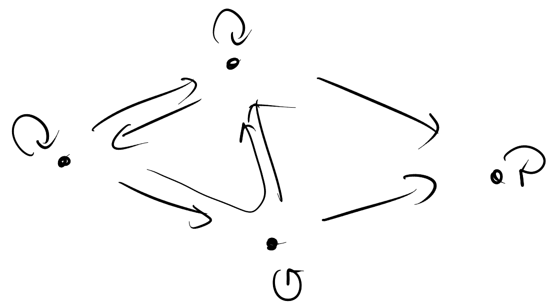
Title: 2-cats & graphical Presentation

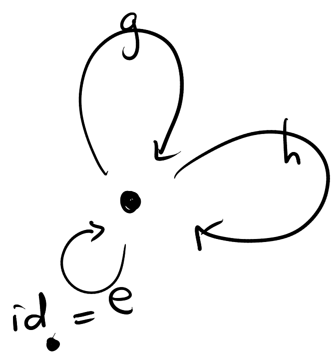
- **Def** (Category)



- **Example**

0. Let G: finite group.

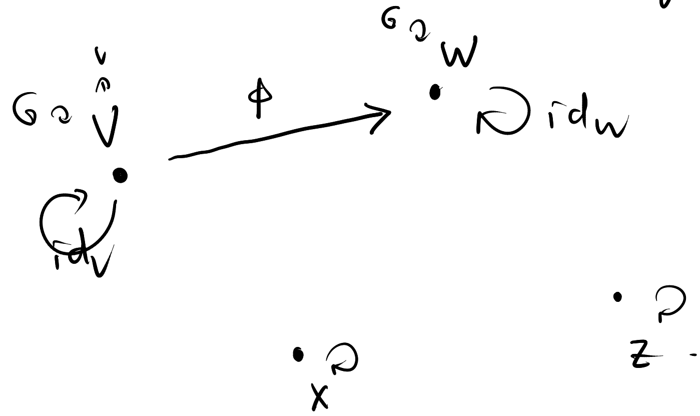
1. $\boxed{\mathbb{B}G:}$



$Mor(\mathbb{B}G) = G$

$\phi(g.v) = g.\phi(v)$

2. $\boxed{Rep(G)}$



has more structures than a category.

$\cdots \quad \otimes \quad \cdots \qquad V, W \overset{\otimes}{\rightsquigarrow} V \otimes W$

$$g(v \otimes w) = (gv \otimes gw)$$

$$\mathrm{Rep}\,G \times \mathrm{Rep}\,G \xrightarrow{-\otimes-} \mathrm{Rep}\,G$$

$$(V \otimes W) \otimes U \overset{\alpha}{\xrightarrow{\sim}} V \otimes (W \otimes U)$$

$$(v \otimes w) \otimes u \mapsto v \otimes (w \otimes u)$$

- <u>Def</u> (monoidal category) cf. [EGNO] Tensor Categories

a category C with... $\otimes$, α

$\mathbb{1}, \ell, r$ $\cdots$ $\mathbb{1} \otimes V \overset{\ell}{\xrightarrow{\sim}} V$, $V \otimes \mathbb{1} \overset{r}{\xrightarrow{\sim}} V$

$$\otimes : C \times C \longrightarrow C$$

$$(V \otimes W) \otimes U \underset{\alpha}{\xrightarrow{\sim}} V \otimes (W \otimes U)$$

$$\alpha : (- \otimes -) \otimes - \longrightarrow - \otimes (- \otimes -)$$

$$\begin{tikzcd}
((V \otimes W) \otimes U) \otimes X \arrow[r, "\alpha", "\sim"'] \arrow[d, "\alpha"', "\wr"] & (V \otimes W) \otimes (U \otimes X) \arrow[r, "\alpha"] & V \otimes (W \otimes (U \otimes X)) \\
(V \otimes (W \otimes U)) \otimes X \arrow[r, "\alpha"'] & V \otimes ((W \otimes U) \otimes X) \arrow[ur, "\alpha"'] \arrow[u, phantom, "="] &
\end{tikzcd}$$

Eg 3. $\mathrm{Rep}\, U_q \mathfrak{g}$. $q \in \mathbb{C}^\times$ .... $q \notin \sqrt{1}$

↑ also has a monoidal structure

$$(V \otimes W) \otimes U \xrightarrow[\alpha]{\sim} V \otimes (W \otimes U)$$

$$(v \otimes w) \otimes u \longmapsto v \otimes (w \otimes u)$$

Grothendieck Ring $(\mathrm{Rep}\, U_q \mathfrak{g})$ all same for all $q \notin \sqrt{1}$.

