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PERIODS OF AUTOMORPHIC FORMS

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Contents

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I. Introduction

Let G be a reductive group over a number field F, and let $H \subset G$ be a subgroup obtained as the fixed point set of an involution θ. Then the following *period integral*

$$\Pi^H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})^1} \varphi(h)\ dh$$

converges absolutely for any cusp form φ on $G(\mathbb{A})$. Our first goal in this paper is to develop a method for defining and computing $\Pi^H(\varphi)$ where φ is a more general automorphic form such as an Eisenstein series. In this case, the integral need not converge and we define it by means of a regularization procedure. Our second goal is to use this regularized period to obtain explicit formulas for the (convergent) period $\Pi^H(\Lambda^T E)$ where $\Lambda^T E$ is a truncated Eisenstein series.

Before describing the contents of this article, let us recall some motivation. The periods $\Pi^H(\Lambda^T E)$ are of interest because they appear in the relative trace formula (RTF) in a role analogous to that played by inner products of truncated Eisenstein series in the Arthur-Selberg trace formula. They arise when one computes the contribution of the continuous spectrum to the relative trace formula (cf. [J2], [J3] for an overview of the RTF). The relative trace formula itself has been used in a variety of applications, beginning with the work of Jacquet-Lai on periods of Hilbert modular forms ([JL], [LA1], also [FH]) and Jacquet on Waldspurger's theorems [J1] (cf. also [GU]). It provides a general tool for studying distinguished representations. Recall that a cuspidal representation (π, V_π) of G is said to be *distinguished by* H if $\Pi^H(\varphi) \neq 0$ for some $\varphi \in V_\pi$. In some cases, periods of this type have an interesting topological or arithmetic interpretation (cf. [HLR], [BR]). Actually, in applications, the notion of period integral must be suitably refined: in general, it may have the form:

$$\int_{ZH(F)\backslash H(\mathbb{A})} \varphi(h)\chi(h)\ dh$$

where Z is a central subgroup of the ambient group and χ is a character of $H(\mathbb{A})$ trivial on $H(F)$. In many cases, it should be possible to characterize the H-distinguished cuspidal representations as images with respect to a functorial transfer to G from a third group G'. General results of this type, which are known in many cases, should eventually follow from a comparison of suitable relative trace formulas on G and G'. The RTF thus provides a link between certain cases of the functoriality conjecture and period integrals.

A candidate for G' over the algebraic closure of F was proposed in [JLR]. In [G] this has been refined to a description of G' as an F-group. Suppose, for example, that $G = \mathrm{Res}_{E/F}\, H$ where E/F is a quadratic extension and θ is the involution induced by Galois conjugation relative to E/F. This is the case we focus on in this article (although our methods generalize; we intend to deal with the general case in a future article). If H is also simple and split, then G' will be either H itself or the unique quasi-split outer form of H that splits over E, according as -1 lies in the Weyl group of H or not ([G]). Thus if $H = GL(n)_{/F}$, then G' is the quasi-split unitary group $U(n)$ (see also [F1] and [F2]) and if $H = Sp(n)_{/F}$, then $G' = Sp(n)_{/F}$. The relative trace formula in this situation is based on the equality of geometric and spectral expansions of the integral

$$\int_{H(F)\backslash H(\mathbb{A})^1} \int_{N(E)\backslash N(\mathbb{A}_E)} K_f(h,n)\ \psi(n)\ dh\ dn \tag{1}$$

where $K_f(x,y)$ is the automorphic kernel attached to a function f on $G(\mathbb{A})$, N is the unipotent radical of the standard Borel subgroup, and ψ is a non-degenerate character of $N(E)\backslash N(\mathbb{A}_E)$. In the spectral development of (1), an orthonormal basis $\{\varphi\}$ for the space of cusp forms contributes

$$\sum_{\{\varphi\}} \Pi^H(\rho(f)\varphi)\overline{W_\varphi(e)}$$

where $W_\varphi(e)$ is the value at e of the Fourier coefficient of φ with respect to ψ. To evaluate the contribution of the continuous spectrum, we are led to truncate the continuous part of the kernel in the first variable and this leads to an integral involving the truncated periods $\Pi^H(\Lambda^T E)$ where E is an Eisenstein series. Of course, it is at this stage in the development of the ordinary Arthur-Selberg trace formula that the inner products of truncated Eisenstein series appear. Unlike the Arthur-Selberg situation, the integral (1) itself is absolutely convergent. Nevertheless, truncation is needed to obtain an explicit formula. See [GJR] for an example where the formulas for $\Pi^H(\Lambda^T E)$ are used in this context. That work considers the case $G = GL(3)$ and compares (1) with the Kuznetzov trace formula for the group $G' = U(3)$. The comparison is used in [FGJR] to characterize H-distinguished representations and to prove that a cuspidal L -packet on $U(3)$ (which is not a non-tempered A-packet) contains a generic element.

We now describe our results in greater detail. The first section deals with the meromorphic continuation of the integral of an exponential polynomial over a cone. This is used in Section IV to show that in the case $G = \mathrm{Res}_{E/F} H$ with E/F quadratic, it is possible to define a regularized period $\Pi^H(\varphi)$ for automorphic forms whose exponents satisfy a certain mild restriction. The regularization is based on Arthur's truncation operators ([A2]) and their associated combinatorics. However, in the context of a pair (G, H), there is more than one way to define the truncation. For example, one can apply Arthur's truncation operator Λ^T relative to G (cf. [LE], [LA2]) or one can restrict φ and apply Arthur's truncation Λ^T operator relative to H. While the former is the more natural of these two, we use a third, *mixed* truncation Λ^T_m, intermediate between them (in fact, this was used in [JL] in the $GL(2)$ case). This truncation appears to be best suited to the study of period integrals. The next step is to compute the period of a truncated Eisenstein series. In §8, we obtain a general formula for periods of the mixed truncation

$$\int_{H(F)\backslash H(\mathbb{A})^1} \Lambda^T_m\varphi(h)\ dh$$

in terms of the regularized periods of the constant terms of φ.

The last two sections of the article are devoted to more explicit results for $H = GL(n)_{/F}$. In Section VI, we compute the regularized period $\Pi^H(E(g,\varphi,\lambda))$ for all cuspidal Eisenstein series (Theorem 23) and show, in particular, that it vanishes unless $E(g,\varphi,\lambda)$ is induced from cuspidal representation $\sigma_1 \otimes \sigma_2$ of a parabolic subgroup of type $(\frac{n}{2}, \frac{n}{2})$ with $\sigma_1^* \approx \overline{\sigma_2}$. We then obtain an explicit formula expressing $\Pi^H(\Lambda^T_m E(g,\varphi,\lambda))$, for any cuspidal Eisenstein series, in terms of certain linear functionals $J(\xi,\varphi,\lambda)$ which we call *intertwining periods.* They are defined for λ in a suitable cone by an absolutely convergent integral

$$J(\xi,\varphi,\lambda)=\int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})}\int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} e^{\left\langle \lambda+\rho_P, H_{P_E}(\eta h)\right\rangle}\ \varphi(m\eta h)\ dm\ dh.$$

Here $\xi,\eta\in GL_n(E)$ are elements such that $\eta\bar{\eta}^{-1}=\xi$, $H_\eta=H\cap\eta^{-1}P_E\eta$ where P_E is the parabolic subgroup of G from which the Eisenstein series is induced, and $M_\eta=\eta H\eta^{-1}\cap M_{P_E}$ where M_{P_E} is the Levi factor of P_E. The name "intertwining period" seems appropriate for several reasons. First, the map $\varphi\to J(\xi,\varphi,\lambda)$ is an $H(\mathbb{A})$-invariant functional on the induced representation $\mathrm{Ind}_P^G\sigma\otimes e^\lambda$ and so, by Frobenius reciprocity, defines an intertwining operator. Furthermore, the J-functionals have several properties in common with the standard intertwining operators $M(s,\lambda)$. Our explicit formula for the period integral

$$\int_{H(F)\backslash H(\mathbb{A})^1}\Lambda_m^T E(h,\varphi,\lambda)\ dh \tag{2}$$

is

$$\sum_{(Q,s)\in\mathcal{G}(P,\sigma)} v_Q\ \frac{e^{\left\langle (s\lambda)_Q, T\right\rangle}}{\prod_{\alpha\in\Delta_Q}\left\langle (s\lambda)_Q\,,\alpha^\vee\right\rangle} J(\xi_Q, M(s,\lambda)\varphi,(s\lambda)_{P'}^Q), \tag{3}$$

where Q and s range over certain sets of parabolic subgroups and Weyl group elements, respectively, and v_Q is a certain volume (see Section VIII for other unexplained notation). This formula is clearly analogous to Langlands' formula [A2] for the inner product of cuspidal Eisenstein series induced from parabolic subgroups P and P', which expresses the inner product $(\Lambda^T E(\lambda,\varphi),\Lambda^T E(\mu,\psi))$ as a sum

$$\sum_Q\sum_{\substack{s\in\Omega(P,Q)\\ t\in\Omega(P',Q)}}\frac{e^{\langle s\lambda+t\mu,T\rangle}}{\prod_{\alpha\in\Delta_Q}\langle s\lambda+t\mu,\alpha^\vee\rangle}(M(s,\lambda)\varphi, M(t,\mu)\psi). \tag{4}$$

Like the standard intertwining operators, the J-functionals can be meromorphically continued and satisfy a set of functional equations. These are described in Section VII, again for the case $H=GL(n)_{/F}$. The functional equations take the form

$$J(\xi,\varphi,\lambda)=J(s\xi s^{-1}, M(s,\lambda)\varphi, s\lambda)$$

for elements s belonging to a certain subset of the Weyl group which we describe combinatorially in §17. For example, if $n=2$ and $\xi=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, then $H_\eta=T$ is a torus isomorphic to E^* in H. In this case, the regularized period $\Pi^H(E(g,\varphi,\lambda))$ is equal to $J(\xi,\varphi,\lambda)$. For the choice $\varphi\equiv 1$,

$$J(\xi,\varphi,\lambda)=\int_{T(F)\backslash GL_2(\mathbb{A}_F)} e^{\langle\lambda+1, H(\eta h)\rangle}\ dh$$

and we have the following explicit formula:

$$J(\xi,\varphi,\lambda)=\frac{1}{2}\mathrm{vol}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}))\frac{L(\lambda,1_F)}{L(\lambda+1,\omega_{E/F})}$$

where $\omega_{E/F}$ is the order two character attached to E/F and $L(\lambda,1_F)$ is the zeta-function of F with archimedean Euler factors. The functional equation in this case reduces to

$$J(\xi,\varphi,\lambda)=m(\xi,\lambda)J(\xi,\varphi,-\lambda)$$

where $m(\xi,\lambda) = L(\lambda, 1_E)/L(\lambda+1, 1_E)$. In §20 we show that the intertwining period factors as a product of local intertwining periods and we compute the local period in the unramified case in terms of Asai L-functions.

It is formula (3) which will be useful for applications to the relative trace formula. For example, if $H = GL(2)$, then (2) is equal to

$$\frac{e^{\lambda T}}{\lambda}\int_{\mathbf{K}} \varphi(k)dk \; + \frac{e^{-\lambda T}}{-\lambda}\int_{\mathbf{K}} M(\xi,\lambda)\varphi(k)dk + \int_{T(F)\backslash GL_2(\mathbb{A}_F)} e^{\langle\lambda+1, H(\eta h)\rangle} \; \varphi(\eta h)dh$$

for a suitable normalization of measures. Here ξ is the non-trivial Weyl group element, $\eta\bar{\eta}^{-1} = \xi$, and $\mathbf{K}$ is a maximal compact subgroup of $GL_2(\mathbb{A}_F)$. This special case was first found by Jacquet and Lai.

It is possible to compute (2) using Λ^T for G in place of Λ^T_m. This calculation, involving multi-dimensional residues and contour shifts, follows the main lines of the traditional proof of Langlands inner product formula [A2], although it is substantially more complicated. We note that a similar analysis is attempted in [F3]. In the case $(GL(n)_{/E}, GL(n)_{/F})$, one finds that for any automorphic form φ, the period integrals of $\Lambda^T_m\varphi$ and $\Lambda^T\varphi$ are the same.

The regularization of integrals is also useful in providing a more conceptual approach to the formulas for inner products of truncated Eisenstein series that occur in the Arthur-Selberg trace formula. In Section V, we describe the results one obtains by applying the above methods to the case of arbitrary G and $\theta = id$, so that $H = G$. The problem is to define a regularization of the integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} \varphi(g)\; dg. \tag{5}$$

As in the case of period integrals, this is possible for automorphic forms φ whose exponents satisfy a certain mild restriction. The regularized integral can then be used to derive a formula for the (convergent) integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T\varphi(g)\; dg.$$

These results can be extended to yield a formula for the inner product of truncated automorphic forms

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T\varphi(g)\; \Lambda^T\psi(g)\; dg = \sum_{P\subset G} (-1)^{d(P)-d(G)} \int^{*}_{P(F)\backslash G(\mathbb{A})^1} \psi_P(g)\varphi_P(g)\hat{\tau}_P(H(g)-T)dg. \tag{6}$$

The terms on the right are *regularized* inner products of the constant terms of φ and ψ and hence they are denoted with a star. This should be seen as a generalization of Langlands' formula for the inner product of truncated cuspidal Eisenstein series ([L], [A2]). In fact, Langlands' formula (4) follows directly from (6), the standard formula for the constant term of an Eisenstein series and Lemma 16. Thus (6) provides a conceptual proof of Langlands' formula. Note that (6) is a formula for the period of a truncated automorphic form on $G\times G$ relative to the period subgroup G embedded diagonally. We omit the proofs in Section V because they are nearly identical, step-by-step, with those in Section IV. It is also possible to extend this method to give a new proof of Arthur's asymptotic formula [A4] for the inner product of truncated non-cuspidal Eisenstein series, where the main term is what is anticipated by the cuspidal case ([LP]).

Period integrals have been investigated from different points of view in recent years in a number of works by several authors, of which we mention the following sample: [F2], [F4], [FJ], [GRS], [JI], [JR], [JY], [MA], [SA]. A regularized integral for $GL(2)$ was developed by Zagier [Z], who gave a variety of applications, and was reformulated by Casselman [C]. As mentioned above, our general version is based on the combinatorics of Arthur's truncation operators. We are grateful to J. Bernstein and W. Casselman for pointing out the relevance of regularization to our problem. We are also grateful to J. Bernstein for helpful conversations, and for providing the additional key insight that, in many cases, the regularized integral of an Eisenstein series vanishes for simple representation-theoretic reasons (Lemma 16).

II. Integrals over cones

Let V be a real finite-dimensional vector space of dimension n. Let V^* be the space of complex linear forms on V and $V^*_{\mathbb{R}}$ the space of real linear forms. We denote by $S(V^*)$ the symmetric algebra of V^*, that is, the space of polynomial functions on V. Likewise we denote by $S(V)$ the symmetric algebra of $V \otimes \mathbb{C}$. An *exponential polynomial* function on V is a function of the form:

$$f(x) = \sum_{1 \leq i \leq r} e^{\langle \lambda_i, x \rangle} P_i(x)$$

where the λ_i are distinct elements of V^* and the $P_i(x)$ are non-zero elements of $S(V^*)$. The λ_i are then uniquely determined and called the exponents of f. We recall the proof of uniqueness.

For $X \in V$ we denote by D_X the corresponding vector field:

$$D_X g(x) = \frac{d}{dt} g(x + tX)\bigg|_{t=0}.$$

For $\lambda \in V^*$ we have:

$$D_X e^{\langle \lambda, x \rangle} = \langle \lambda, X \rangle e^{\langle \lambda, x \rangle}.$$

We can define the operators D_X for $X \in S(V)$ by multiplicativity. Suppose that P is in $S(V^*)$. There is an integer d (total degree of P) such that for all $X \in V$:

$$D_X^d P = 0.$$

Now let $g(x) = e^{\langle \lambda, x \rangle} P(x)$. Then

$$(D_X - \langle \lambda, X \rangle) g(x) = e^{\langle \lambda, x \rangle} D_X P(x).$$

Thus

$$(D_X - \langle \lambda, X \rangle)^d g(x) = 0.$$

We will need the analogous fact for finite difference operators. Denote by Δ_X the finite difference operator defined by:

$$\Delta_X f(x) = f(x + X) - f(x).$$

Then:

$$\left(\Delta_X - (e^{\langle \lambda, X \rangle} - 1)\right) e^{\langle \lambda, x \rangle} P(x) = e^{\langle \lambda, x \rangle} e^{\langle \lambda, X \rangle} \Delta_X P(x).$$

Thus if d is the total degree of P, then

$$\left(\Delta_X - (e^{\langle \lambda, X\rangle} - 1)\right)^d e^{\langle \lambda, x\rangle} P(x) = 0.$$

Now if f has the form

$$f(x) = \sum_{1\le i\le r} e^{\langle \lambda_i, x\rangle} P_i(x),$$

as above, choose $X \in V$ such that the numbers $\langle \lambda_i, X\rangle$ are distinct. By the Chinese remainder theorem, for each i, there exists a polynomial Q_i in one variable such that

$$Q_i(D_X) f(x) = e^{\langle \lambda_i, x\rangle} P_i(x). \tag{7}$$

Similarly, if the numbers $e^{\langle \lambda_i, X\rangle}$ are distinct there is for each i a polynomial in one variable R_i such that

$$R_i(\Delta_X) f(x) = e^{\langle \lambda_i, x\rangle} P_i(x).$$

This shows that the λ_i are uniquely determined by f.

By a *cone* in V we shall mean a closed subset of the form

$$\mathcal{C} = \{x \ \in V : \langle \mu_i, x\rangle \ge 0\} \tag{8}$$

where $\{\mu_i\}$ is a basis of $V^*_{\mathbb{R}}$. Let e_i be the dual basis of V. We will say that $\lambda \in V^*$ is *negative with respect to* $\mathcal{C}$ if $\mathrm{Re}\,\langle \lambda, e_j\rangle < 0$ for each $j = 1, ..., n$. We will say that λ is *non-degenerate with respect to* $\mathcal{C}$ if $\langle \lambda, e_j\rangle \ne 0$ for each $j = 1, ..., n$. Our goal is to define the regularized integral of an exponential polynomial function over a cone.

Lemma 1. *The function*

$$f(x) = \sum_i e^{\langle \lambda_i, x\rangle} P_i(x)$$

is integrable over $\mathcal{C}$ if and only if λ_i is negative with respect to $\mathcal{C}$ for all i.

Proof. The condition is clearly sufficient. Moreover it is necessary if $r = 1$. In general, suppose that f is integrable over $\mathcal{C}$. Choose X such that the numbers $e^{\langle \lambda_i, X\rangle}$ are distinct, and, in addition, the numbers $\langle \mu_j, X\rangle$ are positive. Then $X + \mathcal{C}$ is contained in $\mathcal{C}$. It follows that the function $\Delta_X f$ is integrable over $\mathcal{C}$. Hence the functions $R_i(\Delta_X) f$ are also integrable over $\mathcal{C}$. Thus each term in the sum over i is integrable over $\mathcal{C}$ and our assertion follows. □

To define the regularized integral over $\mathcal{C}$ we study the integral

$$I_{\mathcal{C}}(\lambda)(f) := \int_{\mathcal{C}} f(x) e^{-\langle \lambda, x\rangle} dx.$$

The integral converges absolutely for λ in the open set

$$U = \{\lambda \in V^* : \mathrm{Re}\,\langle \lambda_i - \lambda, e_j\rangle < 0 \ \text{ for all } 1 \le i \le r; 1 \le j \le n\}.$$

We analytically continue this integral as follows. First, in the one-variable case, we have for $\mathrm{Re}(\lambda) > 0$ and any polynomial P in one variable:

$$\int_0^{+\infty} e^{-\lambda x} P(x) dx = \sum_{m\ge 0} \frac{(D^m P)(0)}{\lambda^{m+1}}.$$

Of course the sum on the right is finite. In general, fix an index k. Let $\mathcal{C}_k$ be the intersection of $\mathcal{C}$ and the hyperplane $V_k = \{x | \langle \mu_k, x\rangle = 0\}$. In the integral we can write $x = \langle \mu_k, x\rangle e_k + y$ with $y \in V_k$ and then, for a suitable choice of the Haar measures:

$$I_{\mathcal{C}}(\lambda)(f) = \sum_{1\le i\le r} \sum_{m\ge 0} \frac{1}{\langle \lambda - \lambda_i, e_k\rangle^{m+1}} \int_{\mathcal{C}_k} e^{-\langle \lambda-\lambda_i, y\rangle} (D^m_{e_k}) P_i(y) dy.$$

This formula gives the analytic continuation of $I_{\mathcal{C}}(\lambda)(f)$ to the tube domain U_k defined by:

$$\operatorname{Re}\langle \lambda_i - \lambda, e_j\rangle < 0 \quad \text{for} \quad j \ne k\,,\ 1 \le i \le r.$$

The analytic continuation is a meromorphic function with hyperplane singularities, the singular hyperplanes being:

$$H_{k,i} = \{\lambda | \langle \lambda, e_k\rangle = \langle \lambda_i, e_k\rangle\}\,,\ 1 \le i \le r.$$

Since this is true for all $1 \le k \le n$, the function $I_{\mathcal{C}}(\lambda)(f)$ has analytic continuation to V^* by Hartogs' lemma. The continuation is a meromorphic function with hyperplane singularities. The singular hyperplanes are the hyperplanes $H_{i,k}$, $1 \le k \le n\,,\ 1 \le i \le r$. Of course, we could also use induction on n and the above formula to obtain an explicit expression for the integral.

