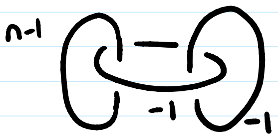
Week 10, 10/24 and 10/26

Tuesday, October 24, 2023 12:31 PM

Announcements: Exercises due 10/31 on Canvas
Dr. Hom will not be here next week, so no class on 10/31 and 11/2
In meantime, take a look at the Kirby problem list
homepages.warwick.ac.uk/~masaw/ftp/kirby_list.pdf
- Solving open Kirby problem means no homework in this class :)



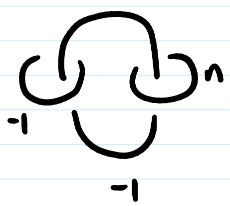
linking matrix: $\begin{bmatrix} n-1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

effect of handleslide on linking matrix:

add 2^{nd} col to 1^{st}: $\begin{bmatrix} n & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

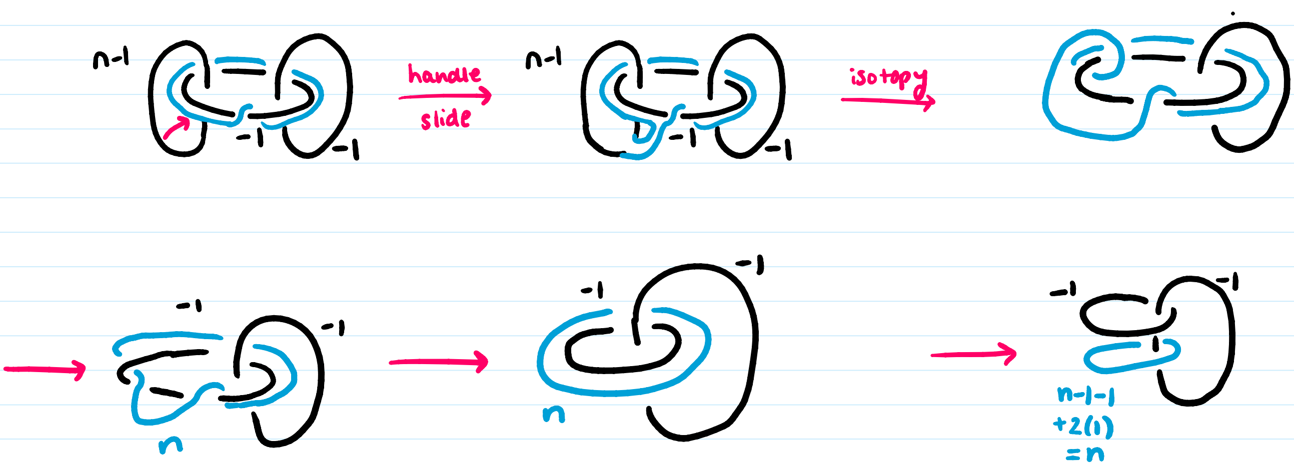
add 2^{nd} row to 1^{st}

blowing up in a different way gives



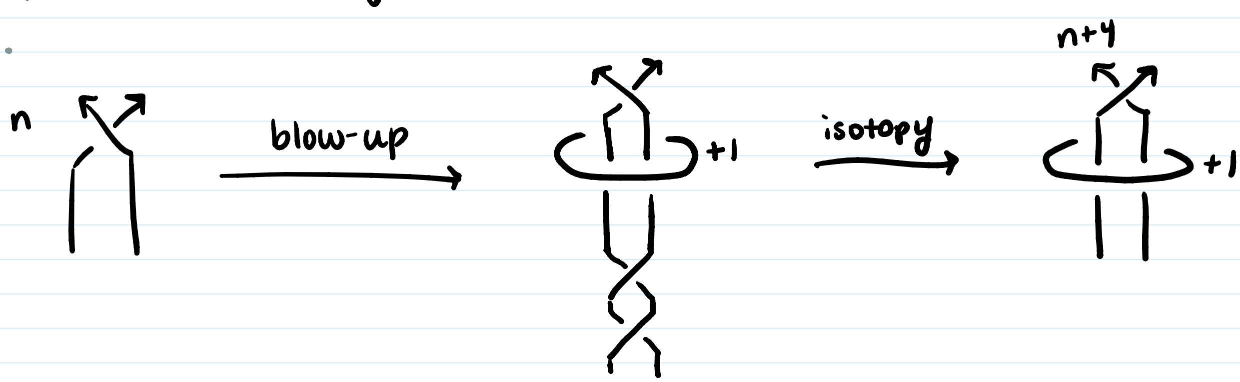
linking matrix: $\begin{bmatrix} n & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

by doing a handleslide:



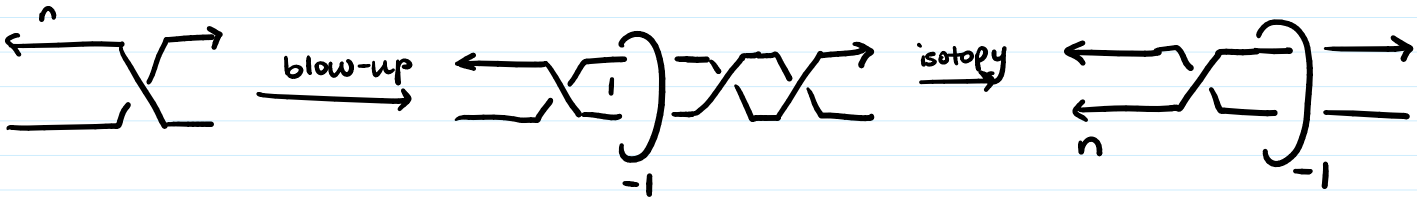
Ways to change crossings:

I.



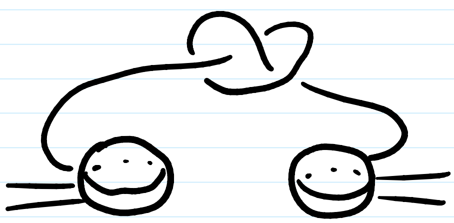
II

II.



