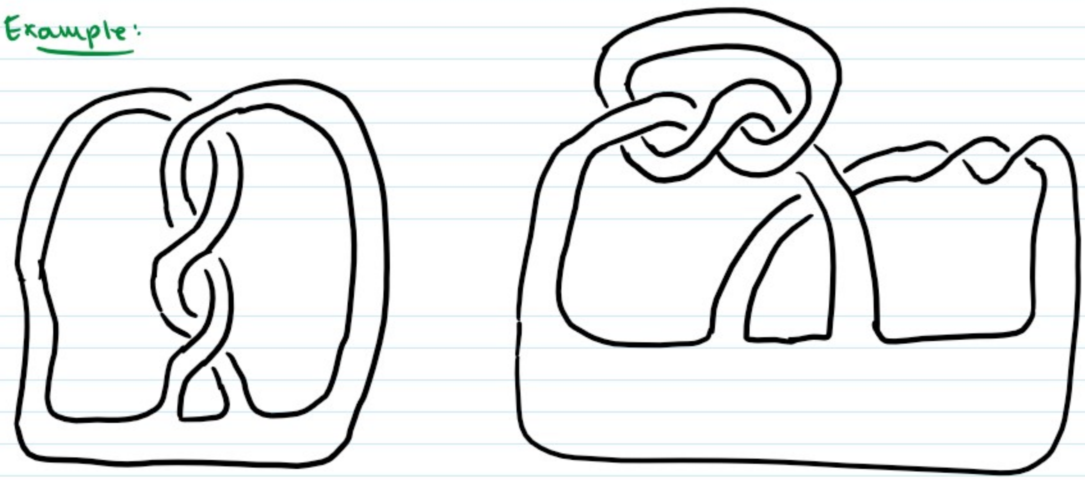
are diffeomorphic.

$\cong$ S^3 $\qquad$ $\cong$ Poincaré Sphere

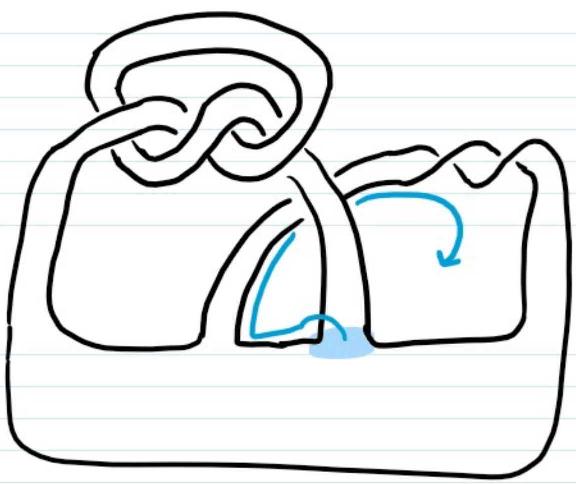
We saw S^3 and the Poincare Sphere have different Rokhlin invariants, hence are NOT diffeomorphic

Fact: Every Seifert surface is isotopic to a surface obtained by attaching bands to a disk
(But the bands can be attached in complicated ways)

Example:



Exercise: perform the indicated 1-handleslide



Remark:

We can slide the foot of a band over another band w/out changing the isotopy type of its boundary w/out changing the isotopy type of its boundary