Lemma 2. *Suppose that f is absolutely integrable on $\mathcal{C}$. Then $I_{\mathcal{C}}(\lambda)(f)$ is holomorphic at 0 and $I_{\mathcal{C}}(0)(f)$ is the integral of f over $\mathcal{C}$.*

Proof. By the previous lemma, if f is integrable over $\mathcal{C}$, then each exponent λ_i is negative with respect to $\mathcal{C}$. Thus the domain of convergence U of the integral $I(\lambda)(f)$ contains the point 0 and our assertion follows. □

Lemma 3. *The function $I_{\mathcal{C}}(\lambda)(f)$ is holomorphic at $\lambda = 0$ if and only if for all i, λ_i is non-degenerate with respect to $\mathcal{C}$, i.e., $\langle \lambda_i, e_k\rangle \ne 0$ for every pair (i, k).*

Proof. If the condition of the lemma is satisfied, then 0 does not belong to any of the hyperplanes $H_{i,k}$. Thus the function is holomorphic at 0. To show conversely that the condition is necessary, it suffices to show that each hyperplane $H_{i,k}$ is indeed a singular hyperplane for the function $I_{\mathcal{C}}(\lambda)(f)$. Assume that some $H_{i_0,k}$ is not singular. Let $c = \langle \lambda_{i_0}, e_k\rangle$. Then

$$\sum_{\{i|\lambda_i(e_k)=c\}} \sum_{m\ge 0} \frac{1}{\langle \lambda - \lambda_i, e_k\rangle^{m+1}} \int_{\mathcal{C}_k} e^{-\langle \lambda-\lambda_i, y\rangle} (D^m_{e_k}) P_i(y) dy$$

must be identically zero for $\lambda \in U_k$. In turn this implies:

$$\sum_{\{i|\lambda_i(e_k)=c\}} \int_{\mathcal{C}_k} e^{-\langle \lambda-\lambda_i, y\rangle} (D^m_{e_k}) P_i(y) dy = 0$$

for each m. By the Fourier inversion formula this implies:

$$\sum_{\{i|\langle\lambda_i, e_k\rangle=c\}} e^{-\langle \lambda-\lambda_i, y\rangle} (D^m_{e_k}) P_i(y) = 0$$

for all $y \in V_k$ and all m. Note that in this formula the linear forms λ_i have distinct restrictions to the hyperplane V_k. It follows that for each index i with $\langle \lambda_i, e_k\rangle = c$ we have:

$$(D^m_{e_k}) P_i(y) = 0$$

for each $m \geq 0$ and each $y \in V_k$. This relation implies $P_i = 0$ for each i with $\langle \lambda_i, e_k \rangle = c$. This is a contradiction and the lemma follows. □

We observe that we can obtain the analytic continuation of the integral in a more direct way. This is useful if one wishes to study how the integral depends on parameters. In one variable we have the formula:

$$\int_0^\infty e^{-\lambda x} D^m f(x)dx = \sum_{0 \leq u \leq m-1} \lambda^{m-1-u}(D^u f)(0) + \lambda^m \int_0^\infty e^{-\lambda x} f(x)dx.$$

In general if T is a polynomial in one variable and $T(X) = \sum_i t_i X^i$ we set:

$$\tilde{T}(X, \lambda) = \sum_i t_i \sum_{0 \leq u \leq i-1} \lambda^{i-1-u} X^u.$$

Then

$$\int_0^\infty e^{-\lambda x}(T(D)f)(x)dx = \tilde{T}(D, \lambda)f(0) + T(\lambda) \int_0^\infty e^{-\lambda x} f(x)dx.$$

In particular, if $T(D)f = 0$, then

$$\int_0^\infty e^{-\lambda x} f(x)dx = -\frac{\left(\tilde{T}(D, \lambda)f\right)(0)}{T(\lambda)}.$$

This formula gives the analytic continuation of the integral. If $T(0) \neq 0$, then the integral is holomorphic at $\lambda = 0$. If $T(0) = 0$, then $T(X) = XS(X)$ where S is another polynomial. The value of the numerator at $\lambda = 0$ is $S(D)f(0)$. Thus 0 is a pole unless $S(D)f(0) = 0$. Since $DS(D)f = 0$ this is equivalent to $S(D)f = 0$.

The generalization to higher dimensions is straightforward. Suppose that f is an exponential polynomial function and for $k = 1, ..., n$ there exists a polynomial T_k in one variable such that

$$T_k(D_{e_k})f = 0.$$

Then

$$I_{\mathcal{C}}(\lambda)(f) = -\int_{\mathcal{C}_k} \frac{\left(\tilde{T}_k(D_{e_k}, \langle \lambda, e_k \rangle)f\right)(y)}{T_k(\langle \lambda, e_k \rangle)} dy.$$

Using induction on n we obtain:

$$I_{\mathcal{C}}(\lambda)(f) = (-1)^n \frac{\left[\prod_k \tilde{T}_k(D_{e_k}, \langle \lambda, e_k \rangle)f\right](0)}{\prod_k T_k(\langle \lambda, e_k \rangle)}.$$

This can be used to give the analytic continuation of $I_{\mathcal{C}}(\lambda)(f)$. The singular hyperplanes have the form $\lambda(e_k) = z_k$ where $1 \leq k \leq n$ and z_k is a root of the polynomial T_k. Thus if $T_k(0) \neq 0$ for $1 \leq k \leq n$, then $I_{\mathcal{C}}(\lambda)(f)$ is holomorphic at $\lambda = 0$.

Note that if 0 is a root of T_k, then $T_k(X) = XS_k(X)$ where S_k is another polynomial. As in the one-variable case the hyperplane $\lambda(e_k) = 0$ is then singular unless $S_k(D_{e_k})f = 0$.

Although it is not needed in the sequel, we remark that with this approach, it is easy to see that under appropriate assumptions the value of the integral will depend holomorphically on a parameter if f does. To that end, consider the following situation. Let A be a finitely generated subalgebra of $S(V)$ (containing 1). Suppose that $S(V)$ is finitely generated as an A-module. Next suppose that $\Omega \subset \mathbb{C}^q$ is a connected open set and for each $s \in \Omega$ we are given a character χ_s of A, that is, a homomorphism $\chi_s : A \to \mathbb{C}$, depending holomorphically on s. This means that the values of χ_s on a set of generators of A are holomorphic functions of s in Ω. Next, let $f_s(x)$ be a smooth function on $V \times \Omega$. We assume it is holomorphic in s and satisfies

$$D_X f_s(x) = \chi_s(X) f_s(x)$$

for all $X \in A$. For each k with $1 \leq k \leq n$, let B_k be the algebra generated by e_k and A. Because A is Noetherian it is a finitely generated A-module. It follows that there is a polynomial $T_k \in A[X]$, whose coefficient of the term of highest degree is 1, such that $T_k(e_k) = 0$. The polynomial T_k has the form:

$$T_k(X) = \sum a_i X^i$$

with $a_i \in A$. We set

$$T_k(X; s) = \sum \chi_s(a_i) X^i.$$

For each $s \in \Omega$ this is a polynomial with complex coefficients and

$$T_k(D_{e_k}; s) f_s = 0.$$

If Ω' is a relatively compact open subset of Ω, one can use this system of differential equations to obtain a majorization of f_s on C, uniform for $s \in \Omega'$, and show that there is an open set U defined by inequalities of the form $\mathrm{Re}(\langle \lambda, e_i \rangle) >> 0$ such that the integral $I_{\mathcal{C}}(\lambda)(f_s)$ converges uniformly on compact subsets of $U \times \Omega'$. To the polynomial $T_k(X; s)$ we associate as before the polynomial $\tilde{T}_k(X, \lambda; s)$. Then the formula

$$I_{\mathcal{C}}(\lambda)(f_s) = (-1)^n \frac{\left[\prod_k \tilde{T}_k(D_{e_k}, \langle \lambda, e_k \rangle; s) f_s\right](0)}{\prod_k T_k(\langle \lambda, e_k \rangle; s)}$$

gives the analytic continuation of the integral as a meromorphic function of (λ, s). We have the following more precise result.

Proposition 4. *With the previous notations assume that there is a point $s_0 \in \Omega$ such that any character χ of $S(V)$ which extends χ_{s_0} takes a non-zero value on the vectors e_k. Then $I_{\mathcal{C}}(\lambda)(f_s)$ is holomorphic at $(0, s_0)$.*

Proof. Since $S(V)$ is integral over A, every character of A extends (in finitely many ways) to $S(V)$ (by the going up theorem). Consider the set I_k of polynomials $P \in A[X]$ such that $P(e_k) = 0$. It is a (finitely generated) ideal. Let $\{T_\alpha\}$ be a set of generators of I_k. An extension χ of χ_{s_0} to B_k is determined by a complex number z such that $T_\alpha(z; s_0) = 0$ for all α and then $\chi(e_k) = z$. Our assumption is that such an extension does not take the value 0 on e_k. Thus there must be at least one index α such that $T_\alpha(0; s_0) \neq 0$. Re-label such a polynomial $T_k(X; s)$. We thus have

$$T_k(D_{e_k}; s) f_s = 0$$

for all s and $T_k(0, s_0) \neq 0$. Again, the formula

$$I_{\mathcal{C}}(\lambda)(f_s) = (-1)^n \frac{\left[\prod_k \tilde{T}_k(D_{e_k}, \langle \lambda, e_k \rangle; s) f_s\right](0)}{\prod_k T_k(\langle \lambda, e_k \rangle; s)}$$

does show that $I_{\mathcal{C}}(\lambda)(f_s)$ is holomorphic at $(0, s_0)$. ∎

We remark that the coefficient a_k of the term of highest degree of T_k need not be 1. Moreover we may have $\chi_{s_0}(a_k) = 0$.

After these preliminaries we define the regularized integral. Denote the characteristic function of a set $\mathcal{Y}$ by $\tau^{\mathcal{Y}}$. Suppose that $\mathcal{C}$ is the cone defined relative to a basis $\{\mu_i\}$ of $V_{\mathbb{R}}^*$ as in (8) and let $\{e_i\}$ denote the dual basis. Let $T \in V$. As before, let

$$f(x) = \sum_{1 \le i \le r} e^{\langle \lambda_i, x \rangle} P_i(x) \tag{9}$$

be an exponential polynomial. For $\lambda \in V^*$ such that $\lambda_i - \lambda$ is negative with respect to $\mathcal{C}$ for all $i = 1, ..., r$, set

$$\begin{aligned} \widehat{f}(\lambda; \mathcal{C}, T) &= \int_V f(x) \tau^{\mathcal{C}}(x - T)\ e^{-\langle \lambda, x \rangle} dx \\ &= \sum_{1 \le i \le r} \int_V P_i(x) \tau^{\mathcal{C}}(x - T)\ e^{-\langle \lambda - \lambda_i, x \rangle} dx \\ &= \sum_{1 \le i \le r} e^{\langle \lambda_i - \lambda, T \rangle} I_{\mathcal{C}}(\lambda)(P_i(\bullet + T) e^{\langle \lambda_i, \bullet \rangle}). \end{aligned} \tag{10}$$

The integrals are absolutely convergent and, as we have seen, $\widehat{f}(\lambda; \mathcal{C}, T)$ extends a meromorphic function on V^*. We will say that the function $f(x)\tau^{\mathcal{C}}(x - T)$ is *#-integrable* if $\widehat{f}(\lambda; \mathcal{C}, T)$ is regular at $\lambda = 0$. In this case, we define the #-integral

$$\int_V^{\#} f(x) \tau^{\mathcal{C}}(x - T)\ dx$$

to be the value $\widehat{f}(0; \mathcal{C}, T)$. Otherwise, we say that the #-integral does not exist. By Lemma 3, the #-integral exists if and only if each exponent λ_j is non-degenerate with respect to $\mathcal{C}$.

Suppose that $V = W_1 \oplus W_2$ is a decomposition of V as a direct sum, and let $\mathcal{C}_j$ be a cone in W_j. Write $T = T_1 + T_2$ and write $x = w_1 + w_2$ relative to $V = W_1 \oplus W_2$. If the exponents λ_j of f are non-degenerate with respect to $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, then the function

$$w_2 \longrightarrow \int_{W_1}^{\#} f(w_1 + w_2) \tau_1^{\mathcal{C}}(w_1 - T_1)\ dw_1$$

is defined and is an exponential polynomial, and it follows by analytic continuation that

$$\int_V^{\#} f(x) \tau^{\mathcal{C}}(x - T)\ dx$$

is equal to

$$\int_{W_2}^{\#} \left(\int_{W_1}^{\#} f(w_1 + w_2) \tau_1^{\mathcal{C}}(w_1 - T_1)\ dw_1 \right)\ \tau_2^{\mathcal{C}}(w_2 - T_2)\ dw_2. \tag{11}$$

Furthermore, we have

Lemma 5. *Let f be as in* (9)*. Then the function*

$$T \longrightarrow \int_V^{\#} f(x) \tau^{\mathcal{C}}(x - T)\ dx$$

is an exponential polynomial with the same exponents as f.

Proof. It is clear from (10) that $\widehat{f}(\lambda;\mathcal{C},T)$ is an exponential polynomial as a function of T with the same exponents as f. Hence the same is true of its value at $\lambda = 0$. □

The domain of the #-integral can be extended in various ways. Without addressing this general question, we shall consider #-integrals of the following slightly more general type of function which will be adequate for our later needs. Suppose that $V = W_1 \oplus W_2$ is a decomposition of V as a direct sum. Let $g(x)$ be a compactly supported function on W_1 and let $\mathcal{C}_2 \subset W_2$ be a cone in W_2. Writing $T = T_1 + T_2$ and $x = w_1 + w_2$ as above, we consider functions of the form

$$g(w_1 - T_1)\ \tau_2^{\mathcal{C}}(w_2 - T_2) \tag{12}$$

which, for convenience, we will call functions of type (C).

It is clear that the integral

$$\widehat{F}(\lambda) = \int_V f(x)\ g(w_1 - T_1)\ \tau_2^{\mathcal{C}}(w_2 - T_2)\ e^{-\langle\lambda,x\rangle}\ dx$$

converges absolutely for an open set of λ whose restriction to W_2 is negative with respect to $\mathcal{C}_2$. Furthermore, $\widehat{F}(\lambda)$ has a meromorphic continuation to V^*. As before, we say that $f(x)g(x-T_1)\tau_2^{\mathcal{C}}(x-T_2)$ is *#-integrable* if $\widehat{F}(\lambda)$ is regular at $\lambda = 0$. If so, we denote the value $\widehat{F}(0)$ by

$$\int_V^{\#} f(x)\ g(w_1 - T_1)\ \tau_2^{\mathcal{C}}(w_2 - T_2)\ dx. \tag{13}$$

Note that the function

$$w_2 \longrightarrow \int_{W_1} f(w_1 + w_2)g(w_1 - T_1)\ \ dw_1$$

is an exponential polynomial on W_2 and (13) is equal to the iterated integral

$$\int_{W_2}^{\#} \left(\int_{W_1} f(w_1 + w_2)g(w_1 - T_1)\ dw_1\right)\ \tau_2^{\mathcal{C}}(w_2 - T_2)\ dw_2. \tag{14}$$

In the next lemma, for $i = 1, ..., r$ let $V = W_{i1} \oplus W_{i2}$ be a direct sum decomposition and let $\mathcal{C}_{i2}$ be a cone in W_{i2}. Let g_i be a compactly supported function on W_{i1} and set $G_i(w_1 + w_2) = g_i(w_1)\tau_{i2}^{\mathcal{C}}(w_2)$ for $w_j \in W_{ij}$.

Lemma 6. *Let $\mathcal{C}$ and $\mathcal{C}^*$ be cones in V. Assume that $\mathcal{C}_{i2}, \mathcal{C} \subset \mathcal{C}^*$ for all i. Assume further that*

$$\tau^{\mathcal{C}}(x) = \sum_{i=1}^{r} a_i G_i(x)$$

for some constants a_i. Let f be as in (9) *and assume that each of the integrals*

$$\int_V^{\#} f(x)G_i(x - T)\ dx$$

exists for $i = 1, \ldots, r$. Then $f(x)\ \tau^{\mathcal{C}}(x - T)$ is #-integrable and

$$\int_V^{\#} f(x)\ \tau^{\mathcal{C}}(x - T)\ dx = \sum a_i \int_V^{\#} f(x)G_i(x - T)\ dx.$$

Proof. In the open set of λ such that $\lambda_j - \lambda$ is negative with respect to $\mathcal{C}^*$ for all j, we have an equality of absolutely convergent integrals

$$\int_V f(x)\, \tau^{\mathcal{C}}(x)\, e^{-\langle \lambda, x\rangle}\, dx = \sum a_i \int_V f(x) G_i(x-T)\, e^{-\langle \lambda, x\rangle}\, dx.$$

The assertion now follows by analytic continuation. □

For future reference, we note the explicit formula

$$\int_V^{\#} e^{\langle \lambda, x\rangle} \tau^{\mathcal{C}}(x-T)\, dx = (-1)^n v(e_1, \ldots, e_n) \frac{e^{\langle \lambda, T\rangle}}{\prod_{j=1}^n \langle \lambda, e_j\rangle} \tag{15}$$

where, as above,

$$\mathcal{C} = \{\sum_{j=1}^n a_j e_j : a_j \geq 0\}$$

and $v(e_1, \ldots, e_n)$ is the volume of the parallelepiped $\{\sum_{j=1}^n a_j e_j : 0 \leq a_j \leq 1\}$. In the case $V = \mathbb{R}$ we have

$$\int_T^{\infty} e^{\lambda t}\, dt = -\frac{e^{\lambda T}}{\lambda}$$

and hence for any cone $\mathcal{C} \subset \mathbb{R}$,

$$\int_V^{\#} e^{\langle \lambda, x\rangle} (1 - \tau^{\mathcal{C}}(x-T))\, dx = -\int_V^{\#} e^{\langle \lambda, x\rangle} \tau^{\mathcal{C}}(x-T)\, dx, \tag{16}$$

since $1 - \tau^C$ is the characteristic function of the cone $-\mathcal{C}$.

III. Automorphic forms and truncation

Let G be a connected, reductive algebraic group over a number field F with adele ring $\mathbb{A}$. In this section, we fix some notation and recall some definitions and results connected with Arthur's truncation operators.

1. **Roots, coroots, etc.** Fix a minimal F-parabolic subgroup P_0 of G and a Levi decomposition $P_0 = M_0 N_0$. An F-parabolic subgroup P is called *standard* if it contains P_0. If P is standard, we write M_P for the unique Levi factor of P containing M_0 and N_P for the unipotent radical of P. A Levi factor of the form M_P will be called a standard Levi subgroup. Since we deal only with standard subgroups, we shall usually drop the word standard and use the term *parabolic subgroup* or *Levi subgroup* to denote a standard F-parabolic subgroup or standard Levi subgroup.

We now recall some standard definitions and notations ([A1], [LR]). Let P be a parabolic subgroup. We write T_P for the maximal split torus in the center of M_P, and T'_P for the maximal quotient split torus of M_P. We set

$$\widetilde{\mathfrak{A}}_P = X_*(T_P) \otimes \mathbb{R} = X_*(T'_P) \otimes \mathbb{R},$$

where $X_*(T)$ is the lattice of 1-parametric subgroups in a torus (the two vector spaces on the right are canonically isomorphic), and set

$$d(P) = \dim \widetilde{\mathfrak{A}}_P.$$

The two descriptions of $\widetilde{\mathfrak{A}}_P$ show that if $Q \subset P$ is a parabolic subgroup contained in P, then there is a canonical injection $\widetilde{\mathfrak{A}}_P \longrightarrow \widetilde{\mathfrak{A}}_Q$ and surjection $\widetilde{\mathfrak{A}}_Q \longrightarrow \widetilde{\mathfrak{A}}_P$. We obtain canonical decomposition

$$\widetilde{\mathfrak{A}}_Q = \widetilde{\mathfrak{A}}_Q^P \oplus \widetilde{\mathfrak{A}}_P.$$

In particular, $\widetilde{\mathfrak{A}}_G$ is a summand of $\widetilde{\mathfrak{A}}_P$ for all P. Set $\mathfrak{A}_P = \widetilde{\mathfrak{A}}_P/\widetilde{\mathfrak{A}}_G$ and $\mathfrak{A}_Q^P = \widetilde{\mathfrak{A}}_P^Q/\widetilde{\mathfrak{A}}_G$. Then we have

$$\mathfrak{A}_Q = \mathfrak{A}_Q^P \oplus \mathfrak{A}_P.$$

In particular, $\mathfrak{A}_P$ is canonically identified as a subspace of $\mathfrak{A}_Q$. Set $\mathfrak{A}_0 = \mathfrak{A}_{P_0}$ and $\mathfrak{A}_0^P = \mathfrak{A}_{P_0}^P$. Then we also have

$$\mathfrak{A}_0 = \mathfrak{A}_0^P \oplus \mathfrak{A}_P$$

for all P. We write $\mathfrak{A}_0^*$, $\mathfrak{A}_P^*$, etc. for the dual spaces (over $\mathbb{R}$).

We now define the standard bases of the above spaces and their duals. Let Δ_0 and $\widehat{\Delta}_0$ be the subsets of simple roots and simple weights in $\mathfrak{A}_0^*$, respectively. We write $\Delta_0^\vee$ for the basis of $\mathfrak{A}_0$ dual to $\widehat{\Delta}_0$, and $\widehat{\Delta}_0^\vee$ for the basis of $\mathfrak{A}_0$ dual to Δ_0. Thus, $\Delta_0^\vee$ is the set of coroots and $\widehat{\Delta}_0^\vee$ is the set of coweights.

For every P, let $\Delta_P \subset \mathfrak{A}_0^*$ be the set of non-trivial restrictions of elements of Δ_0 to $\mathfrak{A}_P$. For each $\alpha \in \Delta_P$, let $\alpha^\vee$ be the projection of $\beta^\vee$ to $\mathfrak{A}_P$, where β is the root in Δ_0 whose restriction to $\mathfrak{A}_P$ is α. Set $\Delta_P^\vee = \{\alpha^\vee : \alpha \in \Delta_P\}$. Then we may also define their dual bases. Namely, we denote the dual basis of Δ_P by $\widehat{\Delta}_P^\vee$ and the dual basis of $\Delta_P^\vee$ by $\widehat{\Delta}_P$.

If $Q \subset P$, we write Δ_Q^P to denote the subset of $\alpha \in \Delta_Q$ appearing in the action of T_Q in the unipotent radical of $Q \cap M_P$. Then $\mathfrak{A}_P$ is the subspace of $\mathfrak{A}_Q$ annihilated by Δ_Q^P. Let $(\Delta_Q^P)^\vee = \{\alpha^\vee : \alpha \in \Delta_Q^P\}$. We then define $(\widehat{\Delta}^\vee)_Q^P$ and $\widehat{\Delta}_Q^P$ to be the bases dual to Δ_Q^P and $(\Delta_Q^P)^\vee$, respectively.