Last time: dotted circle notation for 1-handle

Ex:

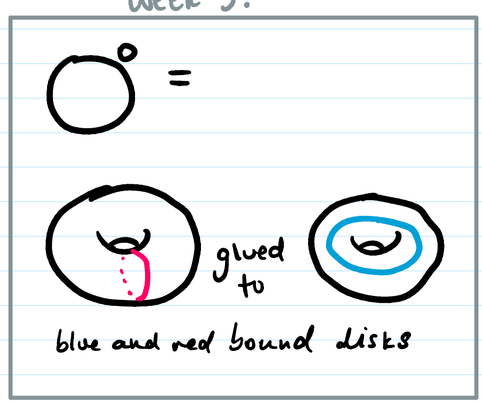


Example:

4-d 1-h
(two 3balls)

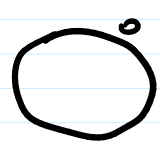
dotted circle notation

week 3:



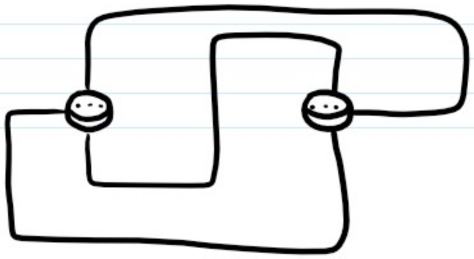
this helps us visualize 3-mfd boundary:

$$\partial((0\text{-h})\cup(1\text{-h})) = \partial(S^1\times D^3) = S^1\times S^2 =$$

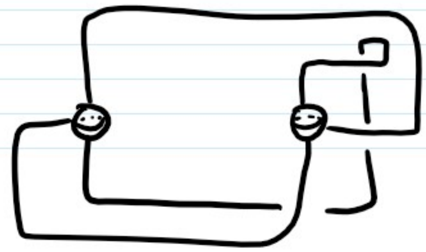


0-surgery on unknot

Example:

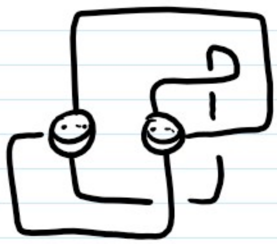


isotopy →

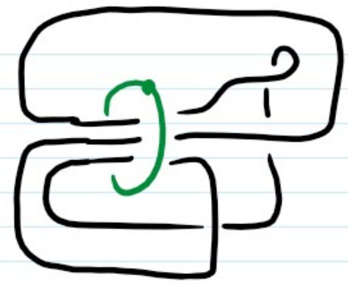


blackboard framing

isotopy →



$\xrightarrow[\text{circle}]{\text{dotted}}$

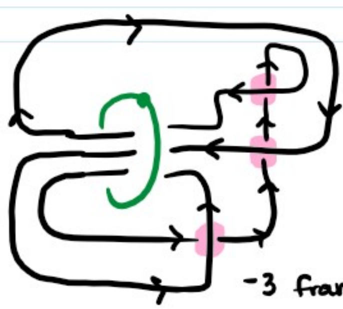


still blackboard framed

Computing blackboard framing:

+1

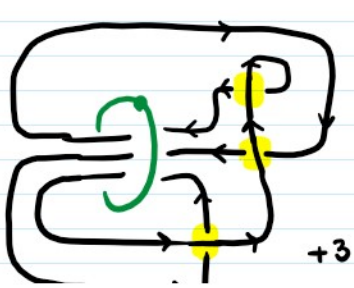
−1



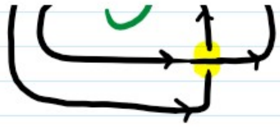
−3 framing

OR:

Exercise: Check that these are equivalent



+3 framing

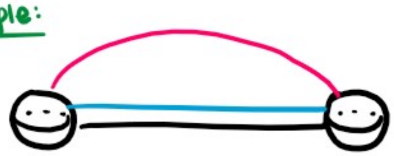


+3 framing

Note: passing from pair-of-3-balls to dotted circle notation requires a choice of arc connecting the two balls

arc

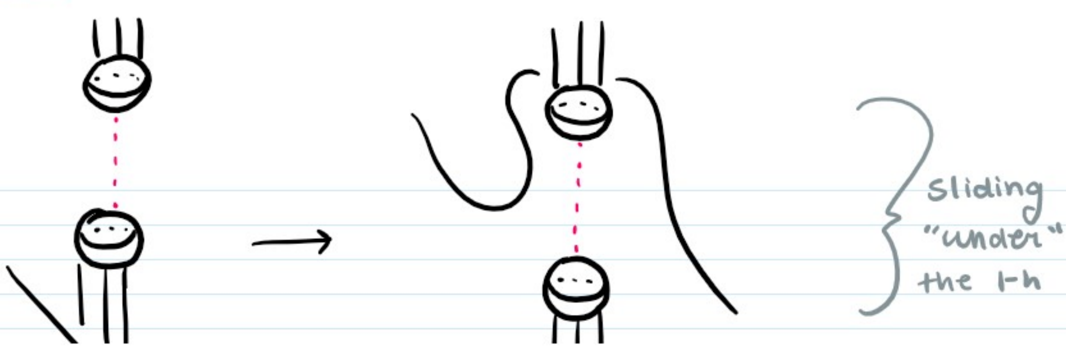
Example:

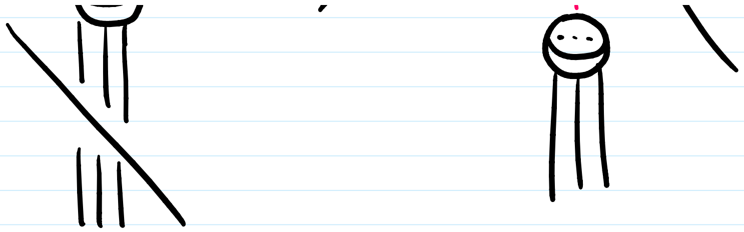


Exercise: Describe how to compute $H_1(\partial X)$ from a Kirby diagram for X where 1-handles are in dotted circle notation

hint: first consider when X has no 3-handles

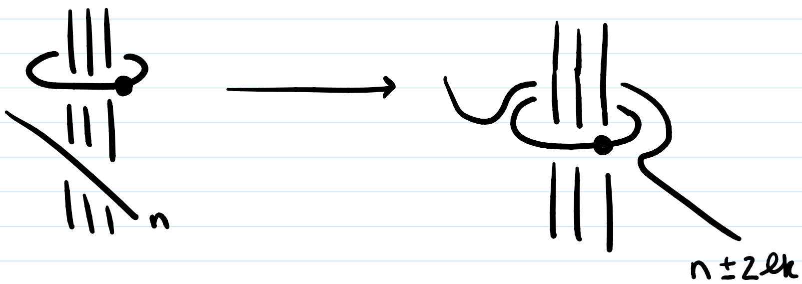
Example:





the 1-h

In dotted circle notation:



formally this is similar to doing a handleslide over
the dotted circle notation.

but really it's going "under" the 1-handle

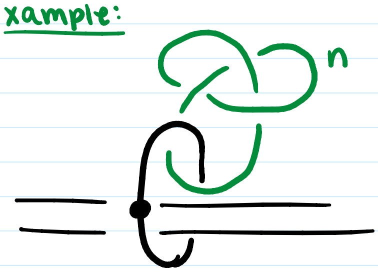
Exercise: check that new framing is $n \pm 2lk$
(because it crosses the ⋮ arc)

Kirby Calculus in dotted circle notation:

2-3 cancelling pair is the same
(as long as we allow slides under 1-h's)

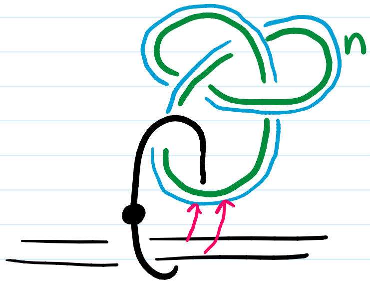
1-2 cancelling pair via example

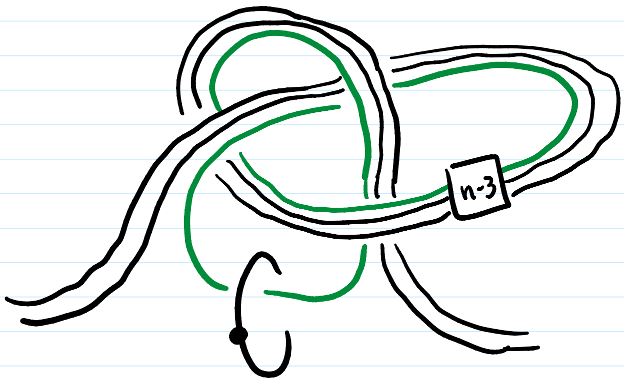
Example:



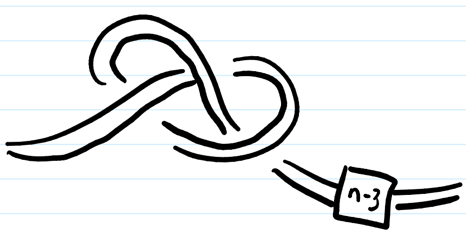
can't cancel trefoil until other 2-handles are slid over trefoil

↓ handleslide



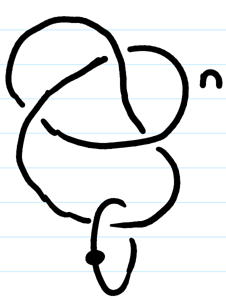


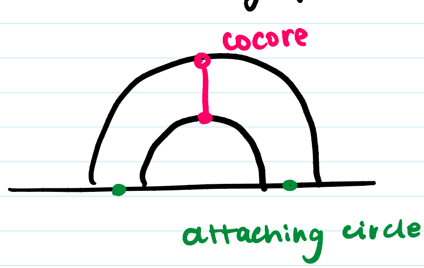
$\downarrow$ cancellation



Exercise: What happens to framings?

Remark: How to see it is a cancelling pair:



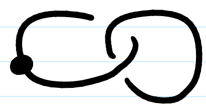


- attach n-framed 2-handle first

- disk D spanned by dotted circle
- can identify D with cocore of 2-h

(since dotted circle is a meridian of attaching circle for 2-h $\Rightarrow$ removing νD is same as removing 2-h)

Example:

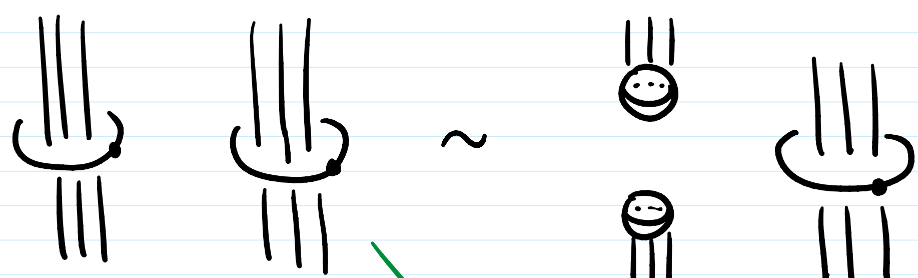


2-h can be identified with νD

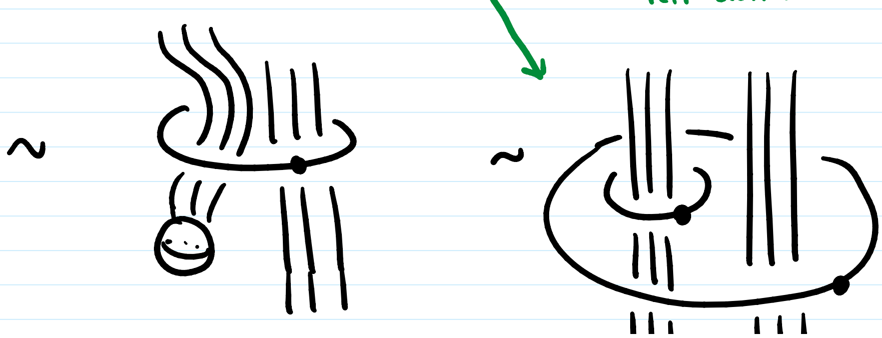
Exercise:

Show that if the attaching circle for a 2-h intersects attaching disk D of a dotted circle in a unique point, then there is a sequence of handle moves that puts 1- and 2-h's in this position

1-handle slide :



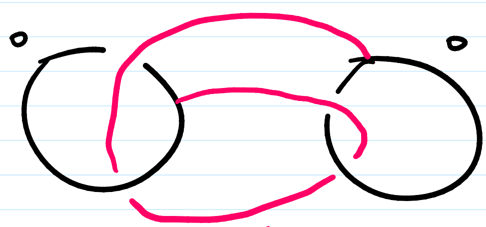
Formally: this is same as slide right dotted circle over left dotted circle



Note: Except the one handle doing the "moving" is the dotted circle on left

Remark: dotted circles need to bound disks so we always need a union of dotted circles to remain an unlink

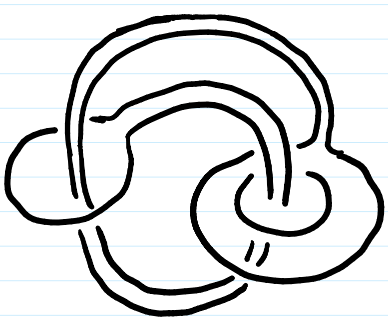
Nonexample:



0-framed 2-h's

choose a different arc.

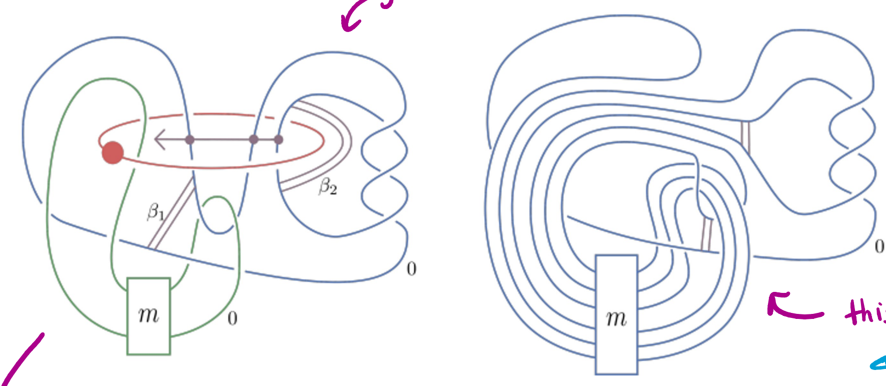
Result:



So this is not a legal move if these were dotted two-handles

From Lisa Piccirillo's research paper "Shake genus and slice genus"

green handle cancels with dotted handle



this is a knot trace

FIGURE 4. A handle diagram for X_m and diffeomorphism to $X_0(K_m)$

isotopy

2 different knots which share the same 0-trace

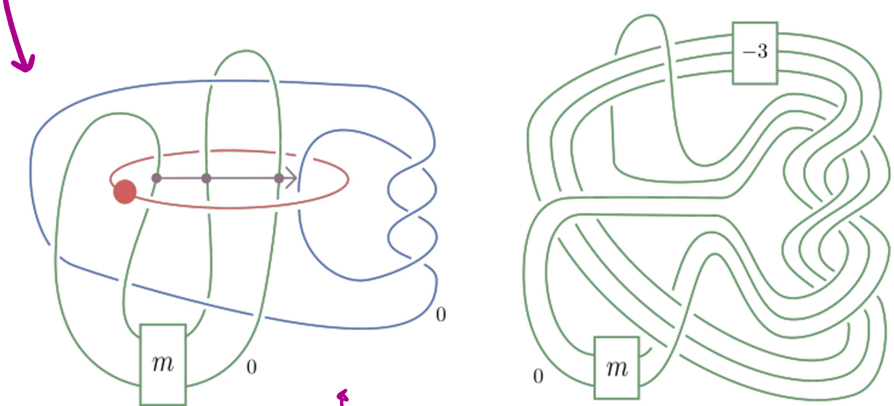
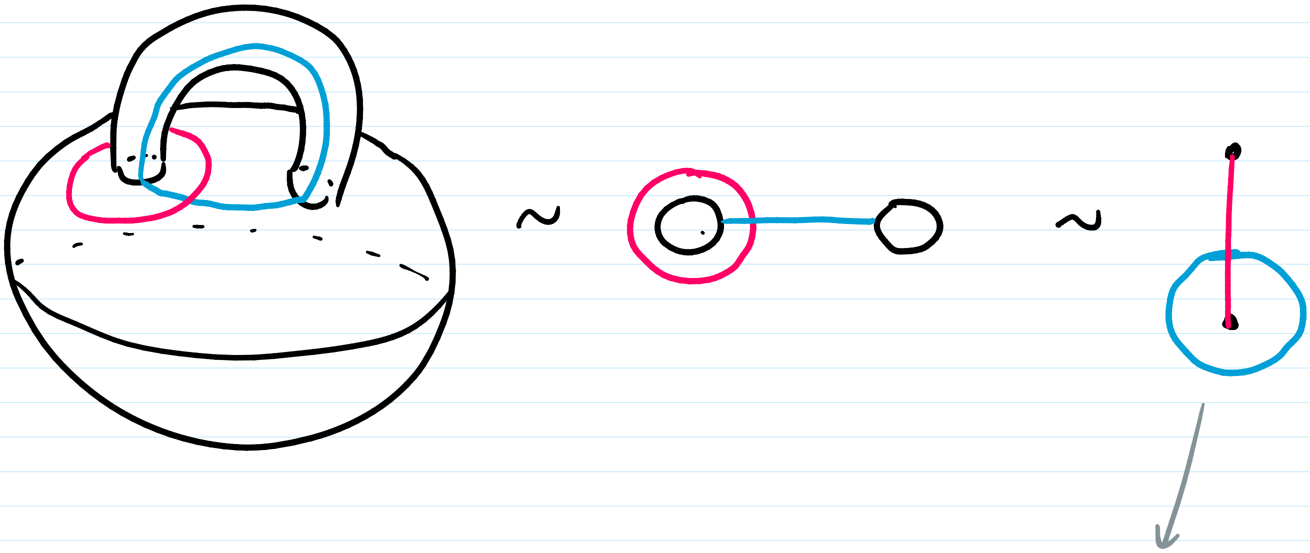