But! With α, $(\mathcal{C}, \otimes, \alpha)$ ...

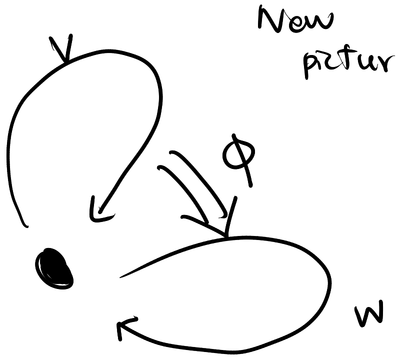
as monoidal cats .. they are different.

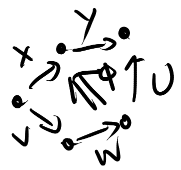
α: monoidal str. --- F-symbols --- 6j-symbols .

§ Another perspective for monoidal cats.

So by mimicking $\mathbb{B}G$, a monoidal cat can be presented as a 2-cat.

New picture





Old picture

$$V \otimes W \otimes U \xrightarrow{\phi} X \otimes Y$$

Def (2-category [strict]) A 2-cat $\mathcal{C}$ is a 1-cat

w/ $\{\text{obj}\} = C_0$, $\{\text{Mor}\} = C_1$

but also .. $\forall x, y \in C_0$

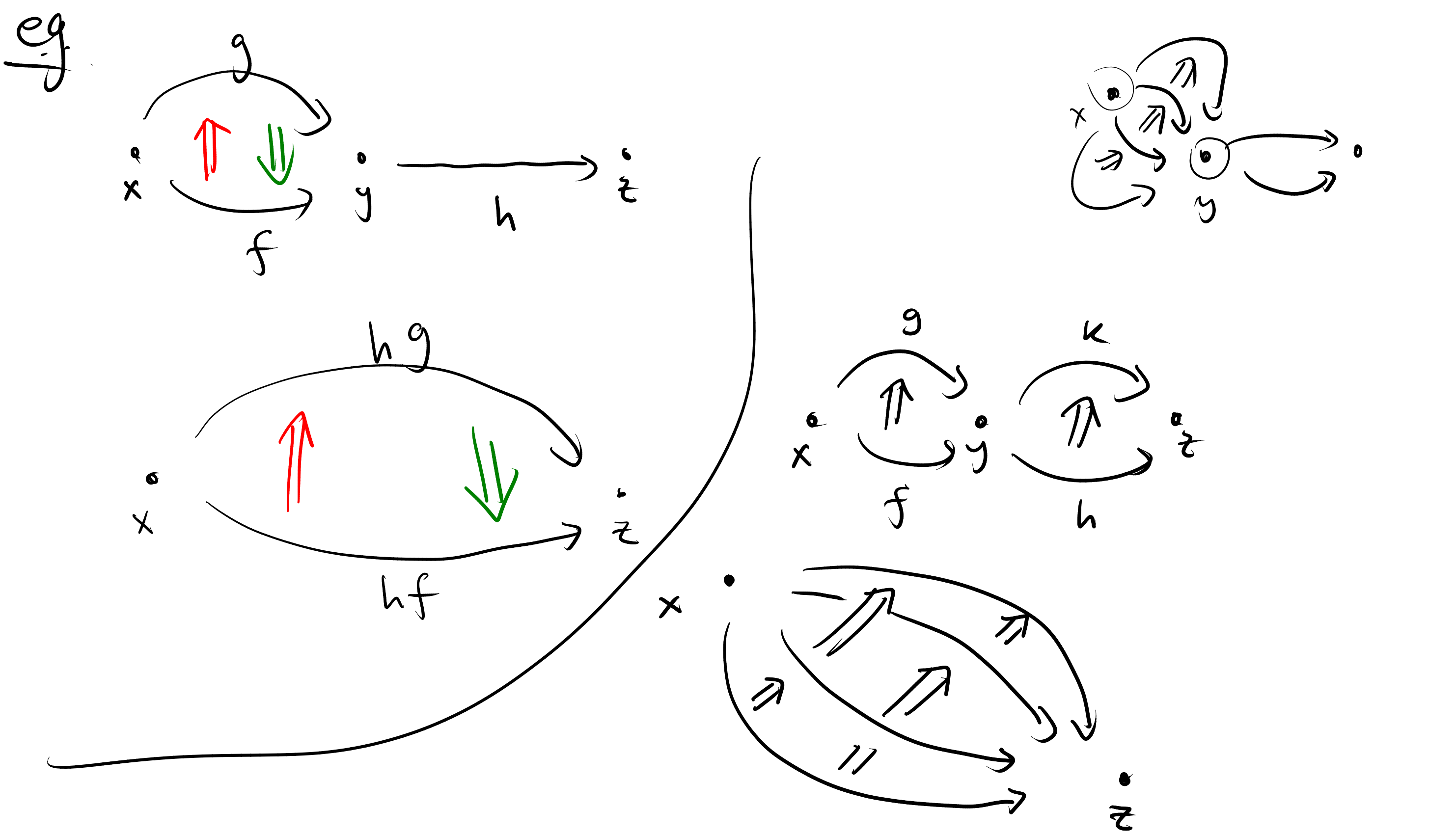
$\mathrm{Mor}_C(x,y)$ is again a 1-category