2. **Inversion formulas.** For convenience, we give an abstract formulation of some inversion relations employed in Arthur's work. Suppose that $\{\tau_P^Q\}, \{\hat{\tau}_P^Q\}$ are sets of constants indexed by pairs of parabolic subgroups $P \subset Q$ satisfying the relations

$$\sum_{Q \subset R \subset P} (-1)^{d(R)-d(P)} \tau_Q^R \hat{\tau}_R^P = \delta_{QP}$$

for any $P \subset Q$.

This set of relations implies (and is equivalent to) the set of relations

$$\sum_{Q \subset R \subset P} (-1)^{d(Q)-d(R)} \hat{\tau}_Q^R \tau_R^P = \delta_{QP}.$$

Indeed,

$$\begin{aligned}
& \sum_{Q \subset R \subset P} (-1)^{d(Q)-d(R)} \hat{\tau}_Q^R \tau_R^P \\
= \quad & \sum_{Q \subset R \subset R' \subset P} (-1)^{d(Q)-d(R)} \hat{\tau}_Q^R \delta_{RR'} \tau_{R'}^P \\
= \quad & \sum_{Q \subset R \subset S \subset R' \subset P} (-1)^{d(Q)-d(R)+d(S)-d(R')} \hat{\tau}_Q^R \tau_R^S \hat{\tau}_S^{R'} \tau_{R'}^P \\
= \quad & \sum_{Q \subset R \subset S \subset R' \subset P} (-1)^{d(Q)-d(R)+d(S)-d(R')} \hat{\tau}_Q^R \tau_R^S \hat{\tau}_S^{R'} \tau_{R'}^P.
\end{aligned}$$

For fixed S, the sum over R vanishes unless $S = Q$ and the sum over R' vanishes unless $S = P$. Hence the sum vanishes unless $Q = P$, in which case it equals $\hat{\tau}_Q^Q \tau_Q^P = 1$.

Given arbitrary sets of constants $\{\alpha_Q^P\}$ and $\{\hat{\alpha}_Q^P\}$, define

$$\beta_Q^P = \sum_{Q \subset R \subset P} (-1)^{d(R)-d(P)} \tau_Q^R \hat{\alpha}_R^P,$$

$$\hat{\beta}_Q^P = \sum_{Q \subset S \subset P} (-1)^{d(Q)-d(S)} \alpha_Q^S \hat{\tau}_S^P.$$

The following lemma is easily verified.

Lemma 7. *The following relations hold:*

1. $\hat{\alpha}_Q^P = \sum_{Q \subset R \subset P} (-1)^{d(R)-d(P)} \hat{\tau}_Q^R \beta_R^P$.
2. $\alpha_Q^P = \sum_{Q \subset R \subset P} (-1)^{d(Q)-d(R)} \hat{\beta}_Q^R \tau_R^P$.
3. *If the constants* $\{\alpha_Q^P\}$ *and* $\{\hat{\alpha}_Q^P\}$ *satisfy*

$$\sum_{Q \subset R \subset P} (-1)^{d(Q)-d(R)} \hat{\alpha}_Q^R \alpha_R^P = \delta_{QP},$$

then

$$\sum_{Q \subset R \subset P} (-1)^{d(R)-d(P)} \beta_Q^R \hat{\beta}_R^P = \delta_{QP}.$$

3. **The standard characteristic functions.** We now extend the linear functionals in Δ_Q^P and $\hat{\Delta}_Q^P$ to elements of the dual space $\mathfrak{A}_0^*$ by means of the canonical projection from $\mathfrak{A}_0$ to $\mathfrak{A}_Q^P$ given by the decomposition $\mathfrak{A}_0 = A_0^Q \oplus A_Q^P \oplus A_P$. Let τ_Q^P be the characteristic function of the subset

$$\{H \in \mathfrak{A}_0 : \langle \alpha, H \rangle > 0 \text{ for all } \alpha \in \Delta_Q^P\}$$

and let $\hat{\tau}_Q^P$ be the characteristic function of the subset

$$\{H \in \mathfrak{A}_0 : \langle \varpi, H \rangle > 0 \text{ for all } \varpi \in \hat{\Delta}_Q^P\}.$$

We recall that $\hat{\tau}_Q^P \geq \tau_Q^P$. The following is a special case of Langlands' combinatorial lemma [A1].

Langlands' Lemma. *If $Q \subset P$ are parabolic subgroups, then for all $H \in \mathfrak{A}_0$ we have*

$$\sum_{Q \subset R \subset P} (-1)^{d(R)-d(P)} \tau_Q^R(H) \hat{\tau}_R^P(H) = \delta_{QP}$$

and

$$\sum_{Q \subset R \subset P} (-1)^{d(Q)-d(R)} \hat{\tau}_Q^R(H) \tau_R^P(H) = \delta_{QP}.$$

For any pair of parabolic subgroups $P \subset Q$ we define functions $\Gamma_P^Q(H, X)$ and $\hat{\Gamma}_P^Q(H, X)$ for H, $X \in \mathfrak{A}_0$ by the formulas

$$\Gamma_Q^P(H, X) = \sum_{Q \subset R \subset P} (-1)^{d(R)-d(P)} \tau_Q^R(H) \hat{\tau}_R^P(H - X),$$

$$\hat{\Gamma}_Q^P(H, X) = \sum_{Q \subset S \subset P} (-1)^{d(Q)-d(S)} \tau_Q^S(H - X) \hat{\tau}_S^P(H).$$

These functions depend only on the projections of H and X onto $\mathfrak{A}_Q^P$. The function Γ_P^G appears in [A3] and we follow [LR] in defining Γ_Q^P, $\hat{\Gamma}_Q^P$ for general $Q \subset P$. By Langlands' Lemma and Lemma 7, we obtain the formulas:

$$\hat{\tau}_Q^P(H-X) = \sum_{Q\subset R\subset P} (-1)^{d(R)-d(P)} \hat{\tau}_Q^R(H)\Gamma_R^P(H,X), \tag{17}$$

$$\tau_Q^P(H-X) = \sum_{Q\subset R\subset P} (-1)^{d(Q)-d(R)} \hat{\Gamma}_Q^R(H,X)\tau_R^P(H),$$

and also

$$\begin{aligned} \sum_{\substack{P \\ S\subset P\subset T}} (-1)^{d(P)-d(T)}\Gamma_S^P(H,X)\hat{\Gamma}_P^T(H,X) &= \delta_{ST}, \\ \sum_{\substack{P \\ S\subset P\subset T}} (-1)^{d(S)-d(P)}\hat{\Gamma}_S^P(H,X)\Gamma_P^T(H,X) &= \delta_{ST}. \end{aligned}$$

Finally, the following relation follows from the definitions:

$$\hat{\Gamma}_Q^P(H,X) = (-1)^{d(Q)-d(R)}\Gamma_Q^P(H-X,-X),$$

and therefore

$$\tau_Q^P(H-X) = \sum_{Q\subset R\subset P} \Gamma_Q^R(H-X,-X)\tau_R^P(H). \tag{18}$$

4. **Automorphic forms.** We shall often write G for $G(F)$, P for $P(F)$, etc., when there is no risk of confusion. To define automorphic forms on G, let $\mathcal{Z}$ be the center of the complexified universal enveloping algebra of G_∞ and fix a good maximal compact subgroup $\mathbf{K} \subset G(\mathbb{A})$. Then the Iwasawa decomposition $G(\mathbb{A}) = N_P(\mathbb{A})M_P(\mathbb{A})\mathbf{K}$ holds. For any $g \in G(\mathbb{A})$, we will say that $g = nmk$ is an Iwasawa decomposition relative to P if $n \in N_P(\mathbb{A})$, $m \in M_P(\mathbb{A})$, and $k \in \mathbf{K}$.

A function

$$\varphi : G\backslash G(\mathbb{A}) \longrightarrow C$$

is called an *automorphic form* if

1. φ is smooth and of moderate growth,
2. φ is right $\mathbf{K}$-finite,
3. φ is $\mathcal{Z}$-finite.

Let $\mathcal{A}(G)$ be the space of automorphic forms on $G\backslash G(\mathbb{A})$.

We also define the space $\mathcal{A}_P(G)$ for any parabolic subgroup $P = MN$. This is the space of smooth, right $\mathbf{K}$-finite functions

$$\varphi : N(\mathbb{A})M\backslash G(\mathbb{A}) \longrightarrow C$$

such that for all $k \in \mathbf{K}$, the function $m \longrightarrow \varphi(mk)$ is an automorphic form on $M(\mathbb{A})$. For $\varphi \in \mathcal{A}_P(G)$ and any parabolic subgroup $Q \subset P$, the constant term is defined in the standard way:

$$\varphi_Q(g) = \int_{N_Q\backslash N_Q(\mathbb{A})} \varphi(ng)dn.$$

The map $\varphi \longmapsto \varphi_Q$ sends $\mathcal{A}_P(G)$ to $\mathcal{A}_Q(G)$.

For each parabolic subgroup P, we have the map

$$H_P : G(\mathbb{A}) \longrightarrow \widetilde{\mathfrak{A}}_P$$

characterized by the following two conditions: (1) $|\chi|(m) = e^{\langle \chi, H_P(m)\rangle}$ for all $m \in M(\mathbb{A})$ and rational characters $\chi \in X^*(M_P)$, and (2) $H_P(nmk) = H_P(m)$ for all $n \in N(\mathbb{A}), m \in M(\mathbb{A})$, and $k \in \mathbf{K}$. We write $H(g)$ for $H_{P_0}(g)$. Then $H_P(g)$ is the projection of $H(g)$ onto $\widetilde{\mathfrak{A}}_P$. The kernel of the restriction of H_P to $M(\mathbb{A})$ is denoted $M(\mathbb{A})^1$.

Set $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. We have an isomorphism $(F_\infty^*)^\ell \simeq T_P(F_\infty)$ for some ℓ. We may embed $\mathbb{R} \hookrightarrow F_\infty$ via $x \longrightarrow 1 \otimes x$ and thus view $\mathbb{R}_+^*$ as a subgroup of F_∞^* in a canonical way. Let A_P for $P \neq G$ and A_G denote the intersections of the image of $(\mathbb{R}_+^*)^\ell$ in $T_P(F_\infty)$ with $G(\mathbb{A})^1$and $Z(\mathbb{A})$, respectively. The map H_P induces an isomorphism of A_P onto $\mathfrak{A}_P$ and we have $M(\mathbb{A}){=}A_G \times A_P \times M(\mathbb{A})^1$. For $X \in \mathfrak{A}_P$, we write e^X for the unique element in A_P such that $H_P(e^X) = X$, and for $\lambda \in \mathfrak{A}_P^*$, we write e^λ for the character $p \to e^{\langle \lambda, H_P(p)\rangle}$ of $P(\mathbb{A})$.

As usual, we denote by $\rho_P \in \mathfrak{A}_P^*$ the unique element such that $e^{2\langle \rho_P, H(m)\rangle} = |\det Ad_N(m)|$ where $Ad_N(m)$ is the adjoint action of m on $Lie(N)$. For suitable normalizations of Haar measure, we have

$$\int_{G(\mathbb{A})^1} f(x)\, dx = \int_{N(\mathbb{A})} \int_{A_P} \int_{M(\mathbb{A})^1} \int_{\mathbf{K}} f(namk)\, e^{-2\langle \rho_P, H(a)\rangle}\, dn\, da\, dm\, dk.$$

5. **Truncation operators.** We recall the definition of Arthur's truncation operators. Let $T \in \mathfrak{A}_0$. Following [A2], we define the truncation of a smooth function φ on $P\backslash G(\mathbb{A})$ by the formula

$$\Lambda^{T,P}\varphi(g) = \sum_{R \subset P} (-1)^{d(R)-d(P)} \sum_{\delta \in R\backslash P} \varphi_R(\delta g)\hat{\tau}_R^P(H(\delta g) - T).$$

The sums over δ are all finite by [A1], Lemma 5.1. Note that $\Lambda^{T,P}\varphi = \Lambda^{T,P}\varphi_P$. For φ invariant under G, Langlands' Lemma immediately yields the inversion formula

$$\varphi(g) = \sum_P \sum_{\delta \in P\backslash G} \Lambda^{T,P}\varphi(g)\tau_P(H(\delta g) - T).$$

From now on, $T \in \mathfrak{A}_0$ will denote a sufficiently regular element in the sense of [A2]. In this case, the function $m \longmapsto \Lambda^{T,P}\varphi(mk)$ is rapidly decreasing on $M_P\backslash M_P(\mathbb{A})^1$ for all $k \in \mathbf{K}$.

IV. Periods relative to quadratic extensions

In this section E/F is a quadratic extension of number fields. We write $\mathbb{A}$ for the adeles of F and $\mathbb{A}_E$ for the adeles of E. Let H be a split, connected reductive group over F and let $G = H_{/E}$. Then $H(F)$ is a subgroup of $G(E)$. Let $\mathbf{K}_F$ and $\mathbf{K}_E$ denote fixed good maximal compact subgroups of $H(\mathbb{A})$ and $G(\mathbb{A}_E)$, respectively. If $P = MN$ is a parabolic subgroup of H, then

$$H(\mathbb{A})^1 = N(\mathbb{A})A_P M(\mathbb{A})^1 \mathbf{K}_F,$$

and a Haar measure on $H(\mathbb{A})^1$ is given by the integral

$$\int_{N(\mathbb{A})} \int_{A_P} \int_{M(\mathbb{A})^1} \int_{\mathbf{K}_F} f(namk)\, e^{-2\langle \rho_P, H(a)\rangle}\, dn\, da\, dm\, dk.$$

There is a bijection between the set of parabolic subgroups of H and G. If P is a parabolic subgroup of H (standard and defined over F), then the corresponding parabolic subgroup of G (also standard and defined over E) is denoted P_E and is characterized by $P_E(E) = P(E)$. Note that $H_E = G$. We identify $\mathfrak{A}_P$ and $\mathfrak{A}_{P_E}$ and their dual spaces. In this identification the roots of H correspond to the roots of G and hence $\rho_P = \rho_{P_E}$. Similarly, the standard characteristic functions do not depend on whether they are defined relative to H or G, i.e., $\tau_Q^P = \tau_{Q_E}^{P_E}, \hat{\tau}_Q^P = \hat{\tau}_{Q_E}^{P_E}$, etc. We have

$$H_{P_E}(a) = 2H_P(a)$$

for $a \in H(\mathbb{A})$. We also write H_E for the height function on G.

For $\varphi \in \mathcal{A}(G)$, we write

$$\varphi_{P_E}(g) = \int_{N(E)\backslash N(\mathbb{A}_E)} \varphi(ng)dn$$

for the constant term of φ relative to the parabolic subgroup P_E in G. We also fix Siegel domains for G and for the parabolic subgroups P_E of G. Thus we choose a compact subset $\Theta \subset N_0(\mathbb{A}_E)M_0(\mathbb{A}_E)^1$ and $T_0 \in \mathfrak{A}_0$ such that $\langle \alpha, T_0 \rangle << 0$ for all $\alpha \in \Delta_0$, and we let $\mathcal{S}^{P_E}$ be the set of elements pak where $p \in \Theta$, $k \in \mathbf{K}_E$, and $a \in A_{P_0}$ satisfies

$$\langle \alpha, H_E(a) \rangle \geq \langle \alpha, T_0 \rangle \quad \text{for all} \quad \alpha \in \Delta_0^P.$$

We write $\mathcal{S}$ for $\mathcal{S}^G$. Let $\mathcal{S}^H$ be the Siegel domain for H consisting of elements pak such that $p \in \Theta \cap H(\mathbb{A})$, $k \in \mathbf{K}_F$, and $a \in A_{P_0}$. By a basic result of reduction theory, we may (and shall) choose Θ and T_0 such that $G(\mathbb{A}_E) = P_E(E)\mathcal{S}^{P_E}$. We may also assume that $H(\mathbb{A}) = H(F)\mathcal{S}^H$.

6. **Mixed truncation.** For $T \in \mathfrak{A}_0$, we define the *mixed truncation* of φ by the formula

$$\Lambda_m^{T,P}\varphi(h) = \sum_{R \subset P} (-1)^{d(R)-d(P)} \sum_{\delta \in R\backslash P} \varphi_{R_E}(\delta h)\hat{\tau}_R^P(H_E(\delta h) - T).$$

This is halfway between truncation on G (since we use constant terms relative to G) and truncation on H (since the sums only involve δ lying in H). We write Λ_m^T for $\Lambda_m^{T,H}$. Thus,

$$\Lambda_m^T\varphi(h) = \sum_{R \subset H} (-1)^{d(R)-d(G)} \sum_{\delta \in R\backslash H} \varphi_{R_E}(\delta h)\hat{\tau}_R(H_E(\delta h) - T).$$

The following inversion formula holds, as before, for all $h \in H(\mathbb{A})$:

$$\varphi(h) = \sum_{P \subset H} \sum_{\delta \in P\backslash H} \Lambda_m^{T,P}\varphi(\delta h)\ \tau_P(H_E(\delta h) - T). \tag{19}$$

Proposition 8. *Assume that T is sufficiently regular. Then for fixed h, the function $m \longrightarrow \Lambda_m^{T,P}\varphi(mh)$ is rapidly decreasing on $M\backslash M(\mathbb{A})^1$.*

Proof. We need only check that each step of the argument in [A2], Section 1, applies. It suffices to treat the case $M = H$. If $P_1 \subset P_2$ are parabolic subgroups, set

$$\phi_{12}(h) = \sum_{\substack{P \\ P_1 \subset P \subset P_2}} (-1)^{d(P)-d(G)}\phi_{P_E}(h).$$

As in [A2],

$$\Lambda_m^{T,H}\varphi(h) = \sum_{P_1 \subset P_2} \sum_{\delta \in P_1\backslash H} F_H^1(\delta h, \frac{1}{2}T)\sigma_1^2(H_E(\delta h) - T)\phi_{12}(\delta h)$$

(the definitions of the characteristic functions F^1 and σ_1^2 are in [A1]; F_H^1 denotes that F^1 is taken with respect to H). The argument of [A2], pp. 96–97, applied to the group H yields the bound

$$\sum_{\delta \in P_1 \backslash H} F_H^1(\delta h, \frac{1}{2}T)\sigma_1^2(H_E(\delta h) - T) \leq C_1 ||h||^{N_1}$$

for all $h \in H(\mathbb{A})$, for some constants $C_1, N_1 > 0$. On the other hand, the argument of [A2], pp. 92–96, for the group G shows that for any $N > 0$ there exists $C > 0$ such that

$$|\phi_{12}(\delta g)| \leq C||g||^{-N}$$

for all $\delta \in G(E)$ and $g \in G(\mathbb{A})^1$ with $F_G^1(\delta g, T)\sigma_1^2(H_E(\delta g) - T) = 1$. However, it is clear from the definition that if $F_H^1(\delta h, \frac{1}{2}T) = 1$, then $F_G^1(\delta h, T) = 1$. It follows that $\Lambda_m^{T,H}\varphi(h)$ is rapidly decreasing on $H(\mathbb{A})^1$. □

7. **The period of an automorphic form.** We now define the regularized period of an automorphic form. The first step is to define a certain integral over $P\backslash H(\mathbb{A})^1$ where P is a parabolic subgroup of H. Let $\tau_k(X)$ be a function of type (C) (as defined in Section II) on $\mathfrak{A}_P$ that depends continuously on $k \in \mathbf{K}_E$, i.e., we assume that there is a decomposition $\mathfrak{A}_P = W_1 \oplus W_2$ such that τ_k has the form

$$g_k(w_1 - T_1)\ \tau_{\mathcal{C}_{2k}}(w_2 - T_2)$$

where the compactly supported function g_k varies continuously in the L^1-norm and linear inequalities defining the cone $\mathcal{C}_{2k}$ vary continuously. Let f be a function on $P_E(E)N_E(\mathbb{A}_E)\backslash G(\mathbb{A}_E)$ of the form

$$f(namk) = \sum_{j=1}^{k} \phi_j(m,k)\alpha_j(H_{P_E}(a),k)e^{\langle \lambda_j + \rho_{P_E}, H_E(a)\rangle} \tag{20}$$

for $n \in N(\mathbb{A})$, $a \in A_P$, $m \in M(\mathbb{A})^1$ and $k \in \mathbf{K}_E$, where for all j,

(a) $\lambda_j \in \mathfrak{A}_P^*$ and $\alpha_j(X,k)$ is a continuous family of polynomials on $\mathfrak{A}_P$ such that for all $k \in \mathbf{K}_E$, $\alpha_j(X,k)e^{\langle \lambda_j, X\rangle}\tau_k(X)$ is #-integrable;

(b) $\phi_j(m,k)$ is absolutely integrable on $M\backslash M(\mathbb{A})^1 \times \mathbf{K}_F$.

In this case, we define the #-integral

$$\int^{\#}_{P\backslash H(\mathbb{A})^1} f(h)\ \tau_k(H_{P_E}(h))\ dh \tag{21}$$

by

$$\sum_{j=1}^{k} \int_{\mathbf{K}_F} \left(\int_{M\backslash M(\mathbb{A})^1} \phi_j(mk)\ dm \right) \left(\int^{\#}_{\mathfrak{A}_P} \alpha_j(2X,k)e^{\langle \lambda_j, 2X\rangle}\tau_k(2X)dX \right)\ dk.$$

Recall that $H_{P_E}(a) = 2H_P(a)$ for $a \in A_P$ and therefore, with our definition of the exponents λ_j, no shift by ρ_P appears in the #-integral over $\mathfrak{A}_P$.

We write $\mathcal{E}_P(f)$ for the set of distinct exponents $\{\lambda_j\}$ occurring in (20). This set is uniquely determined by f, but the functions α_j and ϕ_j are not. However, if we denote by e^X the element in A_P such that $H_P(e^X) = X$, then the function

$$X \longrightarrow \int_{M\backslash M(\mathbb{A})^1} f(e^X mk)\ dm$$

is an exponential polynomial on $\mathfrak{A}_P$ and (21) is equal to

$$\int_{\mathbf{K}_F} \int_{\mathfrak{A}_P}^{\#} \left(\int_{M\backslash M(\mathbb{A})^1} f(e^X mk)\ dm \right) e^{-2\langle \rho_P, X\rangle} \tau_k(2X) dX\ dk.$$

This shows, in particular, that (21) is independent of the decomposition (20). If each of the exponents λ_j are negative with respect to $\mathcal{C}_{2k}$ for all $k \in \mathbf{K}_F$, then the ordinary integral

$$\int_{P\backslash H(\mathbb{A})^1} f(h)\ \tau_k(H_{P_E}(h))\ dh$$

is absolutely convergent and its value coincides with the #-integral by Lemma 2.