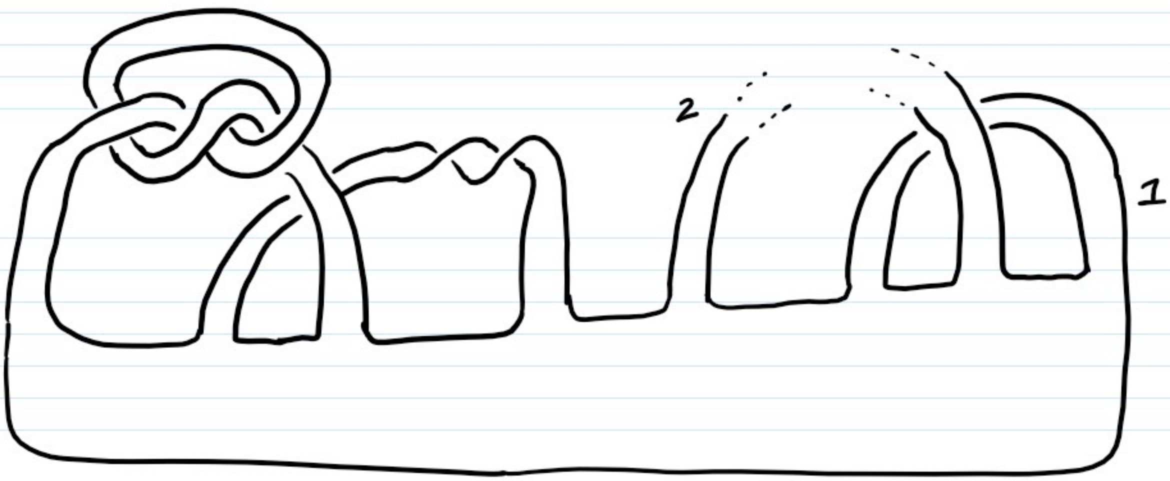
Remark:

We can add two bands:

① One is untwisted & unknotted

② One is twisted (full-twists), knotted, & can link other bands

Example:



This is called stabilization

↳ it increases the genus of the Seifert surface

defn: two Seifert surfaces for an oriented link L are Seifert equivalent if ∃ a sequence of stabilizations taking one to the other

Theorem: (J. Levine)

Any two Seifert surfaces for an oriented link are stably equivalent

Any two Seifert surfaces for an oriented link are stably equivalent

Seifert Matrix:

- We will focus on knots
- most of this works (with maybe minor adjustments) with links as well

Seifert matrix $S = (s_{ij})$

$$s_{ij} = lk(x_i, x_j^+)$$

positive push-off

F Seifert surface for K

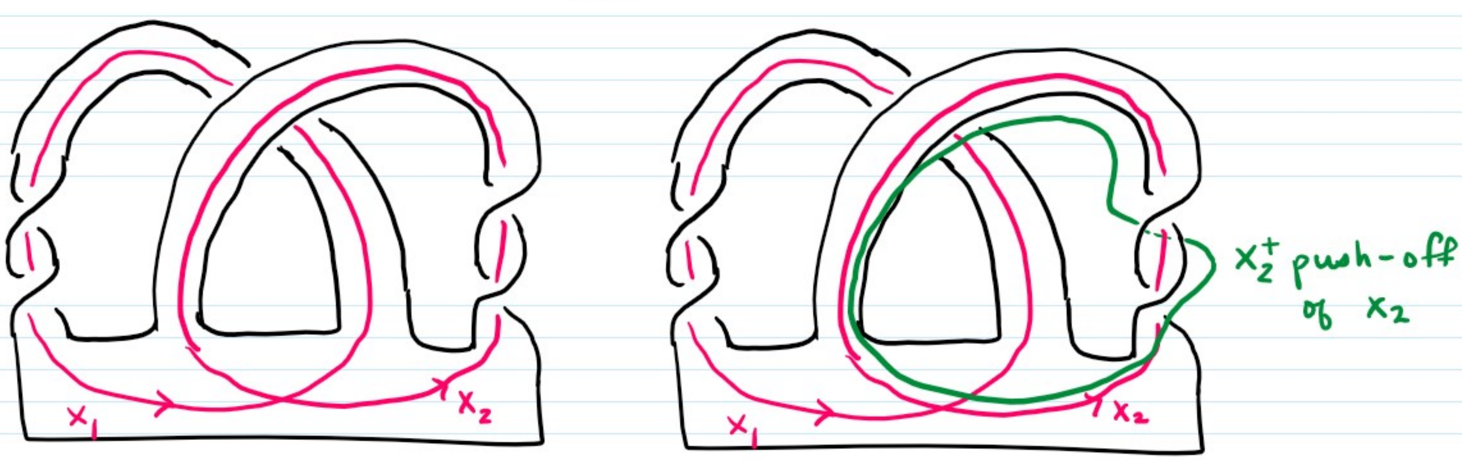
$\{x_i\}$ for $H_1(F; \mathbb{Z})$

Exercise:

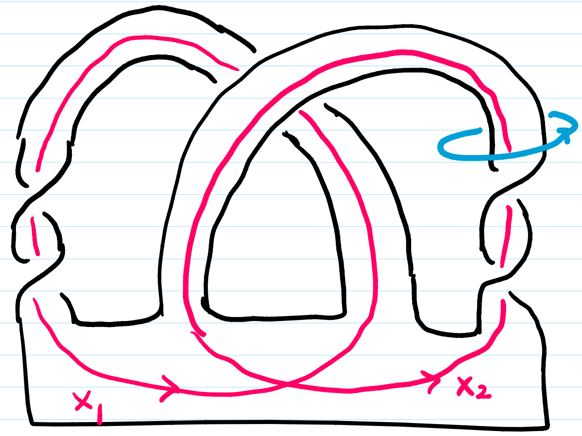
$$lk(x_i, x_j^+) = lk(x_i^-, x_j^+) = lk(x_i^-, x_j)$$

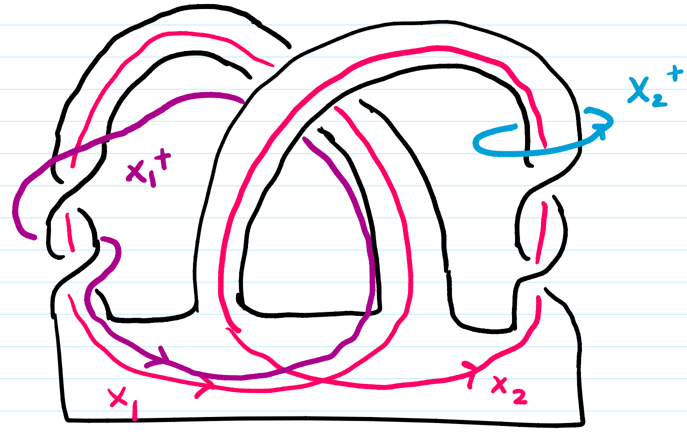
Example:

LHT

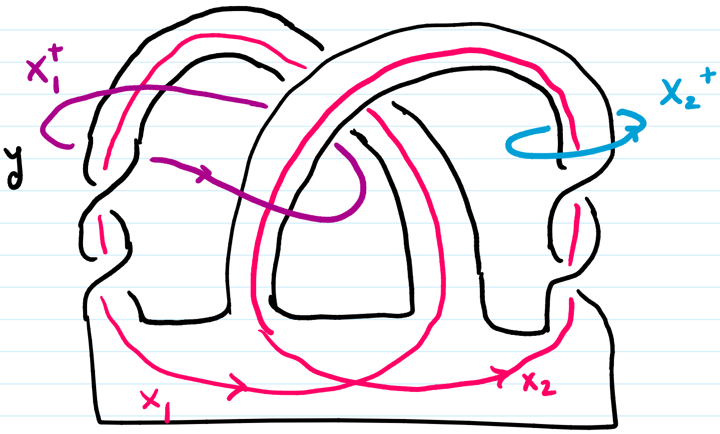


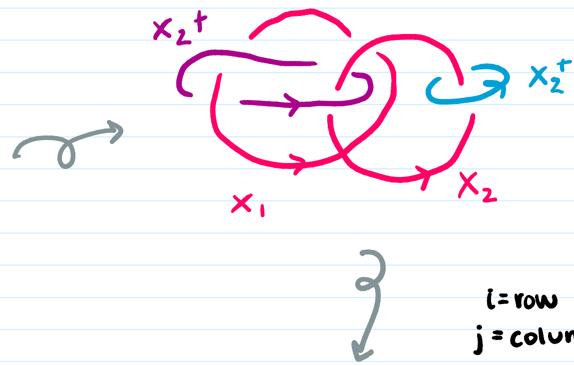
isotopy →





isotopy →





i = row
j = column

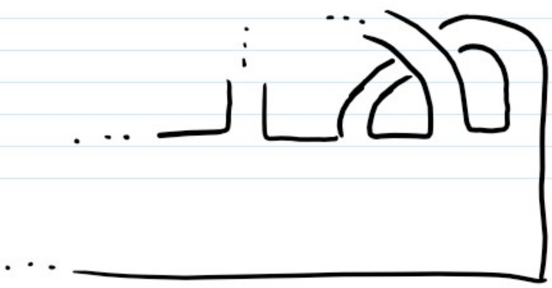
Seifert matrix: $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} lk(x_1, x_1^+) & lk(x_1, x_2^+) \\ lk(x_2, x_1^+) & lk(x_2, x_2^+) \end{bmatrix}$