w/ some extra condition.

$$\underset{z \leftarrow y}{\overset{\text{1-cat}}{\mathrm{Mor}_C(y,z)}} \times \underset{y \leftarrow x}{\overset{\text{1-cat}}{\mathrm{Mor}_C(x,y)}} \xrightarrow{\circ} \underset{z \leftarrow x}{\overset{\text{1-cat}}{\mathrm{Mor}_C(x,z)}}$$

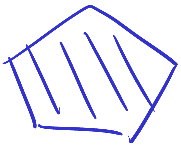
Dictate $\circ$ to be a functor.

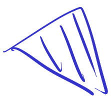
eg



Def (weak 2-category (or a bicategory)) - - - - - - [cf Bénabou Intro to bicats §1.]

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D, \qquad (h(gf)) \;\cancel{=}\; ((hg)f), \qquad (h(gf)) \xrightarrow{\sim} ((hg)f)$$





eg Rep$(U_q \mathfrak{g})$

Rep(G)

any monoidal category

$(\mathrm{Vect}, \otimes)$

...

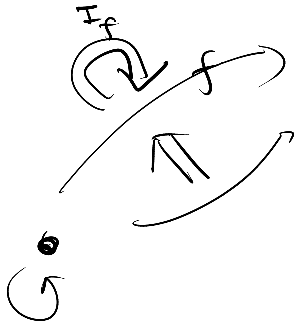
coherence

Thm (cf Gurski's PhD thesis Cor 2.2.7)

Every bicat is biequivalent to a strict 2-category.