We fix a sufficiently regular element $T \in \mathfrak{A}_0^+$. Then the above construction applies to $\Lambda_m^{T,P}\Psi(g)$ and the characteristic function $\tau_P(H_E(g) - T)$ where $\Psi \in \mathcal{A}_{P_E}(G)$. According to [MW], I.3.2, Ψ has a decomposition of the type (20). Namely,

$$\Psi(namk) = \sum_{j=1}^{r} Q_j(H_{P_E}(a))\psi_j(amk) \tag{22}$$

for $n \in N(\mathbb{A}_E), a \in A_{P_E}$, $m \in M(\mathbb{A}_E)^1$ and $k \in \mathbf{K}_E$, where the Q_j are polynomials and $\psi_j \in \mathcal{A}_{P_E}(G)$ satisfies

$$\psi_j(ag) = e^{\langle \lambda_j + \rho_P, H_E(a)\rangle} \psi_j(g)$$

for some exponent $\lambda_j \in \mathfrak{A}_P^*$ for all $a \in A_{P_E}$. By Proposition 8, the functions $m \longrightarrow \Lambda_m^{T,P}\psi_j(mk)$ are rapidly decreasing and hence absolutely integrable over $M\backslash M(\mathbb{A})^1 \times \mathbf{K}_F$. Since τ_P is the characteristic function of the cone spanned by the coweights $\hat{\Delta}_P^\vee$ we see that

$$\int_{P\backslash H(\mathbb{A})^1}^{\#} \Lambda_m^{T,P}\Psi(h)\ \tau_P(H_{P_E}(h) - T)\ dh$$

exists if and only if

$$\langle \lambda_j, \varpi^\vee \rangle \neq 0 \text{ for all } \varpi^\vee \in \hat{\Delta}_P^\vee \text{ and } \lambda_j \in \mathcal{E}_P(\Psi). \tag{23}$$

The same is true for $\Lambda^{T,P}\Psi(h)\tau_P(H_{P_E}(hx) - T)$ for fixed $x \in H(\mathbb{A})$. Indeed, for $h \in H(\mathbb{A})$, let $K(h) \in \mathbf{K}_F$ be any element such that $hK(h)^{-1} \in P_0(\mathbb{A})$. Then

$$H_{P_E}(hx) = H_{P_E}(h) + H_{P_E}(K(h)x)$$

and hence $\tau_P(X - T + H_{P_E}(K(h)x))$ is the characteristic function of a cone depending continuously on h. We set

$$\Pi_P^{H,T}(\Psi) = \int_{P\backslash H(\mathbb{A})^1}^{\#} \Lambda_m^{T,P}\Psi(h)\ \tau_P(H_{P_E}(h) - T)\ dh.$$

For any automorphic form $\varphi \in \mathcal{A}(G)$, we write $\mathcal{E}_P(\varphi)$ for the set of exponents $\mathcal{E}_P(\varphi_P)$. Set

$$\mathcal{A}(G)^* = \{\varphi \in \mathcal{A}(G) : \langle \lambda, \varpi^\vee \rangle \neq 0 \text{ for all } \varpi^\vee \in \hat{\Delta}_P^\vee, \lambda \in \mathcal{E}_P(\varphi),\ P \neq H\}.$$

If $\varphi \in \mathcal{A}(G)^*$, then $\Pi_P^{H,T}(\varphi_P)$ exists for all P, and we can define the *regularized period*

$$\int_{H\backslash H(\mathbb{A})^1}^{*} \varphi(h) dh = \sum_{P\subset H} \Pi_P^{H,T}(\varphi_P).$$

We also denote this integral by $\Pi^{G/H}(\varphi)$. The name and notations are justified by Theorem 9 below.

Let $\mathbb{A}_{Ef}$ be the finite adeles of $\mathbb{A}_E$. For $x \in G(\mathbb{A}_{Ef})$, let $\rho(x)$ denote right translation by $x : \rho(x)\varphi(g) = \varphi(gx)$. The space $\mathcal{A}(G)$ is stable under right translation by $G(\mathbb{A}_f)$. Furthermore, $\rho(x)\varphi$ has the same set of exponents as φ. Indeed, for $k \in \mathbf{K}_E$, write the Iwasawa decomposition of kx as $kx = n'a'm'K(kx)$ and write φ_{P_E} in the form (22):

$$\varphi_{P_E}(namk) = \sum_{j=1}^{r} Q_j(H_{P_E}(a))\; e^{\langle \lambda_j + \rho_{P_E}, H_E(a)\rangle} \psi_j(mk).$$

Since $amkx = n''aa'mm'K(kx)$ for some $n'' \in N_E(\mathbb{A}_E)$, $\varphi_{P_E}(namkx)$ is equal to

$$\sum_{j=1}^{r} Q_j(H_{P_E}(a) + H_{P_E}(a'))\; e^{\langle \lambda_j + \rho_{P_E}, H_E(a) + H_E(a')\rangle} \psi_j(mm'K(kx))$$

and this shows that $\mathcal{E}_P(\rho(x)\varphi) = \mathcal{E}_P(\varphi)$ as claimed. It follows that the space $\mathcal{A}(G)^*$ is invariant under right translation by $G(\mathbb{A}_{Ef})$.

Theorem 9. (i) $\Pi^{G/H}$ *defines an* $H(\mathbb{A}_f)^1$*-invariant linear functional on* $\mathcal{A}(G)^*$.
(ii) $\Pi^{G/H}$ *is independent of the choice of* T.
(iii) *If* $\varphi \in \mathcal{A}(G)$ *is integrable over* $H\backslash H(\mathbb{A})^1$, *then* $\varphi \in \mathcal{A}(G)^*$ *and*

$$\Pi^{G/H}(\varphi) = \int_{H\backslash H(\mathbb{A})^1} \varphi(h)dh.$$

Proof. We first observe that for f defined by (20), we have

$$\int^{\#}_{P\backslash H(\mathbb{A})^1} f(hx)\tau_k(hx)\; dg = \int^{\#}_{P\backslash H(\mathbb{A})^1} f(h)\tau_k(h)\; dh \tag{24}$$

for all $x \in H(\mathbb{A}_f)^1$. To verify this, set $f_\mu(g) = e^{(\mu, H_{P_E}(g))} f(g)$ for $\mu \in \mathfrak{A}_P^*$. If $\langle \operatorname{Re}\mu, \varpi^\vee\rangle << 0$ for all $\varpi^\vee \in \hat{\Delta}_P^\vee$, then

$$\int_{P\backslash H(\mathbb{A})^1} f_\mu(hx)\tau_k(hx)\; dh = \int_{P\backslash H(\mathbb{A})^1} f_\mu(h)\tau_k(h)\; dh,$$

by the invariance of Haar measure, since both sides are absolutely convergent. Both sides have a meromorphic continuation whose value at $\mu = 0$ gives (24).

Now fix $x \in H(\mathbb{A}_f)^1$ and set

$$F_P(g) = (\Lambda_m^{T,P}\rho(x^{-1})\varphi)(gx).$$

Then

$$\int^{\#}_{P\backslash H(\mathbb{A})^1} (\Lambda_m^{T,P}\rho(x^{-1})\varphi)(h)\tau_P(H_{P_E}(h) - T)\; dh$$
$$= \int^{\#}_{P\backslash H(\mathbb{A})^1} F_P(h)\; \tau_P(H_{P_E}(hx) - T)\; dh$$

by (24), and hence we must show that

$$\int^{*}_{H\backslash H(\mathbb{A})^1} \varphi(h)\; dh = \sum_P \int^{\#}_{P\backslash H(\mathbb{A})^1} F_P(h)\; \tau_P(H_{P_E}(hx) - T)\; dh. \tag{25}$$

Let us re-write F_P using the functions Γ_Q^P of §3. As before, for $h \in H(\mathbb{A})$, we let $K(h) \in \mathbf{K}$ be any element such that $hK(h)^{-1} \in P_0(\mathbb{A})$. We claim that

$$F_P(h) = \sum_{\substack{S \\ S \subset P}} \sum_{\eta \in S \backslash P} \Lambda_m^{T,S} \varphi_{S_E}(\eta h) \Gamma_S^P(H_{S_E}(\eta h) - T, -H_{S_E}(K(\eta h)x)). \tag{26}$$

Indeed, by definition,

$$F_P(h) = \sum_{\substack{R \\ R \subset P}} (-1)^{d(R)-d(P)} \sum_{\delta \in R \backslash P} \varphi_{R_E}(\delta h) \hat{\tau}_R^P(H_E(\delta h x) - T)$$

and

$$H_E(\delta h x) - T = H_E(\delta h) - T + H_E(K(\delta h)x).$$

Using (17), we may write $F_P(h)$ as the double sum over all $R \subset S \subset P$ and $\delta \in R \backslash P$ of

$$(-1)^{d(R)-d(S)} \varphi_{R_E}(\delta h) \hat{\tau}_R^S(H_E(\delta h) - T) \Gamma_S^P(H_{S_E}(\delta h) - T, -H_{S_E}(K(\delta h)x)).$$

This equals the sum over $S \subset P$ and $\eta \in S \backslash P$ of

$$\left(\sum_{R \subset S} (-1)^{d(R)-d(S)} \sum_{\delta \in R \backslash S} \varphi_{R_E}(\delta \eta h) \hat{\tau}_R^S(H_E(\delta \eta h) - T) \right) \times \Gamma_S^P(H_{S_E}(\eta h) - T, -H_{S_E}(K(\eta h)x))$$

and (26) follows.

This gives

$$\begin{aligned} &\int_{P \backslash H(\mathbb{A})^1}^{\#} F_P(h) \quad \tau_P(H_{P_E}(hx) - T)\, dh \\ &= \int_{P \backslash H(\mathbb{A})^1}^{\#} \sum_{\substack{S \\ S \subset P}} \sum_{\eta \in S \backslash P} \Lambda_m^{T,S} \varphi_{S_E}(\eta h) \Gamma_S^P(H_{S_E}(\eta h) - T, -H_{S_E}(K(\eta h)x)) \\ &\qquad \times \tau_P(H_{P_E}(hx) - T)\, dh \end{aligned} \tag{27}$$

and our next step is to show that the sum over S can be taken outside of the integral. Recall that the functions $\Gamma_S^P(Z, W)$ depend only on the projections of Z and W onto $\mathfrak{A}_S^P$. According to [A3], Lemma 2.1, for W belonging to a fixed compact subset of $\mathfrak{A}_S^P$, there exists a compact subset $\mathcal{Y} \subset \mathfrak{A}_S^P$ such that the function $Z \longrightarrow \Gamma_S^P(Z, W)$ is the characteristic function of a compact set in $\mathfrak{A}_S^P$ contained in $\mathcal{Y}$. In particular, the function

$$h \longrightarrow \Gamma_S^P(H_{S_E}(h) - T, -H_{S_E}(K(h)x))$$

is supported inside a subset of elements h for which the projection of $H_{S_E}(h)$ onto $\mathfrak{A}_S^P$ lies in a compact set depending only on x. Therefore

$$\sum_{\eta \in S \backslash P} \Lambda^{T,S} \varphi_{S_E}(\eta h) \Gamma_S^P(H_{S_E}(\eta h) - T, -H_{S_E}(K(\eta h)x))$$

is integrable over $M(F) \backslash M(\mathbb{A})^1$. Now let $h = ne^X mk$ be an Iwasawa decomposition of h relative to P with $X \in \mathfrak{A}_P$. There exist polynomials Q_i on $\mathfrak{A}_P$, automorphic forms $\psi_j \in \mathcal{A}_P(G)$, and exponents λ_j in the $\mathcal{E}_P(\varphi)$ such that for all $S \subset P$,

$$\Lambda_m^{T,S}\varphi_{S_E}(h) = \sum_j Q_j(2X)e^{\langle \lambda_j+\rho_P, 2X\rangle}\Lambda_m^{T,S}\psi_j(mk). \tag{28}$$

Since $\Gamma_S^P(Z,W)$ depend only on the projections of Z and W onto $\mathfrak{A}_Q^P$, we may write (27) as the integral over $k \in \mathbf{K}_F$ and sum over j of

$$\begin{aligned}
&\int_{\mathfrak{A}_P}^{\#}\int_{M\backslash M(\mathbb{A})^1}\sum_{\substack{S\\ S\subset P}}\sum_{\eta\in S\backslash P} Q_j(2X)e^{\langle\lambda_j,2X\rangle}\Lambda_m^{T,S}\psi_j(\eta mk)\\
&\quad\times\Gamma_S^P(H_{S_E}(\eta m)-T,-H_{S_E}(K(\eta mk)x))dm\tau_P(2X+H_E(kx)-T)dX\\
=&\int_{\mathfrak{A}_P}^{\#}\sum_j Q_j(2X)e^{\langle\lambda_j,2X\rangle}\tau_P(2X+H_E(kx)-T)dX\int_{M\backslash M(\mathbb{A})^1}\sum_{\substack{S\\ S\subset P}}\sum_{\eta\in S\backslash P}\sum_j\\
&\quad\times\Lambda_m^{T,S}\psi_j(\eta mk)\Gamma_S^P(H_{S_E}(\eta m)-T,-H_{S_E}(K(\eta mk)x))dm
\end{aligned}$$

and since each term in the sum over S is separately integrable over $M\backslash M(\mathbb{A})^1$, we may take the sum over S outside the integral as claimed.

We now claim that

(29)

$$\int_{P\backslash H(\mathbb{A})^1}^{\#}\sum_{\eta\in S\backslash P}\Lambda^{T,S}\varphi_{S_E}(\eta h)\Gamma_S^P(H_{S_E}(\eta h)-T,-H_{S_E}(K(\eta h)x))\tau_P(H_E(hx)-T)\,dh$$

is equal to

$$\int_{S\backslash H(\mathbb{A})^1}^{\#}\Lambda^{T,S}\varphi_{S_E}(h)\Gamma_S^P(H_{S_E}(h)-T,-H_{S_E}(K(h)x))\tau_P(H_E(hx)-T)\,dh. \tag{30}$$

Indeed, (29) is equal to the integral over $k \in \mathbf{K}_F$ and sum over j of

$$\begin{aligned}
&\int_{\mathfrak{A}_P}^{\#}Q_j(2X)e^{\langle\lambda_j,2X\rangle}\tau_P(2X+H_E(kx)-T)dX\\
&\qquad\times\int_{M\backslash M(\mathbb{A})^1}\sum_{\eta\in S\backslash P}\Lambda_m^{T,S}\psi_j(\eta mk)\Gamma_S^P(H_{S_E}(\eta m)-T,-H_{S_E}(K(\eta mk)x))dm
\end{aligned}$$

which can be written

$$\begin{aligned}
&\int_{\mathfrak{A}_P}^{\#}Q_j(2X)e^{\langle\lambda_j,X\rangle}\tau_P(2X+H_E(kx)-T)dX\\
&\qquad\times\int_{S_M\backslash M(\mathbb{A})^1}\Lambda_m^{T,S}\psi_j(mk)\Gamma_S^P(H_{S_E}(m)-T,-H_{S_E}(K(mk)x))dm
\end{aligned}$$

where $S_M = S\cap M$. Expressing the integral over $M_S\backslash M(\mathbb{A})^1$ using the Iwasawa decomposition $m = n'e^{X'}m'k'$ of $M(\mathbb{A})^1$ relative to S_M gives

$$\begin{aligned}
&\int_{\mathfrak{A}_P}^{\#}Q_j(2X)e^{\langle\lambda_j,2X\rangle}\tau_P(2X+H_E(kx)-T)dX\int_{\mathbf{K}_M}\int_{\mathfrak{A}_S^P}^{\#}\int_{M_S\backslash M_S(\mathbb{A})^1}\\
&\times e^{-2\langle\rho_S^P,X'\rangle}\Lambda_m^{T,S}\psi_j(e^{X'}m'k'k)\Gamma_S^P(2X'-T,-H_{S_E}(k'kx))\,dm'dX'dk'
\end{aligned}$$

where $\mathbf{K}_M = \mathbf{K}_F \cap M(\mathbb{A})^1$. Since we are integrating over $\mathbf{K}_F$, we may drop the integral over $\mathbf{K}_M$. However, each function $\Lambda_m^{T,S}\psi_j(e^{X'}m'k)$ has a decomposition analogous to (28) with respect to S and therefore, for fixed k, the function

$$(m', X') \longrightarrow \sum_j \Lambda_m^{T,S}\psi_j(e^{X'}m'k)\Gamma_S^P(2X'-T, -H_{S_E}(kx))\ dm'$$

is a sum of terms, each of which is the product of a function of m' which is absolutely integrable over $M_S\backslash M_S(\mathbb{A})^1$ and a function of X' which itself is equal to an exponential polynomial times the compactly supported function $\Gamma_S^P(2X'-T, -H_{S_E}(kx))$. According to (14), we may combine the integrals over $\mathfrak{A}_S^P$ and $\mathfrak{A}_P$ to a #-integral over $\mathfrak{A}_S$, and we find that (29) is equal to the integral over $k \in \mathbf{K}_F$ and $m' \in M_S\backslash M_S(\mathbb{A})^1$ of

$$\int_{\mathfrak{A}_S}^{\#} e^{-2\langle\rho_P, X\rangle}\Lambda_m^{T,S}\varphi_{S_E}(e^X m'k)\Gamma_S^P(2X-T, -H_{S_E}(kx))\ \tau_P(2X+H_E(kx)-T)dX,$$

and this is equal to (30).

Summing (30) over all S and P such that $S \subset P$, we see that

$$\sum_P \int_{P\backslash H(\mathbb{A})^1}^{\#} F_P(h)\ dh$$

is equal to the sum over parabolic subgroups S of

(31)
$$\sum_{\substack{P \\ P\supset S}} \int_{S\backslash H(\mathbb{A})^1}^{\#} \Lambda_m^{T,S}\varphi_{S_E}(h)\Gamma_S^P(H_{S_E}(h)-T, -H_{S_E}(K(h)x))\tau_P(H_E(hx)-T)dh$$

and (25) will follow if we prove that (31) is equal to

$$\int_{S\backslash H(\mathbb{A})^1}^{\#} \Lambda^{T,S}\varphi_{S_E}(h)\tau_S(H_E(h)-T)dh.$$

The relation (18):

$$\tau_S(Y-X) = \sum_{P\supset S}\Gamma_S^P(Y-X, -X)\tau_P(Y)$$

applied to $Y = H_{S_E}(hx) - T$ and $X = H_{S_E}(hx) - H_{S_E}(h) = H_{S_E}(K(h)x)$ gives

$$\sum_{\substack{P \\ P\supset S}} \Gamma_S^P(H_{S_E}(h)-T, -H_{S_E}(K(h)x))\ \tau_P(H_E(hx)-T) = \tau_S(H_E(h)-T).$$

Thus we need to show that the summation over P in (31) can be taken inside the #-integral.

Let $h = ne^X mk$ with $X \in \mathfrak{A}_S$ be the Iwasawa decomposition of $h \in H(\mathbb{A})$ relative to S. Then

$$\Gamma_S^P(H_{S_E}(h)-T, -H_{S_E}(K(h)x))\ \tau_P(H_E(hx)-T)$$

is equal to

$$\Gamma_S^P(2X-T, -H_{S_E}(kx))\ \tau_P(2X+H_E(kx)-T). \tag{32}$$

Since the subset $\{H_{S_E}(kx) : k \in H(\mathbb{A})\}$ of $\mathfrak{A}_S$ is compact, Lemma 2.1 of [A3] cited above implies that the function $Z \longrightarrow \Gamma_S^P(Z, -H_{S_E}(kx))$ is supported in a fixed compact set independent of k. The cone defining τ_P is the positive Weyl chamber in $\mathfrak{A}_P$ which is contained in the positive Weyl chamber of $\mathfrak{A}_S$. Thus, we may apply

Lemma 6 to take the sum over P inside the integral. This completes the proof of (i).

The proof of (ii) is nearly identical. Suppose that $T' \in \mathfrak{A}_0$ is regular. We have

$$\Lambda_m^{T+T',P}\varphi(h) = \sum_{\substack{Q\\ Q\subset P}} \sum_{\delta\in Q\backslash P} \Lambda_m^{T,Q}\varphi(\delta h)\Gamma_Q^P(H_E(\delta h)-T,T'). \tag{33}$$

Indeed, using the formula

$$\hat{\tau}_R^P(H(\delta h)-T-T') = \sum_{\substack{Q\\ R\subset Q\subset P}} (-1)^{d(Q)-d(P)}\ \hat{\tau}_R^Q(H_E(\delta h)-T)\Gamma_Q^P(H_E(\delta h)-T,T'),$$

we have

$$\begin{aligned}
\Lambda_m^{T+T',P}\varphi(h) &= \sum_{\substack{R\\ R\subset P}} (-1)^{d(R)-d(P)} \sum_{\delta\in R\backslash P} \varphi_{R_E}(\delta h)\hat{\tau}_R^P(H_E(\delta h)-T-T')\\
&= \sum_{\substack{R,Q\\ R\subset Q\subset P}} \sum_{\delta\in Q\backslash P} (-1)^{d(R)-d(Q)} \sum_{\gamma\in R\backslash Q} \varphi_{R_E}(\delta h)\hat{\tau}_R^Q(H_E(\gamma\delta h)-T)\Gamma_Q^P(H_E(\delta h)-T,T')\\
&= \sum_{\substack{Q\\ Q\subset P}} \sum_{\delta\in Q\backslash P} \Lambda_m^{T,Q}\varphi(\delta h)\Gamma_Q^P(H_E(\delta h)-T,T').
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_P \int_{P\backslash H(\mathbb{A})^1}^{\#} \Lambda_m^{T+T',P}\varphi(h)\tau_P(H_E(h)-T-T')\ dh\\
&= \sum_P \int_{P\backslash H(\mathbb{A})^1}^{\#} \sum_{\substack{Q\\ Q\subset P}} \sum_{\delta\in Q\backslash P} \Lambda_m^{T,Q}\varphi(\delta h)\Gamma_Q^P(H_E(\delta h)-T,T')\tau_P(H_E(h)-T-T')\ dh\\
&= \sum_{\substack{Q,P\\ Q\subset P}} \int_{Q\backslash H(\mathbb{A})^1}^{\#} \Lambda_m^{T,Q}\varphi(h)\Gamma_Q^P(H_E(h)-T,T')\tau_P(H_E(h)-T-T')\ dh\\
&= \sum_Q \int_{Q\backslash H(\mathbb{A})^1}^{\#} \Lambda_m^{T,Q}\varphi(g) \sum_{\substack{P\\ P\supset Q}} \Gamma_Q^P(H_E(h)-T,T')\tau_P(H_E(h)-T-T')\ dh
\end{aligned}$$

where the second equality is justified in the same way as the equality of (29) and (30) and the third equality is justified as in the discussion of (31) above. The relation

$$\sum_{\substack{P\\ P\supset Q}} \Gamma_Q^P(H_E(g)-T,T')\tau_P(H_E(g)-T-T') = \tau_Q(H_E(g)-T)$$

follows from (18), and we obtain

$$\sum_Q \int_{Q\backslash H(\mathbb{A})^1}^{\#} \Lambda_m^{T,Q}\varphi(g)\tau_Q(H(g)-T)\ dg = \int_{H\backslash H(\mathbb{A})}^{*} \varphi(g)\ dg$$

as required.