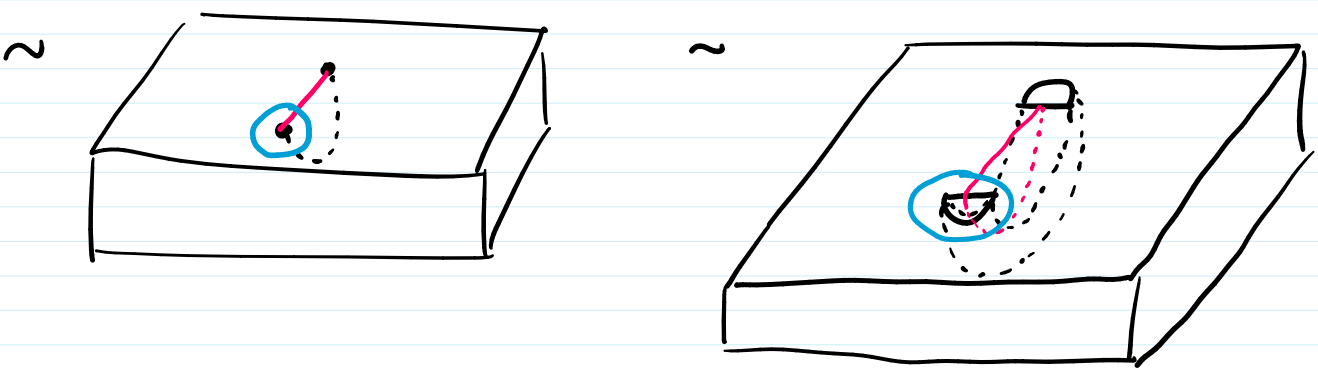
FIGURE 5. The same handle diagram for X_m and diffeomorphism to $X_0(K'_m)$

blue 2-h cancels with dotted handle

Kirby calculus in dotted circle notation:



then dig a tunnel



In 4-dimensions:

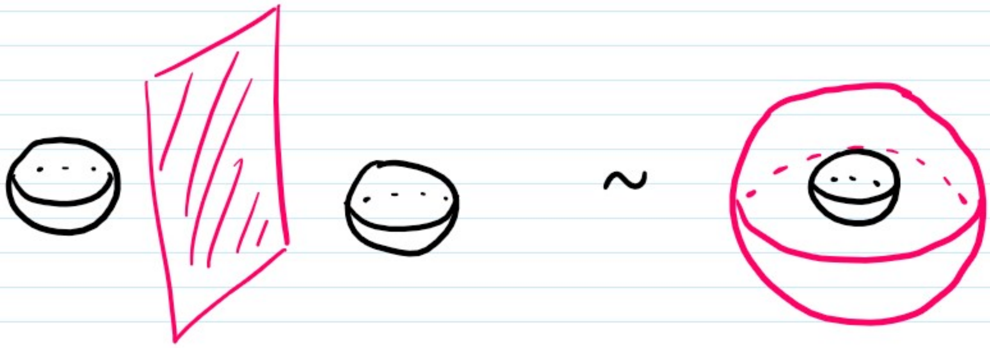
$\partial(S^1 \times D^3) = S^1 \times S^2$

where are the S^1's worth of S^2's in these diagrams?

$\{pt\} \times S^2$

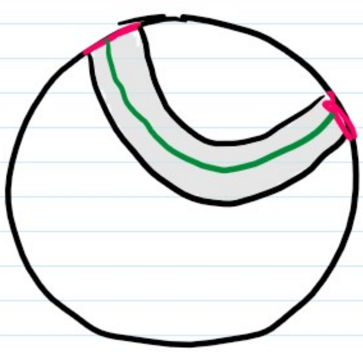
$\{pt\} \times S^2$

☺ @ ∞



all these S^2's are non-separating so they do not bound balls

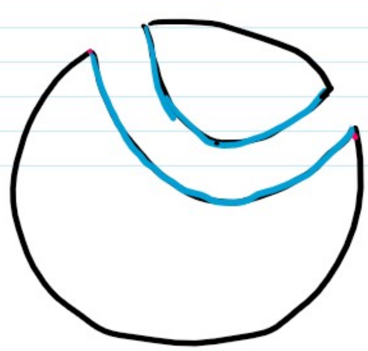
$$\partial\left(D^4 - \nu(\text{pushed in } D^2)\right) = S^1 \times S^2$$