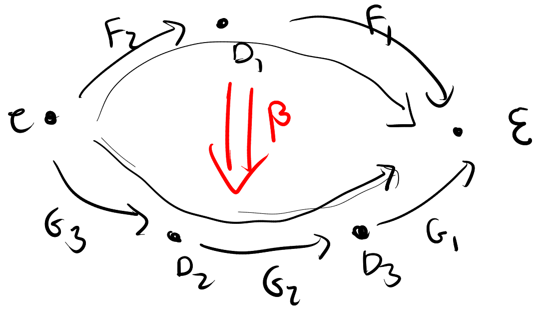
weak 2-category

RMK: This fails for $n > 2$. $(\because \pi_3(S^2) \neq 0)$

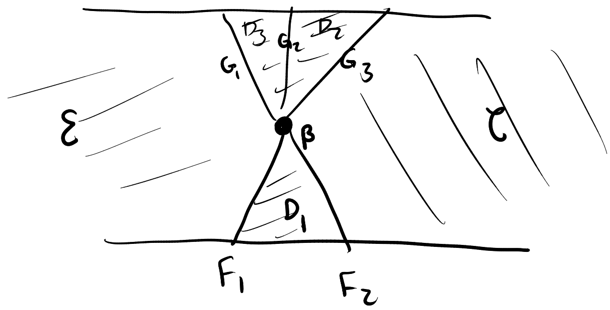


idea Graphical Poincare dual

Old Picture



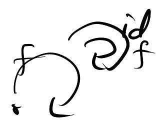
New Picture



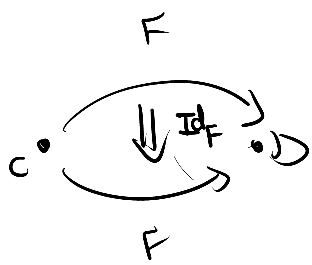
0 cells ⇝ 2 cells

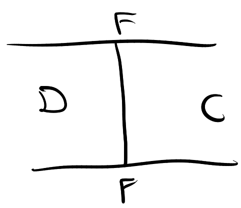
1 cells ⇝ 1 cells

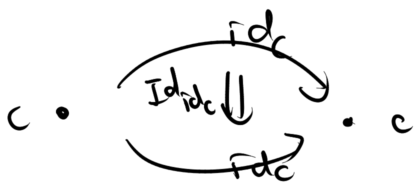
2 cells ⇝ 0 cells

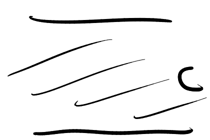


Convention Omit any identities

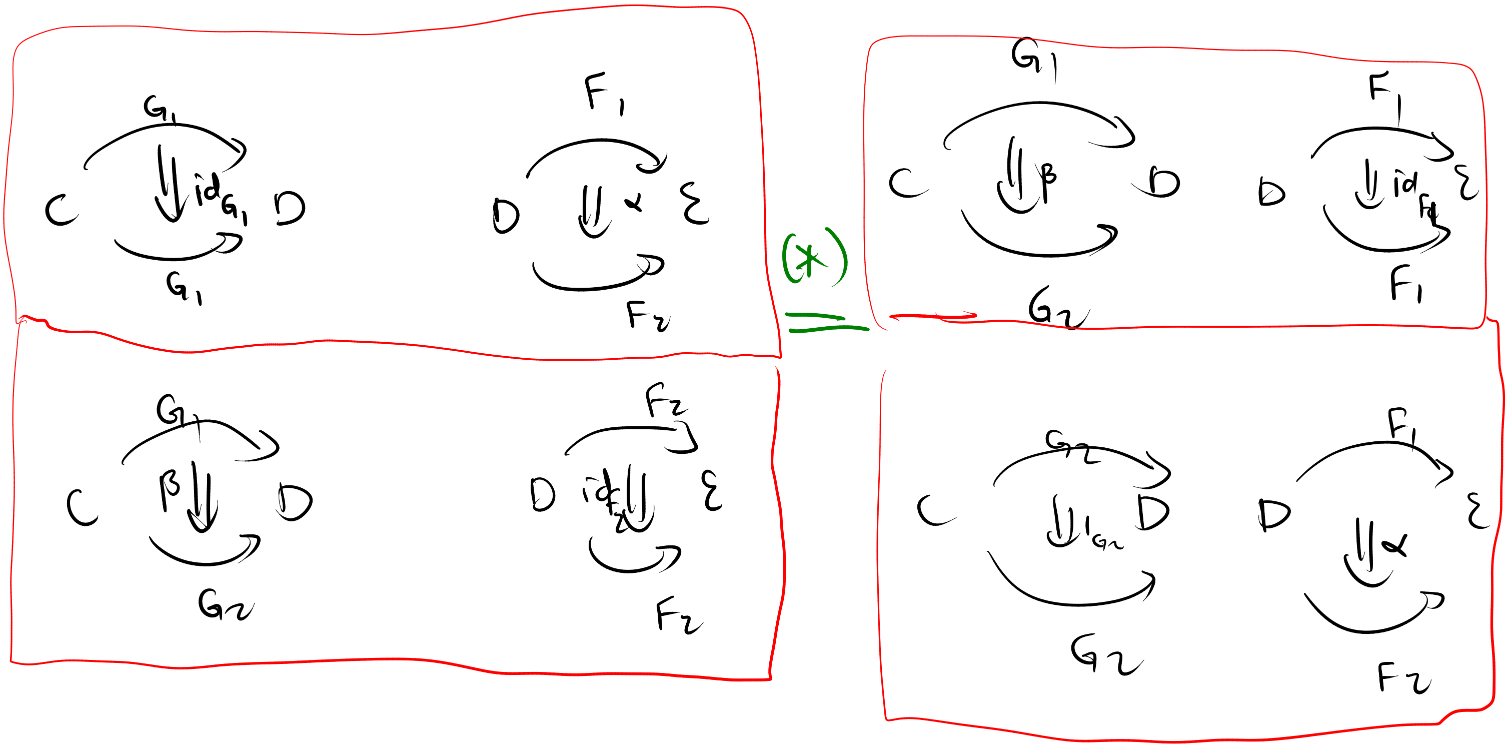








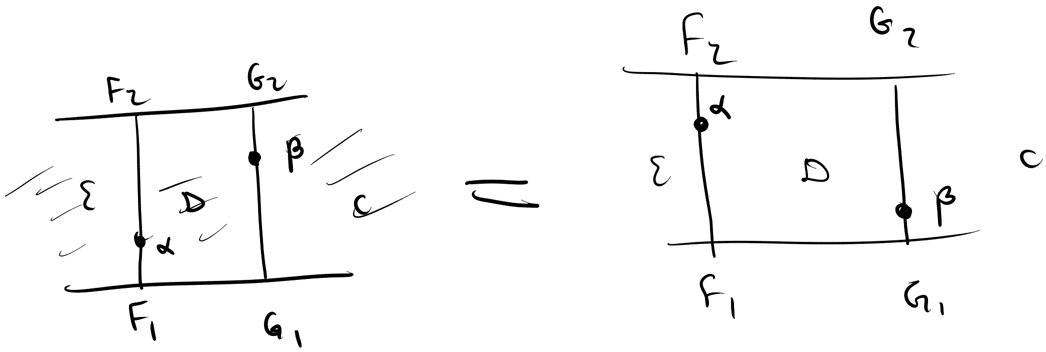
— Why good? ... example



Thm The axioms of a strict 2-cat imply that a planar diag, up to rectilinear isotopy, unambiguously represents a 2-morph!

Cor $(*)$ holds.

proof.



#

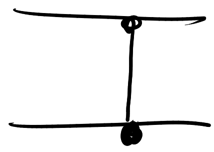