We now prove (iii). By (19), it will suffice to check that if $\varphi \in \mathcal{A}(G)^*$ is integrable over $H\backslash H(\mathbb{A})^1$, then the integral

$$\int_{P\backslash H(\mathbb{A})^1} \Lambda_m^{T,P}\varphi_{P_E}(h)\tau_P(H_E(h)-T)dh \tag{34}$$

converges absolutely and is equal to $\Pi_P^{H,T}(\varphi_P)$. Expand φ_{P_E} as in (22). As shown in [MW], top of p. 50, for all j, there exists a parabolic subgroup $Q \subset P$ and a cuspidal exponent μ of $\varphi_{Q_E}^{cusp}$ such that λ_j is equal to the restriction of μ to $\mathfrak{A}_P$ relative to the decomposition

$$\mathfrak{A}_Q = \mathfrak{A}_Q^P \oplus \mathfrak{A}_P.$$

According to Lemma I.4.11 of [MW], p. 75, if φ is *square-integrable* on $G\backslash G(\mathbb{A})^1$, then the exponent μ can be written in the form $\sum_{\alpha\in\Delta_P} x_\alpha\alpha$ with $x_\alpha < 0$. A nearly identical argument shows that this remains true if φ is assumed to be integrable over $H\backslash H(\mathbb{A})^1$. This says that μ is negative with respect to the cone $\{X \in \mathfrak{A}_P : \tau_P(X) = 1\}$. Therefore the integral $\Pi_P^{H,T}(\varphi_P)$ is absolutely convergent and coincides with the ordinary integral over $P(F)\backslash H(\mathbb{A})^1$as required. □

8. **Period of a truncated automorphic form.** Let $P = MN$ be a parabolic subgroup and $\varphi \in \mathcal{A}_{P_E}(G)$. We may generalize the construction of the previous section to define the regularized integral

$$\int^*_{P\backslash H(\mathbb{A})^1} \varphi(h)\ \tau(H_{P_E}(h)-T)dh \tag{35}$$

where τ is a function of type (C) on $\mathfrak{A}_P$. Suppose that

$$\varphi(namk) = \sum_{j=1}^r Q_j(H_E(a))\psi_j(amk)$$

as in (22) where Q_j and ψ_j are as above. Then we set (35) equal to

$$\sum_{j=1}^r \int_{\mathbf{K}_F} \Pi^{M_E/M}(\psi_j(\ \cdot\ k))dk \int^{\#}_{\mathfrak{A}_P} Q_j(2X)e^{\langle\lambda_j,2X\rangle}\tau(2X-T)\ dX$$

or

$$\int_{\mathbf{K}_F}\int^{\#}_{\mathfrak{A}_P}\left(\int^*_{M\backslash M(\mathbb{A})^1}\varphi(e^X mk)\ dm\right)e^{-2\langle\rho_P,X\rangle}\tau(2X-T)dX\ dk.$$

For $\tau = \tau_P$ this is well-defined provided that the following two conditions are satisfied:

(1*) $\langle\mu,\varpi^\vee\rangle \neq 0$ for all $Q \subset P$, $\varpi^\vee \in (\hat{\Delta}^\vee)_Q^P$, $\mu \in \mathcal{E}_Q(\varphi)$.
(2*) $\langle\lambda,\alpha^\vee\rangle \neq 0$ for all $\alpha \in \Delta_P$, and $\lambda \in \mathcal{E}_P(\varphi)$.

Let $\mathcal{A}(G)^{**}$ be the space of $\varphi \in \mathcal{A}(G)$ such that φ_{P_E} satisfies (1*) (and then also (2*)) for all P.

Theorem 10. *For* $\varphi \in \mathcal{A}(G)^{**}$,

$$\int_{H\backslash H(\mathbb{A})^1} \Lambda_m^T\varphi(h)dh$$

is equal to

$$\sum_{P\subset H}(-1)^{d(P)-d(G)}\int^*_{P\backslash H(\mathbb{A})^1} \varphi_{P_E}(h)\ \hat\tau_P(H_E(h)-T)dh.$$

Proof. By induction on the rank, we may assume that the theorem holds for pairs (M_E, M) where M is the Levi subgroup of a proper parabolic subgroup P of H. We will show below that this induction hypothesis implies that

$$\int^{\#}_{P\backslash H(\mathbb{A})^1}\Lambda^{T,P}_m\varphi_{P_E}(h)\tau_P(H_E(h)-T) \tag{36}$$

is equal to

$$\sum_{\substack{R\\ R\subset P}}(-1)^{d(R)-d(P)}\int^*_{R(F)\backslash H(\mathbb{A})^1}\varphi_{R_E}(h)\hat\tau^P_R(H_E(h)-T)\tau_P(H_E(h)-T)dh. \tag{37}$$

Assuming this, we may sum over P to write

$$\int^*_{H(F)\backslash H(\mathbb{A})^1}\varphi(h)\ dh-\int_{H(F)\backslash H(\mathbb{A})^1}\Lambda^T\varphi(h)\ dh \tag{38}$$

as

$$\sum_{\substack{P\neq G\\ P\supset R}}(-1)^{d(R)-d(P)}\int^*_{R(F)\backslash H(\mathbb{A})^1}\varphi_{R_E}(h)\hat\tau^P_R(H_E(h)-T)\tau_P(H_E(h)-T)dh. \tag{39}$$

For $R\neq H$, Langlands' Combinatorial Lemma gives

$$\sum_{\substack{P\\ R\subset P\neq H}}(-1)^{d(R)-d(P)}\hat\tau^P_R(H_E(h)-T)\tau_P(H_E(h)-T)$$
$$=-(-1)^{d(R)-d(H)}\hat\tau_R(H_E(h)-T)$$

and so the theorem will follow if we check that the summation can be taken inside the integral in (39).

Consider the three cones

$$\begin{aligned}\widehat{\mathcal{C}}_R &= \{X\in\mathfrak{A}_R : \hat\tau_R(X)=1\},\\ \mathcal{C}_P &= \{X\in\mathfrak{A}_P : \tau_P(X)=1\},\\ \widehat{\mathcal{C}}^P_R &= \{X\in\mathfrak{A}^P_R : \hat\tau^P_R(X)=1\}.\end{aligned}$$

The product $\mathcal{C}_P\times\widehat{\mathcal{C}}^P_R$ is contained in $\widehat{\mathcal{C}}_R$. Indeed, $\widehat{\mathcal{C}}_R$ is the positive span of the coroots $\{\alpha^\vee:\alpha\in\Delta_R\}$, $\mathcal{C}_P$ is the positive span of the coweights in $\hat\Delta^\vee_P$ and $\widehat{\mathcal{C}}^P_R$ is the positive span of the coroots $\{\alpha^\vee:\alpha\in\Delta^P_R\}$, so the assertion follows from the fact that all coweights in $\hat\Delta^\vee_P$ are non-negative linear combinations of coroots in $\Delta^\vee_P$. Now let $\lambda\in\mathcal{E}_R(\varphi)$ and for P containing R, write $\lambda=\lambda^P_R+\lambda_P$ relative to the decomposition $\mathfrak{A}_R=\mathfrak{A}^P_R\oplus\mathfrak{A}_P$. By our hypothesis, $\langle\lambda_P,\varpi^\vee\rangle\neq 0$ for all $\varpi^\vee\in\hat\Delta^\vee_P$ and hence λ_P is non-degenerate with respect to $\mathcal{C}_P$. Similarly, $\left\langle\lambda^P_R,\alpha^\vee\right\rangle\neq 0$ for all $\alpha\in\Delta^P_R$ and hence λ^P_R is non-degenerate with respect to $\widehat{\mathcal{C}}^P_R$. Since $\mathcal{C}_P\times\widehat{\mathcal{C}}^P_R\subset\widehat{\mathcal{C}}_R$ for all P, we may apply Lemma 6 to conclude that for any polynomial $Q(X)$,

$$-(-1)^{d(R)-d(G)}\int^{\#}_{\mathfrak{A}_R}Q(X)e^{\langle\lambda,2X\rangle}\hat\tau_R(2X-T)\ dX$$

is equal to

$$\sum_{\substack{P \\ R\subset P\neq G}} (-1)^{d(R)-d(P)} \int_{\mathfrak{A}_R}^{\#} Q(X)e^{\langle\lambda,2X\rangle}\hat{\tau}_R^P(2X-T)\tau_P(2X-T)\ dX.$$

It follows that (39) is equal to

$$-\sum_{R\neq G}(-1)^{d(R)-d(G)}\int_{R(F)\backslash G(\mathbb{A})^1}^{*}\varphi_{R_E}(g)\hat{\tau}_R(H(g)-T)dg$$

and this gives the equality of the theorem.

We now prove the equality of (36) and (37). Write the constant term φ_{P_E} as a sum

$$\varphi_{P_E}(namk)=\sum_{j=1}^{r}Q_j(H_E(a))\psi_j(amk) \tag{40}$$

for $n\in N(\mathbb{A}_E), a\in A_{P_E}$, $m\in M(\mathbb{A}_E)^1$ and $k\in\mathbf{K}_E$, where the Q_j are polynomials and $\psi_j\in\mathcal{A}_{P_E}(G)$ satisfies

$$\psi_j(ag)=e^{\langle\lambda_j+\rho_P,H_E(a)\rangle}\psi_j(g)$$

for some exponent $\lambda_j\in\mathfrak{A}_P^*$ for all $a\in A_{P_E}$. Then (36) is equal to

$$\sum_{j=1}^{r}\left(\int_{\mathbf{K}_F}\int_{M\backslash M(\mathbb{A})^1}\Lambda_m^{T,M}\psi_j(mk)dm\ dk\right)\left(\int_{\mathfrak{A}_P}^{\#}Q_j(2X)e^{\langle\lambda_j,2X\rangle}\tau_P(2X-T)dX\right)$$

where $\Lambda_m^{T,M}$ denotes the mixed truncation with respect to M. Using our induction hypothesis, we may write (36) as the sum over j and $R\subset P$ of $(-1)^{d(R)-d(P)}$ times

$$\left(\int_{\mathbf{K}_F}\int_{R_M(F)\backslash M(\mathbb{A})^1}^{*}(\psi_j)_{(M_R)_E}(mk)\hat{\tau}_R^P(H_E(m)-T)dm\ dk\right) \tag{41}$$

times

$$\left(\int_{\mathfrak{A}_P}^{\#}Q_j(2X)e^{\langle\lambda_j,2X\rangle}\tau_P(2X-T)dX\right)$$

where $R_M=R\cap M$. Choose a decomposition of type (40) for the constant term $(\psi_j)_{R_M}$:

$$(\psi_j)_{(R_M)_E}(namk)=\sum_{j=1}^{r}P_{j,\ell}(H_E(a))\psi_{j,\ell}(amk)$$

for $n\in(N_R\cap M)(\mathbb{A}_E), a\in A_R\cap M(\mathbb{A}_E)^1$, $m\in M_R(\mathbb{A}_E)^1$, $k\in\mathbf{K}_E$, where the $\psi_{j,\ell}$ satisfy

$$\psi_{j,\ell}(ag)=e^{\left\langle\lambda_{j,\ell}+\rho_R^P,H_E(a)\right\rangle}\psi_{j,\ell}(g)$$

for some exponents $\lambda_{j,\ell}\in(\mathfrak{A}_R^P)^*$. Then we may write (41) as a sum over ℓ of

$$\int_{\mathbf{K}_F}\int_{M_R(F)\backslash M_R(\mathbb{A})^1}^{*}\psi_{j,\ell}(mk)dm\ dk\int_{\mathfrak{A}_R^P}^{\#}P_{j,\ell}(2X)e^{\langle\lambda_{j,\ell},2X\rangle}\hat{\tau}_R^P(2X-T)dX.$$

By (11), we may combine the #-integrals over $\mathfrak{A}_R^P$ and $\mathfrak{A}_P$ into a single #-integral over $\mathfrak{A}_R$ and we see that

$$\int_{\mathbf{K}_F}\int^*_{R_M(F)\backslash M(\mathbb{A})^1}(\psi_j)_{(M_R)_E}(mk)\hat{\tau}_R^P(H_E(m)-T)dm\ dk \\ \times\left(\int^{\#}_{\mathfrak{A}_P}Q_j(2X)e^{\langle\lambda_j,2X\rangle}\tau_P(2X-T)dX\right)$$

is equal to the sum over ℓ of

$$\int_{\mathbf{K}_F}\int^*_{M_R(F)\backslash M_R(\mathbb{A})^1}\psi_{j,\ell}(mk)\ dm\ dk \\ \times\int^{\#}_{\mathfrak{A}_R}P_{j,\ell}(2X)Q_j(2X)e^{\langle\mu_{j,\ell},2X\rangle}\hat{\tau}_R^P(2X-T)\tau_P(2X-T)\ dX$$

where $\mu_{j,\ell}=\lambda_j+\lambda_{j,\ell}$, and this equals

$$\int^*_{R(F)\backslash H(\mathbb{A})^1}Q_j(H_{P_E}(g))(\psi_j)_{R_M}(g)\hat{\tau}_R^P(H_E(g)-T)\tau_P(H_E(g)-T)dg.$$

Summing over j gives

$$\int^*_{R(F)\backslash H(\mathbb{A})^1}\varphi_R(g)\hat{\tau}_R^P(H_E(g)-T)\tau_P(H_E(g)-T)dg.$$

This shows that (36) is equal to

$$\sum_{R\subset P}(-1)^{d(R)-d(P)}\int^*_{R(F)\backslash H(\mathbb{A})^1}\varphi_R(g)\hat{\tau}_R^P(H_E(g)-T)\tau_P(H_E(g)-T)dg,$$

as required. □

Corollary 11. *For all $\varphi\in\mathcal{A}(G)^{**}$, the period integral*

$$\int_{H(F)\backslash H(\mathbb{A})^1}\Lambda_m^T\varphi(h)\ dh$$

is an exponential polynomial function of the truncation parameter T whose exponents are contained in those of φ.

Proof. According to the definition of

$$\int^*_{P\backslash H(\mathbb{A})^1}\varphi_{P_E}(h)\ \hat{\tau}_P(H_E(h)-T)dh,$$

the only dependence on T is through #-integrals of the form

$$\int^{\#}_{\mathfrak{A}_P}Q_j(2X)e^{\langle\lambda_j,2X\rangle}\hat{\tau}_P(2X-T)\ dX.$$

The assertion now follows from Lemma 5. □

In fact, one can prove the corollary, without any restrictions on φ by using the relation (33).

9. **Eisenstein series.** Our next goal is to verify that Theorem 10 applies to cuspidal Eisenstein series. To that end, we fix some notation and definitions that will be used in the rest of the article. We work with a reductive group G over a field F.

Let $P = MN$ be a proper parabolic subgroup and let σ be an automorphic subrepresentation of $L^2(M\backslash M(\mathbb{A})^1)$. Let $\mathcal{A}_P(G)_\sigma$ be the subspace of functions $\varphi \in \mathcal{A}_P(G)$ such that φ is left-invariant under A_P and for all $k \in \mathbf{K}$, the function $m \longmapsto \varphi(mk)$ belongs to the space of σ. For $\varphi \in \mathcal{A}_P(G)_\sigma$ and $\lambda \in \mathfrak{A}_P^*$ we write $E(g, \varphi, \lambda)$ for the Eisenstein series which is given, in its domain of absolute convergence, by the infinite series

$$E(g, \varphi, \lambda) = \sum_{\gamma \in P\backslash G} \varphi(\gamma g)\ e^{\langle \lambda + \rho_P, H_P(\gamma g)\rangle}.$$

Let $N_{G(F)}(A_0)$ be the normalizer of A_0 in $G(F)$ and let

$$\Omega = N_{G(F)}(A_0)/C_{G(F)}(A_0)$$

be the Weyl group of G. Recall that a parabolic subgroup Q is said to be *associate* to P if M_Q is conjugate to M_P under Ω. If Q is associate to P, let $\Omega(P, Q)$ be the set of maps $A_P \longrightarrow A_Q$ obtained by restriction of elements $w \in \Omega$ such that $wM_P w^{-1} = M_Q$.

Suppose that Q is associate to P and let w be an element of $\Omega(P, Q)$ with representative $\widetilde{w}$ in $N_{G(F)}(A_0)$. We define the standard intertwining operator

$$M(w, \lambda)\varphi(g) = e^{-\langle w\lambda + \rho_Q, H_Q(g)\rangle} \int_{N_w(\mathbb{A})\backslash N_Q(\mathbb{A})} \varphi(\widetilde{w}^{-1} ng) e^{\left\langle \lambda + \rho_P, H_P(\widetilde{w}^{-1} ng)\right\rangle}\, dn$$

where $N_w = N_Q \cap \widetilde{w} N \widetilde{w}^{-1}$. The operator $M(w, \lambda)$ depends on w but not on the choice of representative $\widetilde{w}$.

Assume that σ is cuspidal. Then the constant term $E_Q(g, \varphi, \lambda)$ relative to a parabolic subgroup Q has a simple expression. If Q does not contain an associate of P, then $E_Q(g, \varphi, \lambda)$ is identically zero. If Q is associate to P, then

$$E_Q(g, \varphi, \lambda) = \sum_{w \in \Omega(P,Q)} M(w, \lambda)\varphi(g) e^{\left\langle w\lambda + \rho_Q, H_Q(g)\right\rangle}.$$

On the other hand, if Q properly contains an associate of P, then [A2]

$$E_Q(g, \varphi, \lambda) = \sum_{Q'} \sum_{\substack{w \in \Omega(P,Q') \\ w^{-1}\alpha > 0 \text{ for } \alpha \in \Delta^Q_{Q'}}} E^Q(g, M(w, \lambda)\varphi, w\lambda) \tag{42}$$

where the sum is over the standard parabolic subgroups $Q' \subset Q$ associate to P and $E^Q(g, \psi, \lambda)$ denotes an Eisenstein series induced from $M_{Q'}$ to M_Q :

$$E^Q(g, \psi, \lambda) = \sum_{\gamma \in Q'\backslash Q} \psi(\gamma g)\ e^{\left\langle \lambda + \rho_{Q'}, H_{Q'}(\gamma g)\right\rangle}.$$

We now return to the situation where $G = H_{/E}$ where E/F is quadratic as above. Consider a cuspidal Eisenstein series $E(g, \varphi, \lambda)$ on G. We shall check that $E(g, \varphi, \lambda)$ belongs to $\mathcal{A}(G)^{**}$ for generic values of the parameter λ. For each Q containing an associate of P, let $\mathcal{O}$ be the open set where E^Q and the intertwining operators are regular. This is a complement of hyperplanes. In $\mathcal{O}$, the exponents of the Eisenstein series along a parabolic subgroup Q are the restrictions to $\mathfrak{A}_Q$ of the linear forms $w\lambda$ with P' associate to P and contained in Q and $w \in \Omega(P, P')$

with $w^{-1}\Delta_{P'}^{Q} > 0$. Thus for $\lambda \in \mathcal{O}$ the Eisenstein series belongs to $\mathcal{A}(G)^{**}$ if for every such Q and such w, and all $\varpi^\vee \in (\hat{\Delta}^\vee)_Q^{Q'}$ with $Q \subset Q'$ we have

$$\langle w\lambda, \varpi^\vee \rangle \neq 0.$$

If we denote this set of such λ by $\mathcal{O}_0$, we see that the Eisenstein series $E(g, \varphi, \lambda)$ belongs to the space $\mathcal{A}(G)^{**}$ for all $\lambda \in \mathcal{O}_0$.