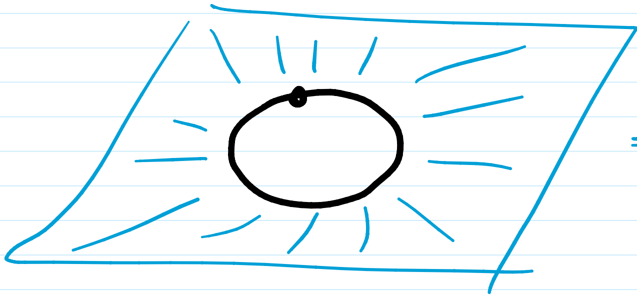
lose this boundary

cut out a nbhd of disk

and gain this boundary

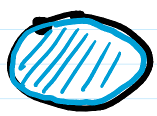


In  this diagram, we can see the S^1's worth of S^2's



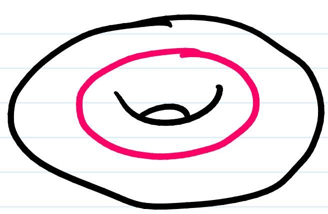
= disk through ∞

disk

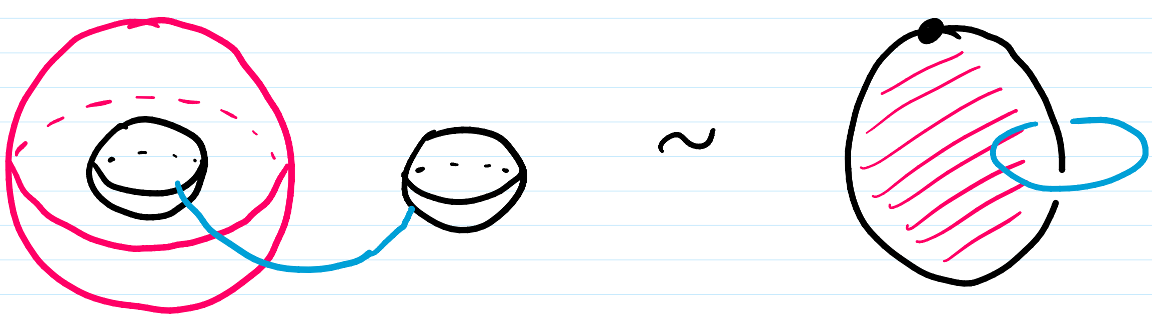


disk

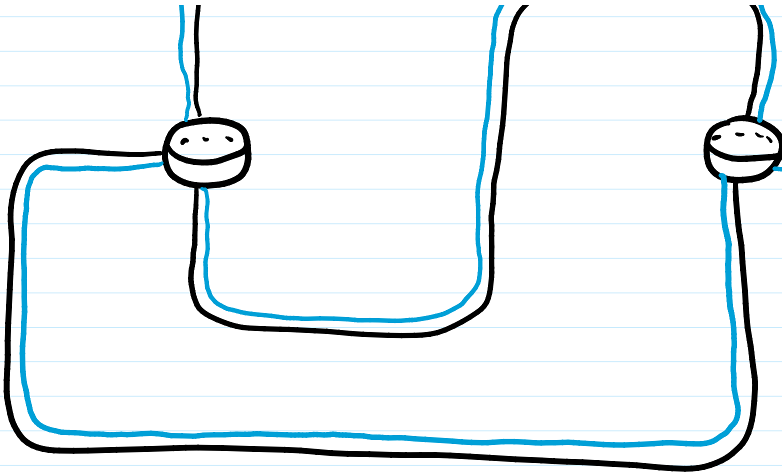
blue disk glued to disk



to get S^2

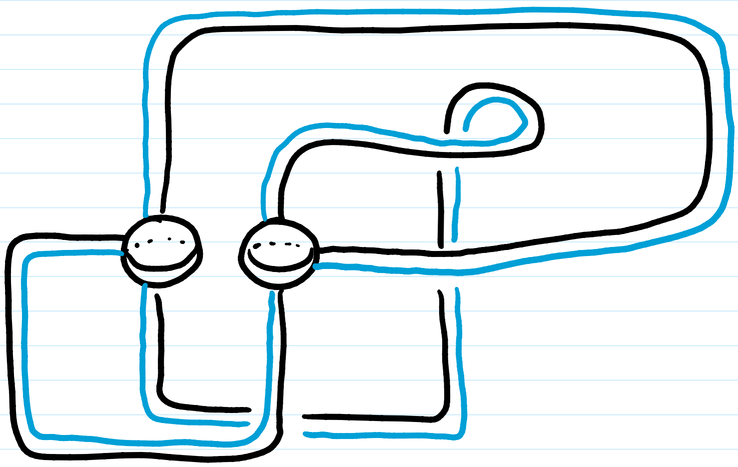


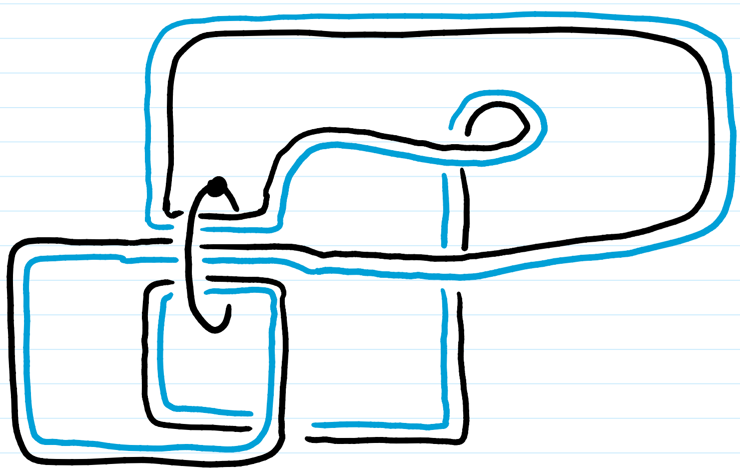
Example:



blackboard framing

move right 3-ball to the left

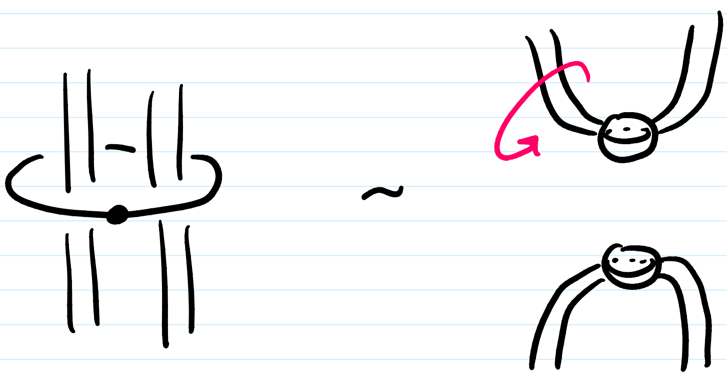




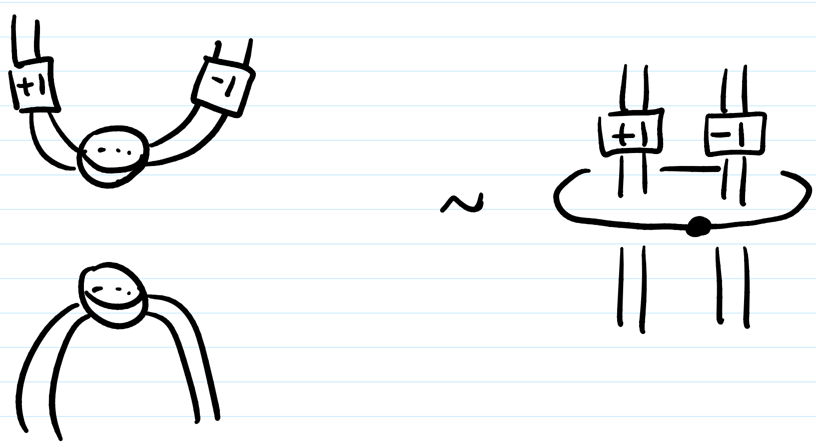
Blow-ups, Blow-downs:

Same as before, as long as $\bigcirc^{\pm 1}$ is geometrically unlinked from dotted circles.