<u>exercise</u> (Eckmann-Hilton argument)

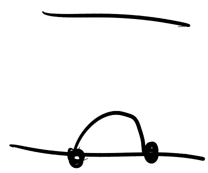
$\pi_{n\geq 1}(X)$ ∴ abelian group

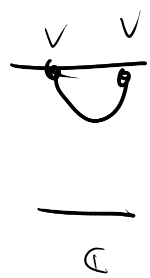
#

§ Temperley-Lieb category

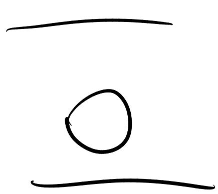
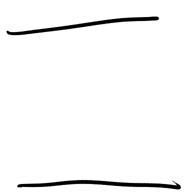
Let $C = (\mathrm{Vect}^{fin}, \otimes)$

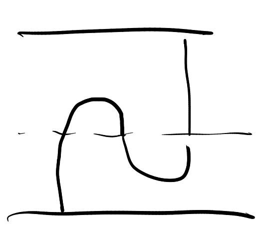
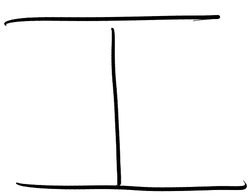
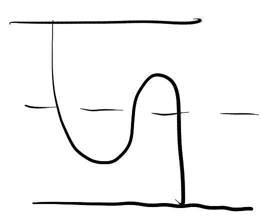
$V := \mathbb{C}^2 = \mathbb{C}\langle e_1, e_2 \rangle$ 

$V \otimes V \xrightarrow{n} \mathbb{C}$: $\begin{cases} e_1 \otimes e_1 \mapsto 0 \\ e_2 \otimes e_2 \mapsto 0 \\ e_1 \otimes e_2 \mapsto -1 \\ -e_2 \otimes e_1 \mapsto -1 \end{cases}$ 

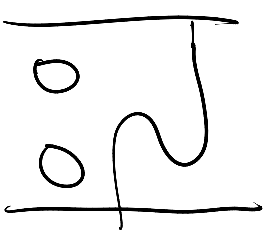
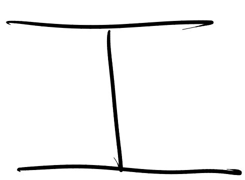
$\mathbb{C} \xrightarrow{u} V \otimes V$ $\quad 1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$ 