Proposition 12. *Let $E(g, \varphi, \lambda)$ be a cuspidal Eisenstein series. Then the period integral of a cuspidal Eisenstein series $\Pi^{G/H}(E(\varphi, \lambda))$ is a meromorphic function of λ with hyperplane singularities. It is holomorphic on $\mathcal{O}_0$.*

Proof. Assume that $E(\varphi, \lambda)$ is induced from $P_E = M_E N_E$. It will suffice to check the claim for each of the integrals

$$\int_{Q\backslash H(\mathbb{A})^1}^{*} \Lambda_m^{T,Q} E_Q(h, \varphi, \lambda)\ \tau_Q(H_E(h) - T)\ dh.$$

By (42), each of these integrals is a sum of terms of the form

$$\int_{Q\backslash H(\mathbb{A})^1}^{*} \Lambda_m^{T,Q} E^Q(h, M(w, \lambda)\varphi, w\lambda)\ \tau_Q(H_E(h) - T)\ dh,$$

which itself can be written as a product of

$$\int_{\mathfrak{A}_Q}^{\#} e^{\langle w\lambda, 2X\rangle} \tau_Q(2X - T) dX$$

and

$$\int_{\mathbf{K}_F \times M_Q\backslash M_Q(\mathbb{A})^1} \Lambda_m^{T,Q} E^Q(mk, M(w, \lambda)\varphi, \lambda)\ dm\ dk. \tag{43}$$

The first factor depends only on the projection $(w\lambda)_Q$ of $w\lambda$ to $\mathfrak{A}_Q$. It can be evaluated explicitly and is clearly meromorphic and holomorphic in $\mathcal{O}_0$. The second factor depends only on the projection $(w\lambda)_0^Q$ of $w\lambda$ to $\mathfrak{A}_0^Q$. The integrand is defined and varies analytically in $\mathcal{O}_0$. According to [MW], Lemma I.2.16, the rate of rapid decrease of Arthur's truncation $\Lambda^T\psi$ is majorized in terms of the rate of slow increase of finitely many derivatives of ψ and hence in terms of the exponents of finitely many derivatives of ψ. The proof applies with little change to the mixed truncation and thus, similarly, the rate of rapid decrease of $\Lambda_m^{T,Q}\psi$ is similarly majorized in terms of the exponents of finitely many derivatives of ψ. In our case, these exponents vary analytically and we may therefore conclude that the integral (43) is uniformly convergent for λ in a compact subset of $\mathcal{O}_0$. Hence (43) is analytic in $\mathcal{O}_0$. ∎

Remark 1. It is not true in general that $\Pi^{G/H}(E(\varphi, \lambda))$ is analytic whenever $E(\varphi, \lambda)$ is analytic. For example, the computations in §20 show that for the Eisenstein series on $GL(2)$ induced from the trivial character and $\varphi \equiv 1$ we have $\Pi^{G/H}(E(\varphi, \lambda)) = \zeta_F(\lambda)/L(\lambda + 1, \omega_{E/F})$ up to a volume factor. The regularized period thus has a pole at $\lambda = 0$. However $E(\varphi, 0) \equiv 0$ by the functional equation.

V. Integral of an automorphic form

In this section, we describe how the considerations of the previous section can be carried over to the case that $H = G$. We omit most of the proofs since they are nearly identical, word for word, as those presented in the previous section.

Let P be a parabolic subgroup of G and let $f \in \mathcal{A}_P(G)$. Then there exists a finite set of distinct exponents $\mathcal{E}_P(f) = \{\lambda_1, ..., \lambda_k\}$ in $\mathfrak{A}_P^*$ such that

$$f(namk) = \sum_{j=1}^{k} \phi_j(mk)\alpha_j(H(a))e^{\langle \lambda_j+\rho_P, H(a)\rangle} \tag{44}$$

for $n \in N(\mathbb{A})$, $a \in A_P$, $m \in M(\mathbb{A})^1$ and $k \in \mathbf{K}$, where for all j, $\alpha_j(X)$ is a polynomial, and $\phi_j(g)$ is an automorphic form in $\mathcal{A}_P(G)$ such that $\phi_j(ag) = \phi_j(g)$ for $a \in A_P$.

The function τ_P is the characteristic function of the cone spanned by the coweights in $\hat{\Delta}_P^\vee$. It follows that if

$$\langle \lambda_j - \rho_P, \varpi^\vee\rangle \neq 0$$

for all $\varpi^\vee \in \hat{\Delta}_P^\vee$ and $\lambda_j \in \mathcal{E}_P(f)$, then the integrals

$$\int_{\mathfrak{A}_P}^{\#} \alpha_j(X)e^{\langle\lambda_j-\rho_P,X\rangle}\tau_P(X-T)dX$$

are defined. Assuming this condition holds, we define the #-integral

$$\int_{P\backslash G(\mathbb{A})^1}^{\#} \Lambda^{T,P}f(g)\ \tau_P(H(g)-T)dg$$

by the formula

$$\sum_{j=1}^{k}\int_{\mathbf{K}}\left(\int_{M\backslash M(\mathbb{A})^1}\Lambda^{T,P}\phi_j(mk)\ dm\right)\left(\int_{\mathfrak{A}_P}^{*}\alpha_j(X)e^{\langle\lambda_j-\rho_P,X\rangle}\tau_P(X-T)dX\right)\ dk.$$

We also denote this expression by $I_P^T(f)$. It coincides with the ordinary integral if $\Lambda^{T,P}f(g)\ \tau_P(H(g)-T)$ is integrable over $P\backslash G(\mathbb{A})^1$.

As before, for $\varphi \in \mathcal{A}(G)$, we write $\mathcal{E}_P(\varphi)$ for $\mathcal{E}_P(\varphi_P)$. Let

$$\mathcal{A}(G)' = \{\varphi \in \mathcal{A}(G) : \langle\lambda-\rho_P,\varpi^\vee\rangle \neq 0 \text{ for all } \varpi^\vee \in \hat{\Delta}_P^\vee, \lambda \in \mathcal{E}_P(\varphi), P \neq G\}.$$

For $\varphi \in \mathcal{A}(G)'$, we define the regularized integral

$$\int_{G\backslash G(\mathbb{A})^1}^{*}\varphi(g)dg = \sum_{P\subset G} I_P^T(\varphi_P).$$

We also denote this integral by $I_G^1(\varphi)$. For $x \in G(\mathbb{A}_f)$, let $\rho(x)$ denote right translation by $x : \rho(x)\varphi(g) = \varphi(gx)$.

Theorem 13. (i) *I_G^1 is independent of the choice of T.*

(ii) *The space $\mathcal{A}(G)'$ is invariant under right translation by $G(\mathbb{A}_f)^1$ and I_G^1 defines an invariant linear functional on $\mathcal{A}(G)'$. Explicitly, $I_G^1(\rho(x)\varphi) = I_G^1(\varphi)$ for all $x \in G(\mathbb{A}_f)$.*

(iii) *If $\varphi \in \mathcal{A}(G)$ is integrable over $G\backslash G(\mathbb{A})^1$, then $\varphi \in \mathcal{A}(G)'$ and $I_G^1(\varphi)$ coincides with the ordinary integral of φ over $G\backslash G(\mathbb{A})^1$.*

Proof. The proof is nearly identical to that of Theorem 9 and therefore we omit it. However, we merely note a slight modification required for part (iii). As in the prove of part (iii) of Theorem 9, it is necessary to check that if $\varphi \in \mathcal{A}(G)$ is integrable over $G\backslash G(\mathbb{A})^1$, then the integral

$$\int_{P\backslash G(\mathbb{A})^1} \Lambda^{T,P}\varphi_P(g)\tau_P(H(g)-T)dg \tag{45}$$

converges absolutely. Expand φ_P as in (44). As shown in [MW], top of p. 50, for all j, there exists a parabolic subgroup $Q \subset P$ and a cuspidal exponent μ of φ_Q^{cusp} such that λ_j is equal to the restriction of μ to $\mathfrak{A}_P$ relative to the decomposition

$$\mathfrak{A}_Q = \mathfrak{A}_Q^P \oplus \mathfrak{A}_P.$$

According to Lemma I.4.11 of [MW], p. 75, if φ is *square-integrable*, the exponent μ can be written in the from $\sum_{\alpha\in\Delta_Q} x_\alpha\alpha$ with $x_\alpha < 0$. This time, we modify the argument to show that if φ is integrable, then

$$\mu - \rho_Q = \sum_{\alpha\in\Delta_Q} x_\alpha\alpha$$

with $x_\alpha < 0$. The restriction of ρ_Q to $\mathfrak{A}_P$ is ρ_P and the set of non-zero restrictions of elements in Δ_Q to $\mathfrak{A}_P$ coincides with Δ_P. It follows that

$$\lambda_j - \rho_P = \sum_{\beta\in\Delta_P} y_\beta\beta$$

with $y_\beta < 0$ and hence

$$\int_{\mathfrak{A}_P} Q_j(X)e^{\langle\lambda_j-\rho_P,X\rangle}\tau_P(X-T)dX$$

converges absolutely. This is all that is required in order that the integral (45) be absolutely convergent. □

10. **Integral of a truncated automorphic form.** We can now define a certain regularized integral for automorphic functions $f \in \mathcal{A}_P(G)$. Assume that f has a decomposition as in (44) and define

$$\int^*_{P\backslash G(\mathbb{A})^1} f(g)\hat{\tau}_P(H(g)-T)dg \tag{46}$$

by

$$\sum_{j=1}^k \int_{\mathbf{K}}\int^*_{M\backslash M(\mathbb{A})^1} \phi_j(mk)\ dmdk\ \left(\int^{\#}_{\mathfrak{A}_P} \alpha_j(X)e^{-\langle\rho_P,X\rangle}\hat{\tau}_P(X-T)dX\right).$$

This is well-defined provided that the following two conditions are satisfied:

(1) $\langle\lambda-\rho_Q,\varpi^\vee\rangle \neq 0$ for all $Q \subset P$, $\varpi^\vee \in (\hat{\Delta}^\vee)_Q^P, \lambda \in \mathcal{E}_Q(\varphi)$.
(2) $\langle\lambda-\rho_P,\alpha^\vee\rangle \neq 0$ for all $\alpha \in \Delta_P$, and $\lambda \in \mathcal{E}_P(\varphi)$.

Let $\mathcal{A}(G)''$ be the space of $\varphi \in \mathcal{A}(G)$ such that (1) (and then also (2)) is satisfied for any P. The following Theorem and Corollary follow by the same proofs as those of Theorem 10 and its Corollary.

Theorem 14. *For all* $\varphi \in \mathcal{A}(G)''$,

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T \varphi(g)\ dg$$

is equal to

$$\sum_{P \subset G} (-1)^{d(P)-d(G)} \int^*_{P(F)\backslash G(\mathbb{A})^1} \varphi_P(g) \hat{\tau}_P(H(g)-T) dg.$$

Corollary 15. *For all* $\varphi \in \mathcal{A}(G)''$, *the integral*

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T \varphi(g)\ dg$$

is an exponential polynomial function of the truncation parameter T.

Again, the corollary can be proved directly and it applies to any $\varphi \in \mathcal{A}(G)$.

11. **Integral of a truncated Eisenstein series.** Let $E(g,\varphi,\lambda)$ be a cuspidal Eisenstein series. As in the proof of Proposition 12, one shows that $I^1_G(E(g,\varphi,\lambda))$ is defined for generic values of the parameter λ and varies analytically in λ.

Lemma 16 (Bernstein's principle)**.** *Let* $P = MN$ *be a proper parabolic subgroup and let* σ *be an irreducible cuspidal representation in* $L^2(M(F)\backslash M(\mathbb{A})^1)$. *Let* $E(g,\varphi,\lambda)$ *be an Eisenstein series where* $\varphi \in \mathcal{A}_P(G)_\sigma$. *Then*

$$I^1_G(E(g,\varphi,\lambda)) = 0$$

for all λ *such that* $E(g,\varphi,\lambda)$ *and* $I^1_G(E(g,\varphi,\lambda))$ *are defined.*

Proof. Suppose that $E(g,\varphi,\lambda)$ and its regularized integral are defined at λ_0. Then there exists an open set $\mathcal{O}$ containing λ_0 such that $I^1_G(E(g,\varphi,\lambda))$ is defined for all $\lambda \in \mathcal{O}$. The map $\varphi \longmapsto I^1_G(E(g,\varphi,\lambda))$ therefore defines a $G(\mathbb{A}_f)^1$-invariant functional on $\mathrm{Ind}^G_P(\sigma \otimes e^\lambda)$ for $\lambda \in \mathcal{O}$. Since there does not exist any such invariant functional for generic values of λ, the function $I^1_G(E(g,\varphi,\lambda))$ must vanish identically. □

Corollary 17. *Let* $P = MN$ *be a parabolic subgroup and let* (M,σ) *be a cuspidal datum. Let* $E(g,\varphi,\lambda)$ *be an Eisenstein series where* $\varphi \in \mathcal{A}_P(G)_\sigma$. *Then*

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T E(g,\varphi,\lambda)\ dg$$

is equal to zero if P *is not a minimal parabolic. If* P *is minimal, then it is equal to*

$$v \sum_{w \in \Omega} \frac{e^{\langle w\lambda - \rho, T\rangle}}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee\rangle} \int_{M\backslash M^1(\mathbb{A}) \times \mathbf{K}} M(w,\lambda)\varphi(mk) dm\ dk$$

where $v = \mathrm{vol}(\{\sum_{\alpha \in \Delta_0} a_\alpha \alpha^\vee : 0 \le a_\alpha < 1\})$.

Proof. For $\lambda \in \mathfrak{A}^*_P$ generic, the integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T E(g,\varphi,\lambda)\ dg$$

is equal to

$$\sum_Q (-1)^{d(Q)-d(G)} \int^*_{Q(F)\backslash G(\mathbb{A})^1} E_Q(g,\varphi,\lambda)\ \hat{\tau}_Q(H(g)-T) dg.$$

Since φ is cuspidal, $E_Q(g,\varphi,\lambda)$ vanishes identically unless Q contains an associate of P. As a function of m, $E_Q(namk,\varphi,\lambda)$ is a sum of Eisenstein series induced from parabolic subgroups associate to P to M_Q. Therefore, by the previous lemma and its defining formula,

$$\int^*_{Q(F)\backslash G(\mathbb{A})^1} E_Q(g,\varphi,\lambda)\ \hat{\tau}_Q(H(g)-T)dg$$

vanishes unless Q is associate to P. If Q is associate to P, then E_Q is a sum of cusp forms on $M_Q(\mathbb{A})$, and the integral of a cusp form over $M_Q(F)\backslash M_Q(\mathbb{A})^1$ is zero unless Q is minimal. In this case, $Q=P$ and the only non-zero term is

$$(-1)^{d(P)-d(G)}\int_{P(F)\backslash G(\mathbb{A})^1} E_P(g,\varphi,\lambda)\ \hat{\tau}_P(H(g)-T)dg.$$

This is equal to

$$(-1)^{d(P)-d(G)}\sum_{w\in\Omega}\left(\int^{\#}_{\mathfrak{A}_P} e^{\langle w\lambda-\rho_P,H\rangle}\hat{\tau}_P(H-T)\ dH\right)$$
$$\times\left(\int_{M\backslash M^1(\mathbb{A})\times\mathbf{K}} M(w,\lambda)\varphi(mk)dm\ dk\right).$$

The lemma follows from the formula

$$\left((-1)^{d(P)-d(G)}\int^{\#}_{\mathfrak{A}_P} e^{\langle w\lambda-\rho_P,H\rangle}\hat{\tau}_P(H-T)\ dH\right)=v\frac{e^{\langle w\lambda-\rho_P,T\rangle}}{\prod_{\alpha\in\Delta_P}\langle w\lambda-\rho_P,\alpha^\vee\rangle}.$$

□

12. **General inner product formula.** The formula in Theorem 14 can be extended to a formula for the inner product

$$\int_{G(F)\backslash G(\mathbb{A})^1}\Lambda^T\psi(g)\ \Lambda^T\varphi(g)\ dg \tag{47}$$

where $\psi,\varphi\in\mathcal{A}(G)$. Recall that Λ^T is a self-adjoint projection [A2] and therefore (47) is equal to

$$\int_{G(F)\backslash G(\mathbb{A})^1}\psi(g)\ \Lambda^T\varphi(g)\ dg.$$

Define a bilinear form on $\mathcal{A}(G)$ by

$$B_G(\psi,\varphi)=\sum_{P\subset G}\int^{\#}_{P\backslash G(\mathbb{A})^1}\psi_P(g)\ \Lambda^{T,P}\varphi_P(g)\ \tau_P(H(g)-T)\ dg.$$

The #-integral is defined, as before, as a sum of products of absolutely convergent integrals over $M(\mathbb{A})^1\times\mathbf{K}$ and #-integrals over $\mathfrak{A}_P$. This reduces to the usual inner product if either φ or ψ is cuspidal. In general, $B_G(\psi,\varphi)$ is well defined if, for all P, $\lambda\in\mathcal{E}_P(\varphi)$, and $\mu\in\mathcal{E}_P(\psi)$, we have $\langle\lambda+\mu,\varpi^\vee\rangle\neq 0$ for all $\varpi^\vee\in\hat{\Delta}_P^\vee$. One shows, as in Section IV that B_G is invariant under $G(\mathbb{A}_f)$ and independent of T.

We use B_G to define a regularized inner product

$$\int^*_{P(F)\backslash G(\mathbb{A})^1}\psi(g)\ \varphi(g)\ \hat{\tau}_P(H(g)-T)\ dg \tag{48}$$

for $\varphi, \psi \in \mathcal{A}_P(G)$. Suppose that

$$\varphi(namk) = \sum_i Q_i(H_P(a))e^{\langle \lambda_i + \rho_P, H_P(a)\rangle}\varphi_i(mk)$$

and

$$\psi(namk) = \sum_j R_j(H_P(a))e^{\langle \mu_j + \rho_P, H_P(a)\rangle}\psi_j(mk)$$

in the usual notation. Then we set (48) equal to

$$\sum_{i,j}\left(\int_{\mathbf{K}} B^1_{M_P}(\psi_j(\bullet k), \varphi_i(\bullet k))dk\right)\left(\int^{\#}_{\mathfrak{A}_P} Q_i(X)R_j(X)e^{\langle \lambda_i+\mu_j, X\rangle}\hat{\tau}_P(X-T)dX\right).$$

This is well-defined provided that

(1′) For all $Q \subset P$, $\lambda' \in \mathcal{E}_Q(\varphi)$, and $\mu' \in \mathcal{E}_Q(\psi)$, we have $\langle \lambda' + \mu', \varpi^\vee\rangle \neq 0$ for all $\varpi^\vee \in (\hat{\Delta}^\vee)^P_Q$.

(2′) $\langle \lambda + \mu, \alpha^\vee\rangle \neq 0$ for all $\alpha \in \Delta_P$.

Again, by arguments parallel to those in the proof of Theorem 10, we obtain the following

Theorem 18. *Let $\psi, \varphi \in \mathcal{A}(G)$. Suppose that ψ_P and φ_P satisfy condition (1′) (and then also (2′)) for all P. Then the integral*

$$\int^*_{P(F)\backslash G(\mathbb{A})^1} \psi_P(g)\varphi_P(g)\hat{\tau}_P(H(g)-T)dg$$

is well-defined for all P and

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T\psi(g)\ \Lambda^T\varphi(g)\ dg$$

is equal to

$$\sum_{P\subset G}(-1)^{d(P)-d(G)}\int^*_{P(F)\backslash G(\mathbb{A})^1} \psi_P(g)\varphi_P(g)\hat{\tau}_P(H(g)-T)\ dg.$$

This theorem may be regarded as a generalization of the Langlands inner product formula. When applied to cuspidal Eisenstein series $E(g, \varphi, \lambda)$ and $E(g, \psi, \lambda)$ (induced from parabolics P and P') we obtain the formula

$$\begin{aligned}(49)\qquad &\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T E(g,\psi,\lambda)\Lambda^T E(g,\varphi,\mu)\ dg \\ &= \sum_Q (-1)^{d(Q)-d(G)}\int^*_{Q(F)\backslash G(\mathbb{A})^1} E_Q(g,\psi,\lambda)E_Q(g,\varphi,\mu)\hat{\tau}_Q(H(g)-T)dg.\end{aligned}$$

As noted before, the constant term $E_Q(g, \psi, \lambda)$ vanishes unless Q contains an associate of P'. On the other hand, if Q properly contains an associate of P or P', then

$$\int^*_{Q(F)\backslash G(\mathbb{A})^1} E_Q(g,\psi,\lambda)E_Q(g,\varphi,\mu)\hat{\tau}_Q(H(g)-T)dg = 0$$

by formula (42) and a variant of Bernstein's principle, viz., non-existence of an invariant pairing, either locally or globally, between $\mathrm{Ind}^Q_P(\sigma'\otimes e^\lambda)$ and $\mathrm{Ind}^Q_{P'}(\tau'\otimes e^\mu)$ for generic λ, $\mu \in \mathfrak{A}_{P'}$ and any representations σ', τ' of $M_{P'}$. Therefore (49) reduces to a sum over parabolic subgroups Q associate to P. Direct calculation of the $*$-integral using formula (42) for the constant terms $E_Q(g,\psi,\lambda)$ immediately yields the Langlands inner product formula [A2], Lemma 4.2.

VI. Regularized periods of a cuspidal Eisenstein series on $GL(n)$

For the rest of the paper, $H = GL(n)_{/F}$ and $G = GL(n)_{/E}$. Our goal in this section is to evaluate

$$\int_{H(F)\backslash H(\mathbb{A})^1} \Lambda_m^T E(h,\varphi,\lambda)\ dh$$

for the case of cuspidal Eisenstein series.

13. **Weyl groups and double cosets.** We take this opportunity to fix some notation that will be used from now on. Let $B = TU$ be the standard Borel subgroup of upper-triangular matrices of H, T the diagonal subgroup, and U the unipotent radical of B. We identify the Weyl groups $N_H(T)/T$ and $N_G(T_E)/T_E$ of H and G, respectively and set $\Omega = N_H(T)/T = N_G(T_E)/T_E$ with length function ℓ. The group Ω is isomorphic to the symmetric group S_n and is naturally identified with the group of permutation matrices in H or G (matrices whose entries are 0 or 1, with a single 1 in each row and column). For any (standard) parabolic subgroup $P = MN$ of H, we let $\Omega_M = N_M(T)/T$ be the Weyl group of the (standard) Levi factor M. In this case, M consists of block diagonal matrices with blocks of size $n_1, ..., n_r$ for some partition $n = n_1 + \cdots + n_r$ and is thus isomorphic to a product $GL(n_1) \times \cdots \times GL(n_r)$.

Let $\Omega(M)$ be the set of elements $w \in \Omega$ such that $M' = wMw^{-1}$ is again a standard Levi subgroup and w is of minimal length in the class $w\Omega_M$. This latter condition is satisfied if and only if $w\Delta_0^M = \Delta_0^{M'}$. Explicitly, an element $w \in \Omega(M)$ is represented by a unique permutation matrix that shuffles the diagonal blocks of M without causing any internal change within each block. The permutation matrix itself is built out of blocks of size $n_j \times n_j$ (not necessarily along the diagonal) which are either the identity matrix or the zero matrix.