Example: twisting 1-handle



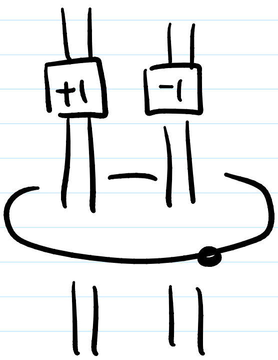
Let's put in a full twist:



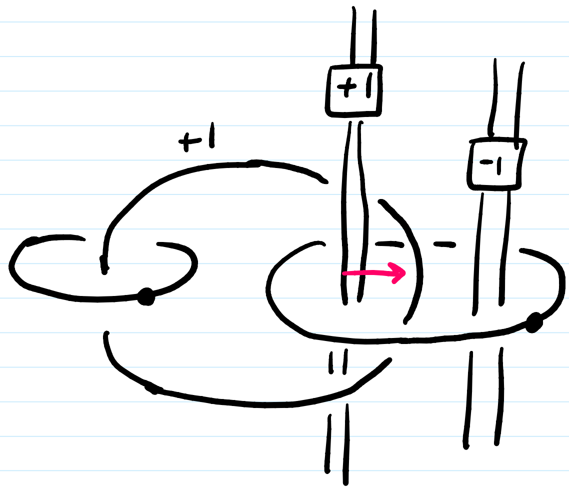
Exercise: use double strand notation to determine the framings

Another way to see it:

Example:

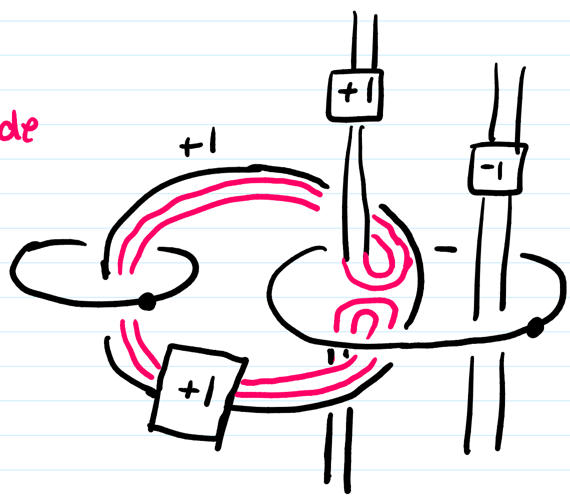


1-2 pair $\sim$



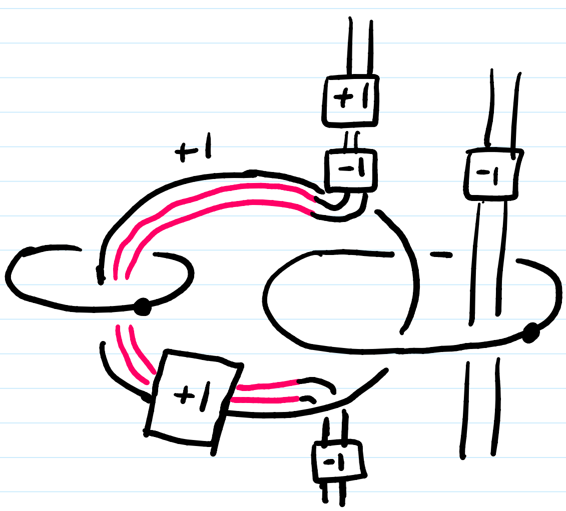
handleslide both strands

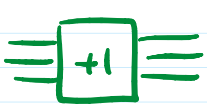
$\sim$

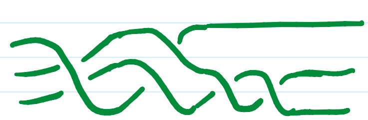


isotopy

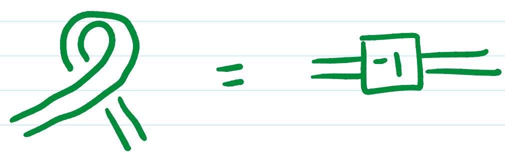
$\sim$



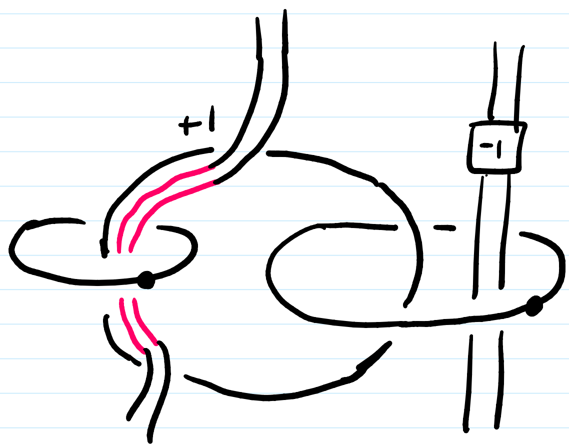
Remark: 