Clearly

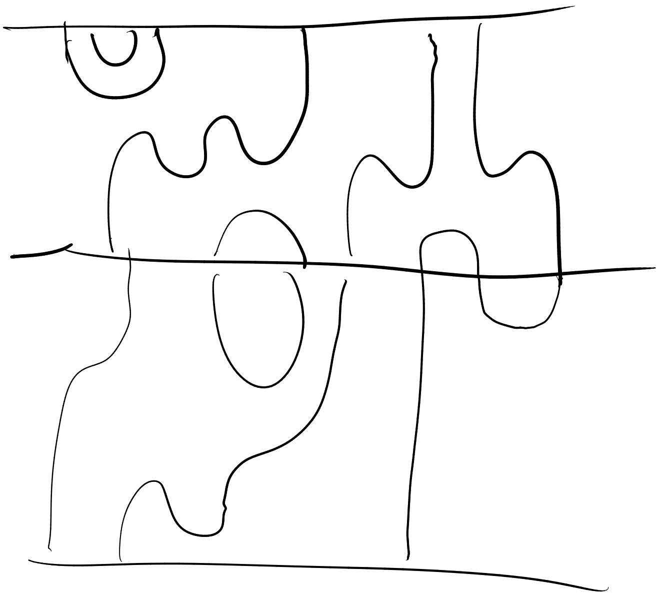
 $= (-2) \cdot$ 

 $=$  $=$ 

Coro

 $= (4) \cdot$ 

also



$V^{\otimes 7}$

$\uparrow$

$V^{\otimes 5}$

$32\left(\underline{\qquad}\right)$
128

§ adjoint functors

Let $\mathbf{Cat}$ be the 2-cat of categories :

Obj : cats

Mor : functors

2-Mor : Nat transfoms!

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "E"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "E^\vee"] \end{tikzcd}$$

Def 1

$$\mathrm{Mor}_{\mathcal{C}}[c, E^\vee(d)] \cong \mathrm{Mor}_{\mathcal{D}}[E(c), d]$$

eg. free constr. / forgetful func

eg. $\mathrm{coind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{ind}_H^G$

eg. $X \xrightarrow{f} Y$

$f_! \quad f_* \quad f^* \quad f^!$

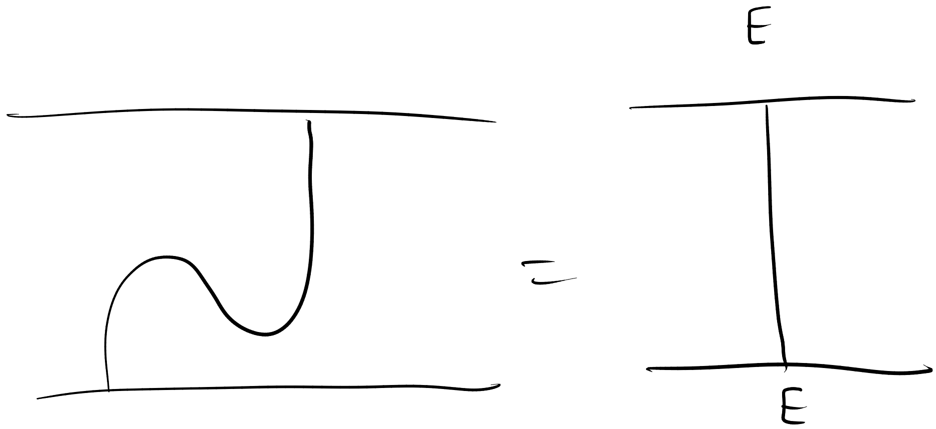
Def 2

$$EE^\vee \overset{\epsilon}{\Longrightarrow} 1_D$$

$$1_C \overset{\eta}{\Longrightarrow} E^\vee E$$

s.t.

$$\begin{cases} \left(E(c) \xrightarrow{E(\eta)} EE^\vee E(c) \xrightarrow{\epsilon_E} E(c)\right) = 1_{E(c)} \\ E^\vee(d) \xrightarrow{\eta_{E^\vee(d)}} E^\vee E E^\vee(d) \xrightarrow{E^\vee(\epsilon)} E^\vee(d) = 1_{E^\vee(d)} \end{cases}$$



$$E \overset{1_E}{\Longrightarrow} E$$

$$E(c) \xrightarrow{1_{E(c)}} E(c)$$