If M' is another standard Levi subgroup, let

$$\Omega(M, M') = \{w \in \Omega(M) :\ wMw^{-1} = M'\}.$$

We also set

$$\Omega^M = \{w \in \Omega : wMw^{-1} = M\}.$$

Then $\Omega(M, M)$ is a subgroup of Ω^M and we have

$$\Omega^M = \Omega(M, M) \ltimes \Omega_M. \tag{50}$$

Let $\Omega_2 = \{\xi \in \Omega : \xi^2 = e\}$ and let

$$\Omega_2(M, M) = \{w \in \Omega(M, M) : w^2 = e\}$$

be the set of involutions in $\Omega(M, M)$. If $\xi \in \Omega_2(M, M)$, we denote the ± 1 eigenspaces of ξ in $\mathfrak{A}_P$ and $\mathfrak{A}_P^*$ by $(\mathfrak{A}_P)_\xi^\pm$ and $(\mathfrak{A}_P^*)_\xi^\pm$, respectively. Let $P_\xi^\pm$ be the projections onto $(\mathfrak{A}_P^*)_\xi^\pm$.

Lemma 19. 1. *In any double coset κ of $B(E)\backslash G(E)/H(F)$ there exists a representative η so that $\eta\bar{\eta}^{-1} \in N_G(T_E)$ and its image in Ω depends only on κ.*

2. *The map*

$$B(E)\backslash G(E)/H(F) \to \Omega \tag{51}$$

defined above is injective and its image is Ω_2.

3. *Let* $\xi \in \Omega_2$. *There exists a constant* C *such that*

$$\left\langle P_\xi^- \varpi, H_E(\eta) \right\rangle \leq C \tag{52}$$

for all $\varpi \in \widehat{\Delta}_0$ *and all* $\eta \in G(E)$ *such that* $\eta\bar{\eta}^{-1}$ *represents* ξ *(i.e.,* $\eta\bar{\eta}^{-1} \in \xi T_E$*).*

Proof. Part 1 is proved in [S], §4 (for the split case, but the proof carries over verbatim). Furthermore, [S] reduces part 2 to the following two statements, which follow from Hilbert's Theorem 90:

(a) Every $\xi \in \Omega_2$ can be represented by $g\bar{g}^{-1}$ for some $g \in G(E)$.
(b) If $\xi \in \Omega_2$, then the action of $\mathbb{Z}/(2)$ on T_E by the involution $t \to \xi\bar{t}\xi^{-1}$ has trivial first cohomology.

To prove part 3, we observe that $\eta = \xi\bar{\eta}$ and hence if $\eta = ank$, then

$$H_E(\eta) = H_E(\xi\bar{\eta}) = \xi H_E(\bar{\eta}) + H_E(\xi\overline{n}).$$

We have $H_E(\bar{\eta}) = H_E(\eta)$ and thus $P_\xi^- H_E(\eta) = \frac{1}{2} H_E(\xi n)$. The result follows from the fact that $\langle \varpi, H_E(\xi n)\rangle$ is bounded for all $\varpi \in \hat{\Delta}_0$. □

We now describe $P(E)\backslash G(E)/H(F)$ for general P. For any such double coset, take a representative η so that $\eta\bar{\eta}^{-1}$ represents $w \in \Omega_2$. Map w to its reduced representative in $\Omega_M\backslash\Omega/\Omega_M$. Denote by ι_P the resulting map from $P(E)\backslash G(E)/H(F)$ to $\Omega_M\backslash\Omega/\Omega_M$. We identify $\Omega_M\backslash\Omega/\Omega_M$ with ${}^M\Omega^M$ – the set of left and right Ω_M-reduced elements in Ω. Let ${}^M\Omega_2^M$ be the set of involutions in ${}^M\Omega^M$. This corresponds to the double cosets in $\Omega_M\backslash\Omega/\Omega_M$ which contain an involution. Alternatively, these are double cosets D which are self-inverse (i.e. $D^{-1} = D$). Indeed, if ξ is the reduced representative in $\Omega_M w \Omega_M$, then ξ^{-1} is the reduced representative of $\Omega_M w^{-1}\Omega_M$.

Proposition 20. *ι_P is a bijection between $P(E)\backslash G(E)/H(F)$ and ${}^M\Omega_2^M$.*

Proof. Choosing another η in the double coset changes $\eta\bar{\eta}^{-1}$ to $p\eta\bar{\eta}^{-1}\bar{p}^{-1}$ for some $p \in P(E)$. Since $P(E)\backslash G(E)/P(E) \longleftrightarrow \Omega_M\backslash\Omega/\Omega_M$, ι_P is well defined. Moreover, it is clear that the image of ι_P consists of the cosets of $\Omega_M\backslash\Omega/\Omega_M$ which contain an involution. Suppose that PgH and $Pg'H$ have the same image under ι_P. Let $\xi, \xi' \in \Omega_2$ be the image of BgH and $Bg'H$ respectively under ι_B. By assumption $\xi' \in \Omega_M\xi\Omega_M$. We first prove that ξ' can be obtained from ξ by successively performing two kinds of operations: conjugation by an element of Ω_M and multiplying by a simple reflection inside M which commutes with the involution. We may suppose, to begin with, that $\xi' \in {}^M\Omega_2^M$. If $\xi \neq \xi'$, then ξ is not reduced and there exists $\alpha \in \Delta^P$ so that $\xi\alpha < 0$, that is, $\ell(\xi s_\alpha) = \ell(\xi) - 1$. If $\xi\alpha \neq -\alpha$, then $(\xi s_\alpha)^{-1}\alpha < 0$ and hence $\ell(s_\alpha \xi s_\alpha) = \ell(\xi) - 2$. If $\xi\alpha = -\alpha$, then $\xi^{-1}s_\alpha\xi = s_{\xi\alpha} = s_\alpha$ and $\xi = (\xi s_\alpha)s_\alpha$. In any case, we can reduce the length of ξ by the operations described above. Next, we show that these operations can be realized by $\xi \mapsto p\xi\bar{p}^{-1}$ for $p \in P(E)$. In other words, if $\eta\bar{\eta}^{-1}$ represents $\xi \in \Omega_2$ and $\xi'' \in \Omega_2$ is obtained by one of the operations above, then ξ'' can be represented by $p\eta\bar{\eta}^{-1}\bar{p}^{-1}$. This is evident for the first operation. For the second one, suppose that s_α commutes with ξ with $\alpha \in \Delta_0^M$. Then $\xi\alpha = \pm\alpha$ and (possibly after interchanging ξ and $\xi'' = s_\alpha\xi$) we may assume that $\xi\alpha = \alpha$. We can also assume that $\eta\bar{\eta}^{-1}$ is actually equal to the permutation matrix of ξ and hence centralizes the rank 1 subgroup M_α corresponding to the root α. Choose $\eta_\alpha \in M_\alpha \subset M$ so that $\eta_\alpha\overline{\eta_\alpha}^{-1}$ represents s_α. Then $\eta_\alpha\eta\bar{\eta}^{-1}\overline{\eta_\alpha}^{-1} = \eta\bar{\eta}^{-1}\eta_\alpha\overline{\eta_\alpha}^{-1}$ represents ξ''. Thus, there exists $g'' \in Pg'H$ such that $\iota_B(Bg''H) = \xi$. By the injectivity part of Lemma 19, $BgH = Bg''H$, hence injectivity of ι_P. □

For any double coset in $P(E)\backslash G(E)/H(F)$ choose a representative η so that $\eta\bar{\eta}^{-1}$ is the permutation matrix $\xi \in {}^M\Omega_2^M$. Let $G^* = \mathrm{Res}_{E/F}H$ and similarly for P^*. Define F-subgroups of G^* by

$$\begin{aligned} H_\eta &= H \cap \eta^{-1}P^*\eta, \\ P_\eta &= \eta H\eta^{-1} \cap P^*, \end{aligned}$$

and set $M_\eta = M^* \cap P_\eta$, $N_\eta = N^* \cap P_\eta$. We observe that

$$\eta H\eta^{-1} = \{x \in G^* : \bar{x} = \xi^{-1}x\xi\}.$$

Hence the groups P_η , M_η, N_η depend only on ξ and we sometimes denote them by P_ξ, M_ξ, N_ξ.

In general, P_η need not equal $M_\eta N_\eta$. For example, consider the parabolic subgroup

$$P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

of $GL(3)$ and the involution $\xi = (23) \in {}^M\Omega_2^M$. Then

$$P_\eta(F) = \left\{ \begin{pmatrix} x & \beta & \overline{\beta} \\ 0 & \alpha & 0 \\ 0 & 0 & \overline{\alpha} \end{pmatrix} : x \in F^* \text{ and } \alpha, \beta \in E^* \right\}$$

and $M_\eta(F)$ is the diagonal subgroup of $P_\eta(F)$, but $N_\eta = \{e\}$. However, we do have $P_\eta = M_\eta N_\eta$ if ξ normalizes M. We now make the following definition.

Definition 1. A representative η as above is called *P-admissible* if $\xi = \eta\bar{\eta}^{-1} \bmod T_E$ lies in Ω^M. We also speak about an admissible double coset in $P(E)\backslash G(E)/H(F)$.

Lemma 21. *Let η be a representative of a double coset in $P(E)\backslash G(E)/H(F)$ such that $\xi = \eta\bar{\eta}^{-1}$ represents an element in ${}^M\Omega_2^M$.*

(1) *In the case $P = B$, every η is B-admissible.*
(2) *If η is P-admissible, then M_η is a twisted F-form of M defined by the Galois action $m \mapsto \xi\bar{m}\xi^{-1}$.*
(3) *If η is P-admissible, then $P_\eta = M_\eta N_\eta$ is a Levi decomposition of P_η. The center of M_η is contained in the center of M_E.*
(4) *ι_P induces a bijection between the P-admissible double cosets in $P(E)\backslash G(E)/H(F)$ and $\Omega_2(M, M)$.*

Proof. Part (1) is immediate from the definition since $\Omega^T = \Omega$. Part (2) is also immediate from the definition. Observe that

$$P_\eta(F) = \{p \in P_E(E) : \overline{p} = \xi^{-1}p\xi\}.$$

Since $\xi \in \Omega^M$ normalizes M, it follows that $P \cap \xi N\xi^{-1} = N \cap \xi N\xi^{-1}$. Therefore, if $p = mn \in P_\eta(F)$ with $m \in M(E)$ and $n \in N(E)$, then $\overline{m} = \xi^{-1}m\xi$ and $\overline{n} = \xi^{-1}n\xi$.

On the other hand, M_η is the fixed point set of an involution and hence is reductive ([S]). This proves the first part of (3). The second claim follows from the fact that the center of $M_\eta(F)$ is contained in that of $M_\eta(E) \simeq M(E)$ since $M_\eta(F)$ is Zariski dense in $M_\eta(E)$. Finally, (4) follows from (50). □

Remark 2. Explicitly, if P corresponds to the partition $(n_1, \dots, n_r)$, then the admissible double cosets correspond to involutions π of $\{1, \dots, r\}$ so that $n_{\pi(i)} = n_i$ and then $M_\eta(F) \simeq \prod_{\pi(i)=i} GL(n_i, F) \times \prod_{\pi(i)<i} GL(n_i, E)$.

14. **Invariant functionals on principal series representations.**

Proposition 22. *Let v be a place of F where E is inert and let τ be a character of the diagonal subgroup $T(E_v)$. Suppose there exists a non-zero H_v-invariant linear functional on the induced representation $\mathrm{Ind}_{B(E_v)}^{G(E_v)} \tau$. Then there exists $\eta \in B(E)\backslash G(E)/H(F)$ such that the restriction of τ to $T_\eta(F_v)$ is trivial.*

Proof. We drop v from the notation in this proof. For $x \in G(E)$, denote by π^x the representation obtained by conjugating π by x. Let δ_η be the modulus function on B_η. According to the geometric lemma in [BZ], Theorem 5.2, the representation $\mathrm{Res}_{H(F)} \mathrm{Ind}_{B(E)}^{G(E)} \tau$ is "glued" from $\mathrm{ind}_{H_\eta(F)}^{H(F)} (\delta_{B_E}^{1/2} \delta_\eta^{-1/2} \tau|_{B_\eta})^\eta$ (induction with compact support) where η ranges over representatives of $B(E)\backslash G(E)/H(F)$ as above. The modulus factors appear because we use normalized induction. We claim that $\delta_{B_E}^{1/2} \delta_\eta^{-1/2} = \delta_\eta^{1/2}$. Since $\delta_B = \delta_{B_E}^{1/2}$ on $B(F)$, it suffices to check that the restriction of δ_B to B_η is δ_η. Suppose that $\eta\bar{\eta}^{-1}$ represents $\xi \in \Omega_2$. Let $\mathfrak{u}_{\{\alpha,\xi\alpha\}} = \{x \in u_\alpha + u_{\xi\alpha} : Ad(\eta\bar{\eta}^{-1})x = \bar{x}\}$ where $\mathfrak{u}_\alpha$ is the root space of α. Then $Lie(U_\eta)$ decomposes as a direct sum of $\mathfrak{u}_{\{\alpha,\xi\alpha\}}$ over orbits $\{\alpha, \xi\alpha\} \subset \Phi^+$ of ξ and $\dim_F \mathfrak{u}_{\{\alpha,\xi\alpha\}} = |\{\alpha, \xi\alpha\}|$. We have $Lie(A_\eta) = (\mathfrak{A}_P)_\xi^+$ where A_η is the split component of T_η and thus for $t \in A_\eta$,

$$\delta_\eta(t) = \prod_{\{\alpha,\xi\alpha\}\subset\Phi^+} \left|\det Ad(t)|_{\mathfrak{u}_{\{\alpha,\xi\alpha\}}}\right| = \prod_{\{\alpha\in\Phi^+:\xi\alpha>0\}} |\alpha(t)| .$$

On the other hand, since $t \in A_\eta$ we have

$$\prod_{\{\alpha\in\Phi^+:\xi\alpha<0\}} |\alpha(t)| = 1$$

because $|\alpha(t)|\,|(-\xi\alpha)(t)| = 1$. Thus:

$$\delta_\eta(t) = \prod_{\alpha\in\Phi^+} |\alpha(t)| = \delta_B(t)$$

as claimed. Thus, if $\mathrm{Ind}_{B(E)}^{G(E)} \tau$ has a non-zero $H(F)$-invariant functional, then at least one of the quotients $\mathrm{ind}_{H_\eta(F)}^{H(F)} (\delta_\eta^{1/2} \tau)^\eta$ must have a non-zero $H(F)$-invariant functional. In this case, the dual, which is isomorphic to $\mathrm{Ind}_{H_\eta(F)}^{H(F)} (\mathrm{mod}_{H_\eta}^{-1/2} (\tau^\eta)^{-1})$, has an $H(F)$-invariant vector. By Frobenius reciprocity this occurs if and only if the restriction of τ^η to $H_\eta(F)$ is trivial as claimed. □

15. **The regularized period.** The next theorem describes the regularized period of a cuspidal Eisenstein series. We write $x \longrightarrow \bar{x}$ for the conjugation of E over F and if π is a representation of $G(\mathbb{A}_E)$ or a Levi subgroup, we write $\bar{\pi}$ for the representation $g \longrightarrow \pi(\bar{g})$. Let P be the parabolic subgroup of H corresponding to the partition $(n_1, \dots, n_r)$ of n. Assume that P is a proper subgroup. If $\sigma =$

$\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$ is a representation of $M_E(\mathbb{A}_E)^1$ and $\lambda \in \mathfrak{A}_P^*$, we write $\sigma(\lambda)$ for the representation that extends σ to $M_E(\mathbb{A}_E)$ by $\sigma(\lambda)(am) = e^{\langle \lambda, H_{P_E}(a)\rangle}\sigma(m)$. We write σ^* for the contragredient of σ.

Theorem 23. *Let $\varphi \in \mathcal{A}_P(G)_\sigma$ where $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$ is a cuspidal representation of $M_E(\mathbb{A}_E)^1$, and let $E(\varphi, \lambda) = E(g, \varphi, \lambda)$ be the associated Eisenstein series. Suppose that $E(\varphi, \lambda)$ is regular at $\lambda = \lambda_0$ and that $\Pi^{G/H}(E(\varphi, \lambda_0))$ is defined and non-zero. Then either $r = 1$ and $\varphi = E(\varphi, \lambda_0)$ is a distinguished cusp form on $G(\mathbb{A}_E)$, or $r = 2$ and $\sigma_2^* = \overline{\sigma_1}$. In this case, if $\langle \mathrm{Re}\,\lambda, \alpha^\vee\rangle >> 0$ for the unique root $\alpha \in \Delta_P$, then the period is given by the following absolutely convergent integral:*

$$\Pi^{G/H}(E(\varphi, \lambda)) = \int_{H_\eta(F)\backslash H(\mathbb{A})^1} e^{\langle \lambda + \rho_P, H_{P_E}(\eta h)\rangle}\ \varphi(\eta h)\ dh$$

where $\eta\bar{\eta}^{-1} = \xi$ is the unique non-trivial element in $\Omega(M, M)$. Furthermore, $\Pi^{G/H}(E(\varphi, \lambda))$ extends to a meromorphic function of λ.

Remark 3. More precisely, it can be proved that the right hand side of the above formula converges absolutely in the domain $\langle \mathrm{Re}\,\lambda - \rho_P, \alpha^\vee\rangle > 0$ ([LP]).

Proof. We may suppose that $r > 1$ and hence that $E(g, \varphi, \lambda)$ is a true Eisenstein series. If $\Pi^{G/H}(E(\varphi, \lambda_0)) \neq 0$, then $\Pi^{G/H}(E(\varphi, \lambda))$ is non-zero for generic $\lambda \in \mathfrak{A}_{P,\mathbb{C}}^*$. This implies that there exists a dense open set $\mathcal{O} \subset \mathfrak{A}_{P,\mathbb{C}}^*$ such that $\mathrm{Ind}_P^G(\sigma(\lambda))$ admits an $H(\mathbb{A}_f)^1$-invariant functional for all $\lambda \in \mathcal{O}$. Choose a place v where E_v is inert and σ_v is unramified, induced from an unramified character $t \mapsto e^{\langle \alpha, H_E(t)\rangle}$ of T_E for some $\alpha \in \mathfrak{A}_{0,\mathbb{C}}^*$. We observe that for $\xi \in \Omega_2$, $T_\xi = \{t \in T(E) : \bar{t} = \xi t \xi^{-1}\}$ and $\mathfrak{A}_\xi^+$ is the Lie algebra of the split component of T_ξ modulo the center. From the previous proposition, we see that for all $\lambda \in \mathcal{O}$, there exists $\xi_\lambda \in \Omega_2$ such that $\left\langle \lambda + \alpha, \mathfrak{A}_{\xi_\lambda}^+ \right\rangle = 0$. Since Ω_2 is a finite set, there exists $\xi \in \Omega_2$ and a subset $\mathcal{O}' \subset \mathcal{O}$ with non-empty interior such that $\xi_\lambda = \xi$ for all $\lambda \in \mathcal{O}'$. Therefore $\mathfrak{A}_\xi^+$ is orthogonal to $\mathfrak{A}_P^*$ and hence $\mathfrak{A}_P^* \subset (\mathfrak{A}^*)_\xi^-$. We claim that P must be a maximal parabolic subgroup of type (m, m) where $m = \frac{n}{2}$ and ξ is a Weyl group element that interchanges the two blocks. Indeed, ξ acts by -1 on $\mathfrak{A}_P^*$ and therefore normalizes M_P, acting as a transposition on its blocks. But if M_P has more than two blocks, then $\mathfrak{A}_P^* \cap (\mathfrak{A}^*)_\xi^+ \neq 0$ since ξ necessarily fixes certain non-scalar elements in the center of M_P.