Remark: If F is built from a disk with bands attached, bands induce a basis for $H_1(F)$

Exercise:

What is the effect on S of sliding band i over j?

Stabilization:



- adds 2 rows and 2 columns

$$S \longmapsto \left[\begin{array}{c|cc} S & | & | \\ & * & 0 \\ & | & | \\ \hline - * - & * & 1 \\ - 0 - & 0 & 0 \end{array}\right]$$

defn:
two integral matrices S_1 and S_2 are S-equivalent if $\exists$ a sequence of stabilizations that can be applied to obtain S_1' and S_2' that are related by $S_1' = U^T S_2' U$ for some invertible matrix U

Theorem:

$$\boxed{\text{Any 2 Seifert matrices for } K \text{ are S-equivalent}}$$

The Alexander Polynomial:

K a knot

S a Seifert matrix

defn:
$$\boxed{\begin{array}{c} \Delta_K(t) = \det\left(t^{1/2} S - t^{-1/2} S^T\right) \\ \text{or sometimes } \det\left(S - t S^T\right) \end{array}}$$

differ by factor of t^n

defn: $\boxed{\begin{array}{c} \Delta_K(t) = \det\left(t^{1/2} S - t^{-1/2} S^T\right) \\ \text{or sometimes } \det\left(S - t S^T\right) \end{array}}$ differ by factor of t^n

Remark:

- It's common to only define the Alexander polynomial $\Delta_K(t)$ up to a factor of $\pm t^n$
- Most common normalizations:
  - make polynomial w/ non-zero constant
  - symmetrize it (so smallest & lowest exponents are the same)

Exercise:

This is a well-defined knot invariant

hint: check that $\det\left(t^{1/2} S - t^{-1/2} S^T\right)$ is unaffected by stabilization

Example: U = unknot

$\Delta_U(t) = 1$

Example: K = LHT

$$S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\det\left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} - t \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right)$$

symmetrized

$\Delta_K(t) = t^2 - t + 1$ or $t - 1 + t^{-1}$

$\Delta_K(t) = t^2 - t + 1$ or $t - 1 + t^{-1}$

Mirror of a knot – change all crossings

example:

K mK

mirror

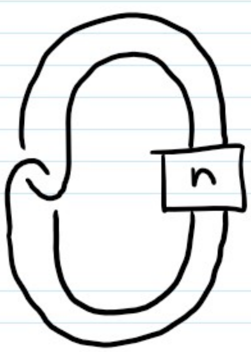
Exercise:

① $\Delta_{mK}(t) = \Delta_K(t)$

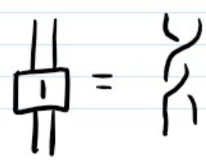
② $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$

Example:

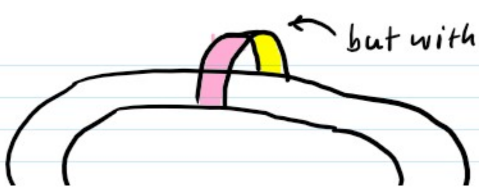
twist knots



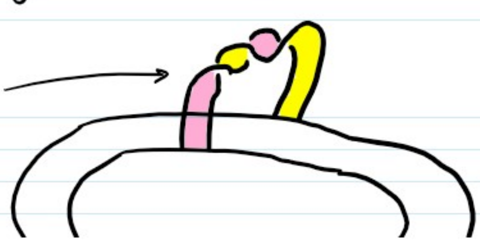
full twists where

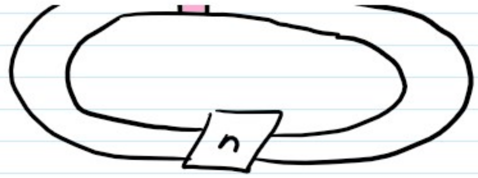


this can always bound genus 1 surface

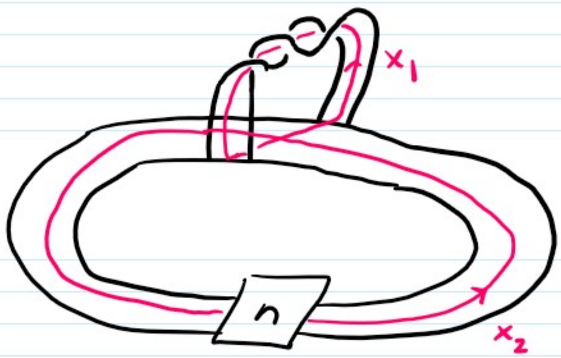


but with a full twist

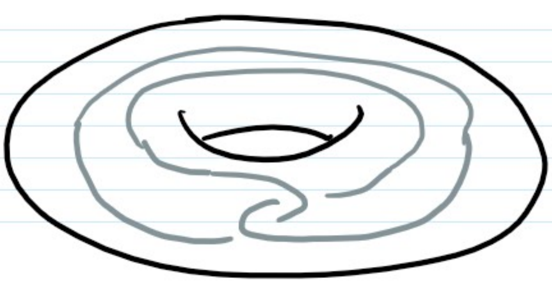




What's the Seifert matrix?



$$S = \begin{pmatrix} -1 & 0 \\ 1 & n \end{pmatrix}$$

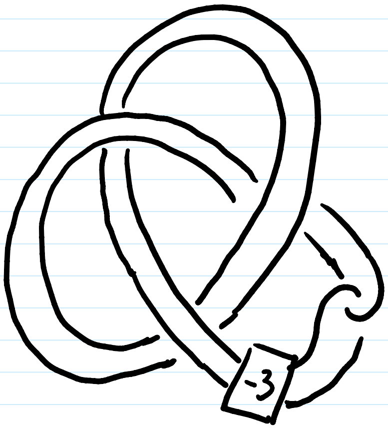


pattern $P \subset S^1 \times D^2$

$$\det(S - tS^T) = -nt^2 - (2n+1)t - n$$

Example:

Whitehead double $\qquad$ $P(K)$ sattelite knot



companion K

writhe(K) = # positive crossings − # negative crossings

$$S = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$$

$\det(S - tS^T)$

$\Delta_K(t) = -t$

$= 1$

Aside:

Whitehead doubles of nontrivial knots are always non-trivial

(check with fundamental group of

the knot complement)

called the knot group

Exercises:

1. $\Delta_K(t) \doteq \Delta_K(t^{-1})$ $\doteq$ ← means up to a factor of t^n

2. Normalize $\Delta_K(t)$ to be a polynomial w/ non-zero constant term

Show $g(K) \geq \frac{1}{2} \deg \Delta_K(t)$

Aside:

$\Delta_K(t) = t^2 - t + 1$ K = trefoil = LHT

other common normalization is

$t - 1 + t^{-1}$ (symmetrized)

↓

multiply by t and get rid of negative exps

Aside:

- $S + S^T$ is easiest way to get a symmetric matrix from S and can take signature of this thing
- can also take signature of Hermitian matrices $\omega S - \bar{\omega} S^T$

  $\omega \in$ unit circle in $\mathbb{C}$

  and signature only jumps at roots of Alexander polyn.

Alexander Module and Infinite Cyclic Covers

(Rolfsen Ch. 7)

$X = S^3 \setminus \nu(K)$

$H_1(X; \mathbb{Z}) = \mathbb{Z}$ (by Alexander duality ?)

$G = \ker\left(\pi_1(X) \longrightarrow H_1(X)\right)$

$\tilde{X}$ covering space of X corresponding to G

deck group is $\mathbb{Z}$ with generator t