= 

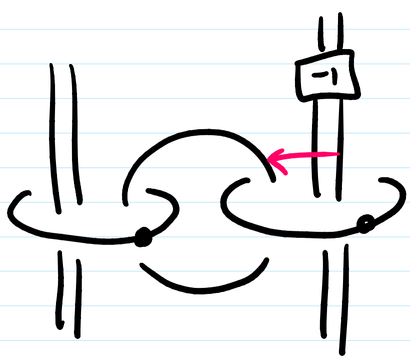
Remark:



isotopy $\sim$

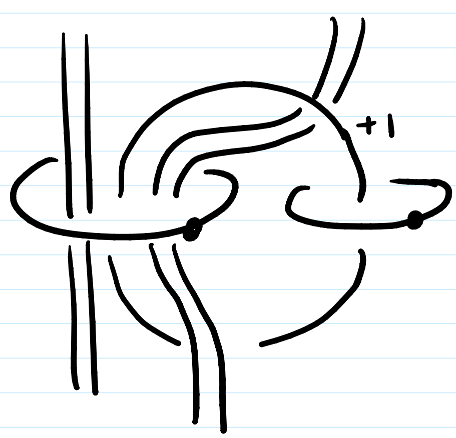


isotopy $\sim$

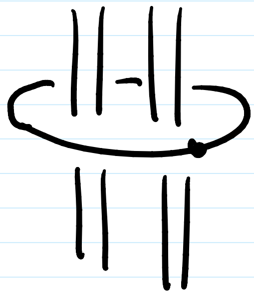


handleslide

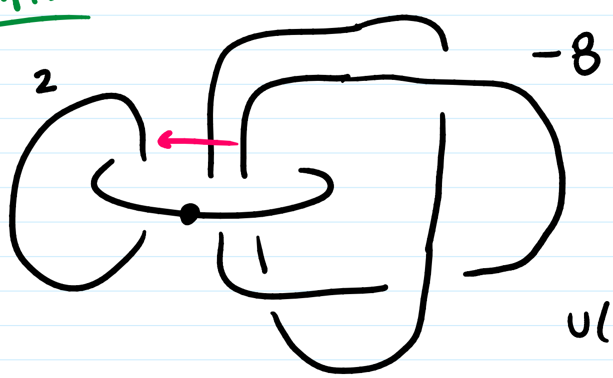
handleslide $\sim$



cancelling
~
1/2 pair

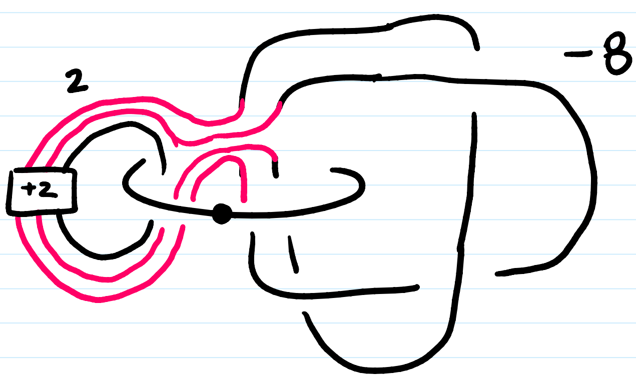


Example:

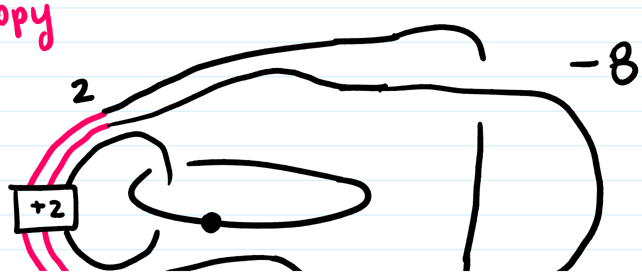


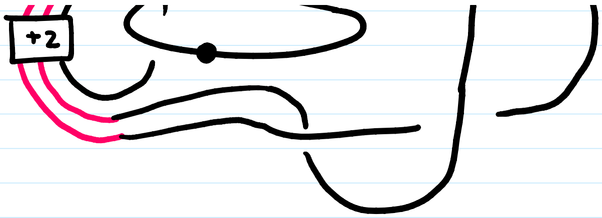
$\cup(3\text{-}h)\cup(4\text{-}h)$

~ handleslide



isotopy
~

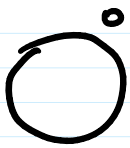




cancelling $\sim$ 1/2 pair

$-8 + 2^2 \cdot 2$

$\cup(3\text{-h}) \cup (4\text{-h})$

$\sim$ 

$\cup(3\text{-h}) \cup (4\text{-h})$ $= S^4$

Rokhlin Invariant:

We can use 4-mfds to get invariants of 3-mfds

Exercise: Let X_1, X_2 be smooth 4-mfds with diffeomorphic $\mathbb{Z}HS^3$ boundaries. Let $\varphi: \partial X_1 \longrightarrow \partial X_2$ be an orientation reversing diffeo and $X = X_1 \cup_\varphi X_2$.
Show that $\sigma(X) = \sigma(X_1) + \sigma(X_2)$

defn: Let Y be a $\mathbb{Z}HS^3$ and W a smooth

defn: Let Y be a $\mathbb{Z}HS^3$ and W a smooth simply connected 4-mfd with $\partial W = Y$ whose intersection form Q_W is even. The **Rokhlin invariant** of Y is

$$\boxed{\mu(Y) = \frac{1}{8}\sigma(W) \bmod 2}$$

Remark:

1) such a W exists since Y can be expressed as an even surgery on a link

2) $\mathbb{Z}HS^3$ guarantees unimodular so Q_W is unimodular since $\partial W = \mathbb{Z}HS^3$ so $\sigma(Q_W)$ is divisible by 8

3) different choice W' gives the same Rokhlin invariant $\mu(Y)$ since

$$\sigma(W \cup_Y -W') = \sigma(W) - \sigma(W') \equiv 0 \bmod 16$$

Exercise:

1) $\mu(-Y) = -\mu(Y)$

2) $\mu(Y_1 \# Y_2) = \mu(Y_1) + \mu(Y_2)$

Examples:

$\mu(S^3) = 0$

$\mu\left(\Sigma(2,3,5)\right) = \frac{1}{8}(8) \bmod 2 = 1$

Poincare Homology sphere

-1-surgery on left handed trefoil

OR