Assume now that P is of type (m, m). Then its associate class consists only of P itself. In particular, the constant term $E_{Q_E}(h, \varphi, \lambda)$ vanishes for all proper parabolic subgroups $Q \neq P$. Therefore

$$\begin{aligned}
\Lambda_m^T E(h, \varphi, \lambda) &= E(h, \varphi, \lambda) - \sum_{\delta \in P\backslash H} E_{P_E}(\delta h, \varphi, \lambda)\ \hat{\tau}_P(H_{P_E}(\delta h) - T), \\
\Lambda_m^{T,P} E(h, \varphi, \lambda) &= E_{P_E}(h, \varphi, \lambda)
\end{aligned}$$

and

$$\begin{aligned}
(53) \qquad \Pi^{G/H}(E(\varphi, \lambda)) &= \int_{H\backslash H(\mathbb{A})^1} \Lambda_m^T E(h, \varphi, \lambda) dg \\
&\quad + \int^*_{P\backslash H(\mathbb{A})^1} E_{P_E}(h, \varphi, \lambda) \hat{\tau}_P(H_{P_E}(h) - T) dh.
\end{aligned}$$

Now assume that $\langle \operatorname{Re}\lambda, \alpha^\vee\rangle >> 0$ where α is the unique root in Δ_P. Using the formula

$$E_{P_E}(h,\varphi,\lambda) = \varphi(h)e^{\left\langle \lambda+\rho_P, H_{P_E}(h)\right\rangle} + M(\xi,\lambda)\varphi(h)e^{\left\langle -\lambda+\rho_P, H_{P_E}(h)\right\rangle}$$

we see that $\Lambda_m^T E(h,\varphi,\lambda)$ is equal to $I(h)+II(h)+III(h)$ where

$$I(h) = \sum_{\delta\in P_E\backslash G - P\backslash H} \varphi(\delta h)e^{\left\langle \lambda+\rho_P, H_{P_E}(\delta h)\right\rangle},$$

$$II(h) = \sum_{\delta\in P\backslash H} \varphi(\delta h)e^{\left\langle \lambda+\rho_P, H_{P_E}(\delta h)\right\rangle}(1-\hat{\tau}_P(H_{P_E}(\delta h)-T)),$$

$$III(h) = -\sum_{\delta\in P\backslash H} M(\xi,\lambda)\varphi(\delta h)e^{\left\langle -\lambda+\rho_P, H_{P_E}(\delta h)\right\rangle}\hat{\tau}_P(H_{P_E}(\delta h)-T).$$

The sum $II(h)+III(h)$ is absolutely integrable over $H\backslash H(\mathbb{A})^1$ and the integral is equal to

$$\begin{aligned} &\int_{P\backslash H(\mathbb{A})^1} \varphi(h)e^{\left\langle \lambda+\rho_P, H_{P_E}(h)\right\rangle}(1-\hat{\tau}_P(H_{P_E}(h)-T))dh \\ - &\int_{P\backslash H(\mathbb{A})^1} M(\xi,\lambda)\varphi(h)e^{\left\langle -\lambda+\rho_P, H_{P_E}(h)\right\rangle}\hat{\tau}_P(H_{P_E}(h)-T)\, dh. \end{aligned}$$

Indeed, using the Iwasawa decomposition, we bound them by

$$\int_{\mathfrak{A}_P}\int_{M(F)\backslash M(\mathbb{A})^1\times\mathbf{K}} e^{\left\langle \operatorname{Re}\lambda-\rho_P, H_{P_E}(X)\right\rangle}|\varphi(mk)|(1-\hat{\tau}_P(H_{P_E}(X)-T))\, dX\, dm\, dk$$

and

$$\int_{\mathfrak{A}_P}\int_{M(F)\backslash M(\mathbb{A})^1\times\mathbf{K}} e^{\left\langle -\operatorname{Re}\lambda-\rho_P, H_{P_E}(X)\right\rangle}|M(\xi,\lambda)\varphi(mk)|\hat{\tau}_P(H_{P_E}(X)-T)\, dX\, dm\, dk.$$

The integrals over $\mathfrak{A}_P$ are finite since $\langle \operatorname{Re}\lambda-\rho_P, \alpha^\vee\rangle > 0$, as are the integrals over $M(F)\backslash M(\mathbb{A})^1\times\mathbf{K}$ since φ and $M(\xi,\lambda)\varphi$ are integrable on $M(F)\backslash M(\mathbb{A})^1$. On the other hand, the relation (16) yields the equality

$$\begin{aligned} &\int_{P\backslash H(\mathbb{A})^1} \varphi(g)e^{\left\langle \lambda+\rho_P, H_{P_E}(h)\right\rangle}(1-\hat{\tau}_P(H_{P_E}(h)-T))dh \\ = \quad &-\int^*_{P\backslash H(\mathbb{A})^1} \varphi(g)e^{\left\langle \lambda+\rho_P, H_{P_E}(h)\right\rangle}\ \hat{\tau}_P(H_{P_E}(h)-T)dh, \end{aligned}$$

and so the contribution of $II(h)+III(h)$ is equal to

$$-\int^*_{P\backslash H(\mathbb{A})^1} E_{P_E}(h,\varphi,\lambda)\hat{\tau}_P(H_{P_E}(h)-T)dh.$$

Now (53) shows that

$$\Pi^{G/H}(E(\varphi,\lambda)) = \int_{H(F)\backslash H(\mathbb{A})^1} I(h)\, dh,$$

where we recall that

$$I(h) = \sum_{\delta\in P_E\backslash G - P\backslash H} \varphi(\delta h)e^{\left\langle \lambda+\rho_P, H_{P_E}(\delta h)\right\rangle}.$$

We know that $I(h)$ is absolutely integrable, but we do not yet know that the series

$$|I|(h) = \sum_{\delta \in P_E\backslash G - P\backslash H} |\varphi(\delta h)| e^{\left\langle \mathrm{Re}\,\lambda + \rho_P, H_{P_E}(\delta h)\right\rangle}$$

is integrable. We will show in fact that it is bounded. Assume that λ is real for simplicity of notation, and set

$$\begin{aligned} I^T(h) &= \sum_{\delta \in P_E\backslash G - P\backslash H} |\varphi(\delta h)| e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta h)\right\rangle} \hat{\tau}_P(H_{P_E}(\delta h) - T), \\ I_T(h) &= \sum_{\delta \in P_E\backslash G - P\backslash H} |\varphi(\delta h)| e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta h)\right\rangle} (1 - \hat{\tau}_P(H_{P_E}(\delta h) - T)). \end{aligned}$$

Then $|I|(h) = I^T(h) + I_T(h)$. We shall bound $I^T(h)$ and $I_T(h)$ separately.

By [A1], Lemma 5.1, the number of $\delta \in P_E\backslash G$ such that $\hat{\tau}_P(H_{P_E}(\delta h) - T) = 1$ is bounded by $C\,\|h\|^N$ for some constants C, $N > 0$. To bound $I^T(h)$, it will therefore suffice to prove the following proposition.

Proposition 24. *For any $M > 0$ there exists a constant $K > 0$ such that for h in the Siegel domain $\mathcal{S}^H$ we have*

$$|\varphi(\delta h)|\, e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta h)\right\rangle} \le K\,\|h\|^{-M}$$

if $\delta \in P_E\backslash G - P\backslash H$ and $\hat{\tau}_P(H_{P_E}(\delta h) - T) = 1$.

We begin the proof of Proposition 24 with the following lemma. We write C_1, C_2, etc. for the *positive* constants that appear in the arguments below. Let $(\mathfrak{A}_0^P)^{*+}$ denote the positive Weyl chamber in $(\mathfrak{A}_0^P)^*$.

Lemma 25. *There exists $\mu \in (\mathfrak{A}_0^P)^*$ and a constant C with the following property. For all $h \in \mathcal{S}^G \cap H(\mathbb{A})^1$ and $\delta \in G(E) - P_E H$ such that $\delta h \in \mathcal{S}^{P_E}$, we have*

$$\langle \alpha, H_{P_E}(\delta h)\rangle \le \langle \mu, H_E(\delta h)\rangle + C.$$

Proof. Let $h \in \mathcal{S}^G \cap H(\mathbb{A})^1$ and $\delta \in G(E) - P_E H$. We do not assume at the outset that $\delta h \in \mathcal{S}^{P_E}$. However, multiplying δ on the left by $P_E(E)$ if necessary, we can use Proposition 20, to assume that $\delta\overline{\delta}^{-1} \in N_G(T_E)$ represents $\varrho \in {}^M\Omega_2^M$. According to Lemma 19, (52), we have $\left\langle P_\varrho^- \varpi, H_E(\delta h)\right\rangle < C_1$ for some constant C_1 where $\hat{\Delta}_P = \{\varpi\}$. There exist a unique $a \in \mathbb{R}$ and $\nu \in (\mathfrak{A}_0^P)^*$ such that $P_\varrho^- \alpha = a\alpha + \nu$. Since $\delta \notin P_E H$, ϱ does not lie in Ω_M and $a > 0$ because $a = \frac{1}{2}\left\langle P_\varrho^- \alpha, \alpha^\vee\right\rangle = \frac{1}{2}\left\langle P_\varrho^- \alpha, P_\varrho^- \alpha^\vee\right\rangle$. Also, $\left\langle P_\varrho^- \varpi, \beta^\vee\right\rangle = \frac{1}{2}\langle \varpi, -\varrho\beta^\vee\rangle \le 0$ for any $\beta \in \Delta^P$, since ϱ is reduced, so that $\nu \in -(\mathfrak{A}_0^P)^{*+}$. Using $\langle \alpha, H_E(\delta h)\rangle < a^{-1}C_1 - a^{-1}\langle \nu, H_E(\delta h)\rangle$ and setting $\kappa = -a^{-1}\nu$, we obtain

$$e^{\left\langle \alpha, H_{P_E}(\delta h)\right\rangle} \le C_2 e^{\langle \kappa, H_E(\delta h)\rangle}. \tag{54}$$

Now we observe that for any $\mu' \in (\mathfrak{A}_0^P)^{*+}$ there exists $\mu'' \in (\mathfrak{A}_0^P)^*$ such that for all $g \in G(\mathbb{A}_E)$ and $p \in P_E$ such that $pg \in \mathcal{S}^{P_E}$ we have

$$e^{\left\langle \mu', H(g)\right\rangle} \le C_3 e^{\left\langle \mu'', H(pg)\right\rangle}. \tag{55}$$

To prove this, write $p^{-1} = \delta' u$ with $\delta' \in M_E(E), u \in N(E)$ and $pg = namk$ where m lies in the Siegel domain of M_E^1. Then

$$e^{\left\langle \mu', H(g) \right\rangle} = e^{\left\langle \mu', H(\delta' m) \right\rangle} \leq C_4 \left\| m \right\|^N \leq C_5 e^{\left\langle \mu'', H(m) \right\rangle}$$

where the first inequality can be found in [A1], equation (5.2) in the proof of Lemma 5.1, and the second inequality for some μ'' is proved in [MW], p. 20 (vi). Now apply (55) with $\mu' = \kappa$ and $g = \delta h$. Since $H_{P_E}(\delta h) = H_{P_E}(p\delta h)$, we obtain

$$e^{\left\langle \alpha, H_{P_E}(p\delta h) \right\rangle} \leq C_6 e^{\left\langle \mu'', H_E(p\delta h) \right\rangle}$$

where $p\delta h \in \mathcal{S}^{P_E}$. □

We now prove Proposition 24. Let μ be as in Lemma 25 and let $h \in \mathcal{S}^G \cap H(\mathbb{A})^1$ and $\delta \in G(E) - P_E H$. Since the conclusion of the proposition does not change if we replace δ by $p\delta$ for $p \in P_E$, we may assume that δh belongs to $\mathcal{S}^{P_E}$. We get

$$|\varphi(\delta h)| \, e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta h) \right\rangle} \leq C_7 \ |\varphi(\delta h)| \, e^{\langle \mu, H(\delta h) \rangle}.$$

The function $m \longrightarrow \varphi(mk)$ is cuspidal for all $k \in \mathbf{K}$ and therefore rapidly decreasing on $M(E) \backslash M(\mathbb{A}_E)^1$. For any $\nu \in (\mathfrak{A}_0^P)^*$ there exists C_8 such that

$$|\varphi(\delta h)| \, e^{\langle \mu, H(\delta h) \rangle} \leq C_8 e^{\langle \nu, H(\delta h) \rangle}$$

and thus

$$|\varphi(\delta h)| \, e^{\langle \lambda + \rho_P, H_{P_E}(\delta h) \rangle} \leq \quad C_9 \quad e^{\langle \nu, H(\delta h) \rangle}.$$

Now, since $\delta h \in \mathcal{S}^{P_E}$,

$$\langle \beta, H_E(\delta h) \rangle > \langle \beta, T_0 \rangle \ \text{ for } \beta \in \Delta_0^M.$$

The hypothesis $\hat{\tau}_P(H_{P_E}(\delta h) - T) = 1$ is equivalent to $\langle \alpha, H_E(\delta h) \rangle \geq \langle \alpha, T \rangle$ and therefore

$$\|\delta h\| \leq C_{10} e^{\langle \zeta, H_E(\delta h) \rangle} e^{k \langle \alpha, H_E(\delta h) \rangle} \tag{56}$$

for some $\zeta \in (\mathfrak{A}_0^P)^*$ and $k > 0$. Again, using Lemma 25, given N, we may choose $\nu \in (\mathfrak{A}_0^P)^*$ so that

$$e^{\langle \nu, H(\delta h) \rangle} \leq C_{11} \left\| \delta h \right\|^{-N}.$$

By [MW], p. 21, (vii), there exists $C_{12} > 0$ such that

$$\|g\| \leq C_{12} \left\| \gamma g \right\| \tag{57}$$

for all $g \in \mathcal{S}^G$ and $\gamma \in G(E)$, and hence

$$|\varphi(\delta h)| \, e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta h) \right\rangle} \leq C_{13} \ \left\| h \right\|^{-N}.$$

This completes the proof of Proposition 24.

It remains to bound $I_T(h)$. In fact we will show that

$$g \rightarrow \sum_{\delta \in P_E \backslash G} |\varphi(\delta g)| e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta g) \right\rangle} (1 - \hat{\tau}_P(H_{P_E}(\delta g) - T))$$

is bounded for λ positive enough. We can assume that $g \in \mathcal{S}^G$. In analogy to (56), there exist $k_1 > 0$ and $\nu \in (\mathfrak{A}_0^P)^*$ such that

$$\|\delta g\| \leq C_{14} e^{\langle \nu, H_E(\delta g)\rangle} e^{-k_1 \langle \alpha, H_{P_E}(\delta g)\rangle}$$

for all $\delta g \in \mathcal{S}^{P_E}$ and $\hat{\tau}_P(H_{P_E}(\delta g) - T) = 0$. Since φ is rapidly decreasing we infer that for any $N_2, k_2 > 0$

(58)

$$|\varphi(\delta g)| e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta g)\right\rangle}(1 - \hat{\tau}_P(H_{P_E}(\delta g) - T)) \leq C_{15} \|\delta g\|^{-N_2} e^{-k_2 \left\langle \alpha, H_{P_E}(\delta g)\right\rangle}$$

provided that λ is positive enough. For $x \in \mathbb{Z}$, $x < T$ is the number of δ such that $\delta g \in \mathcal{S}^{P_E}$ and $x \leq \langle \alpha, H_{P_E}(\delta g)\rangle < x + 1$ is bounded by $C_{16} \|g\|^{N_3} e^{k_3 x}$ for some $N_3, k_3 > 0$. In fact, according to Lemma 5.1 of [A1], this is true also for $x \leq \langle \alpha, H_{P_E}(\delta g)\rangle$. The claim now follows from (58) taking into account (57).

We now compute the integral of $I(h)$. Let $\{\gamma\}$ be a set of representatives for the double cosets $P_E\backslash G/H$ other than the trivial coset $P_E H$. We may choose them so that $\gamma\overline{\gamma}^{-1} \in N_G(T_E)$. We also assume that $\eta \in \{\gamma\}$ is one of the representatives. Set

$$E_\gamma(h, \varphi, \lambda) = \sum_{\delta \in P_E\backslash P_E \gamma H} \varphi(\delta h) e^{\left\langle \lambda + \rho_P, H_{P_E}(\delta h)\right\rangle}.$$

Then

$$I(h) = \sum_{\{\gamma\}} E_\gamma(h, \varphi, \lambda).$$

We have shown that the series

$$|E_\gamma|(h, \varphi, \lambda) = \sum_{\delta \in P_E\backslash P_E \gamma H} |\varphi(\delta h)| e^{\left\langle \operatorname{Re}\lambda + \rho_P, H_{P_E}(\delta h)\right\rangle}$$

are integrable for $\langle \operatorname{Re}\lambda, \alpha^\vee\rangle >> 0$. We will show that

$$\int_{H\backslash H(\mathbb{A})^1} E_\gamma(h, \varphi, \lambda) = 0$$

for $\gamma \neq \eta$.

As before, we let P_γ be the subgroup of H such that

$$P_\gamma(F) = P_E(E) \cap \gamma H(F) \gamma^{-1}$$

and let $R = R_\gamma$ be the unipotent radical of P_γ. Then we have

$$\int_{H\backslash H(\mathbb{A})^1} E_\gamma(h,\varphi,\lambda)\ dh = \int_{P_\gamma(F)\backslash \gamma H(\mathbb{A})^1\gamma^{-1}} \varphi(h\gamma)e^{\langle \lambda+\rho_P, H_{P_E}(h\gamma)\rangle}\ dh$$

$$= \int_{P_\gamma(F)R(\mathbb{A})\backslash \gamma H(\mathbb{A})^1\gamma^{-1}} e^{\langle \lambda+\rho_P, H_{P_E}(h\gamma)\rangle} \left(\int_{R(F)\backslash R(\mathbb{A})} \varphi(uh\gamma)du\right) dh.$$

We will show that the inner integral vanishes due to the cuspidality of φ.

Let $w = \gamma\bar{\gamma}^{-1}$ represent an element in Ω_2 and set $K = P_E \cap w^{-1}P_E w$. Define an involution Ξ on $G(E)$ by the formula $\Xi(x) = w^{-1}\overline{x}w$. Then P_γ is the set of fixed points of Ξ in P_E. Thus $P_\gamma \subset K$ and therefore P_γ is also equal to the set K^Ξ of fixed points of Ξ in K. Now K contains four subgroups

$$K_0 = M_E \cap w^{-1}M_E w, \qquad K_2 = N_E \cap w^{-1}M_E w,$$

$$K_1 = M_E \cap w^{-1}N_E w, \qquad K_3 = N_E \cap w^{-1}N_E w.$$

The subgroups K_1 and K_2 normalize K_3 and the set of commutators $[K_1, K_2]$ is contained in K_3. Hence $U = K_1K_2K_3$ is a unipotent subgroup of K which is normal in K since K_0 normalizes K_1, K_2, and K_3. Furthermore, $K = K_0 \ltimes U$ and $P_\gamma = K_0^\Xi \ltimes U^\Xi$. In particular, $R_\gamma = U^\Xi$.

We claim that for any $u_1 \in K_1$, there exists $u_3 \in K_3$ such that $u_1\Xi(u_1)u_3 \in R$. Indeed, $u_1\Xi(u_1)u_3$ is fixed by Ξ if and only if

$$\Xi(u_3)u_3^{-1} = u_1^{-1}\Xi(u_1)^{-1}u_1\Xi(u_1).$$

This equation has a solution u_3 in K_3 since the right-hand side belongs to K_3 and satisfies the equation $x\Xi(x) = 1$. The coset $u_3K_3^\Xi$ is uniquely determined by u_1. Set $m(u_1) = \Xi(u_1)u_3$. Then the map $u_1 \longrightarrow u_1m(u_1)K_3^\Xi$ defines a bijection $K_1 \longrightarrow R/K_3^\Xi$. This is an isomorphism of algebraic groups whose inverse is the canonical projection onto M_E. It follows that the formula

$$\int_{K_1(\mathbb{A})} \left(\int_{K_3^\Xi(\mathbb{A})} f(u_1m(u_1)t)\ dt\right)\ du_1$$

defines an invariant measure on $R(\mathbb{A})$. Here du_1 and dt are Haar measures on $K_1(\mathbb{A})$ and $K_3^\Xi(\mathbb{A})$, respectively. If f is left-invariant under $R(F)$,

$$\int_{R(F)\backslash R(\mathbb{A})} f(r)\ dr = \int_{K_1(F)\backslash K_1(\mathbb{A})} \left(\int_{K_3^\Xi(F)\backslash K_3^\Xi(A)} f(u_1m(u_1)t)\ dt\right)\ du_1$$

where dr is an invariant measure on $R(F)\backslash R(\mathbb{A})$. If f is a function on $G(\mathbb{A}_E)$ which is left invariant under $N_E(\mathbb{A})$, then this formula simplifies to

$$\int_{R(F)\backslash R(\mathbb{A})} f(r)\ dr = \int_{K_1(F)\backslash K_1(\mathbb{A})} f(u_1m(u_1))\ du_1$$

since $K_3^\Xi \subset N_E$ and then to

$$\int_{R(F)\backslash R(\mathbb{A})} f(r)\ dr = \int_{K_1(F)\backslash K_1(\mathbb{A})} f(u_1)\ du_1$$

since $u_1m(u_1)u_1^{-1} \in N_E(\mathbb{A})$.

The function φ above is $N_E(\mathbb{A})$-invariant, and therefore

$$\int_{H\backslash H(\mathbb{A})^1} E_\gamma(h,\varphi,\lambda)\ dh$$

is equal to

$$\int_{P_\gamma(F)R(\mathbb{A})\backslash\gamma H(\mathbb{A})^1\gamma^{-1}} e^{\left\langle\lambda+\rho_P,H_{P_E}(h\gamma)\right\rangle}\left(\int_{K_1(F)\backslash K_1(\mathbb{A})}\varphi(u_1 h\gamma)du\right)dh.$$

However K_1 is the unipotent radical of a (not necessarily standard) parabolic subgroup of M_E. If γ is not admissible (i.e. $\gamma\neq\eta$ according to Lemma 21), then $K_1\neq 0$. Since φ is cuspidal, the inner integral over K_1vanishes as desired. We remark that the idea that these terms vanish already appears in [F3].

We have shown that

$$\begin{aligned}\Pi^{G/H}(E(\varphi,\lambda)) &= \int_{H(F)\backslash H(\mathbb{A})^1} E_\eta(h,\varphi,\lambda)\ dh\\ &= \int_{H_\eta(F)\backslash H(\mathbb{A})^1} e^{\left\langle\lambda+\rho_P,H_{P_E}(\eta h)\right\rangle}\ \varphi(\eta h)\ dh.\end{aligned}$$

Note that this integral is absolutely convergent. To complete the proof, i.e. to show that $\sigma_2^*=\overline{\sigma_1}$, we rewrite the expression for $\Pi^{G/H}(E(\varphi,\lambda))$ slightly as follows. Let

$$\xi=\begin{pmatrix}0 & 1_m\\ 1_m & 0\end{pmatrix}.$$

We may choose η to be

$$\eta=\begin{pmatrix}1_m & i1_m\\ 1_m & -i1_m\end{pmatrix}$$

where $i\in E-F$ and $i^2\in F$. Then

$$M_\eta=\{m=diag(g,\overline{g}):g\in GL_m(E)\}$$

and $H_\eta=\eta^{-1}M_\eta\eta$. For future reference, we write this as a separate proposition. Henceforth, we assume that $\varphi(ag)=\varphi(g)$ for $a\in A_P$.

Proposition 26. *With the previous notation, we have, for $\langle\operatorname{Re}\lambda,\alpha^\vee\rangle >> 0$,*

$$\Pi^{G/H}(E(\varphi,\lambda))=$$

$$\int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\left\langle\lambda+\rho_P,H_{P_E}(\eta h)\right\rangle}\left(\int_{M_\eta\backslash M_\eta(\mathbb{A})^1}\varphi(m\eta h)\ dm\right)dh, \tag{59}$$

where the integral on the right is absolutely convergent.

□

Example 1. We illustrate the decomposition of the group P_γ used above in the case that P corresponds to $(2,1,1,1)$ and $w=(13)(24)$:

$$P_E=\left\{\begin{pmatrix}\# & \# & * & * & *\\ \# & \# & * & * & *\\ 0 & 0 & \# & * & *\\ 0 & 0 & 0 & \# & *\\ 0 & 0 & 0 & 0 & \#\end{pmatrix}\right\},\ wP_Ew^{-1}=\left\{\begin{pmatrix}\# & * & 0 & 0 & *\\ 0 & \# & 0 & 0 & *\\ * & * & \# & \# & *\\ * & * & \# & \# & *\\ 0 & 0 & 0 & 0 & \#\end{pmatrix}\right\}.$$

Then

$$K = P_E \cap w^{-1} P_E w = \left\{ \begin{pmatrix} K_0 & K_1 & 0 & 0 & K_3 \\ 0 & K_0 & 0 & 0 & K_3 \\ 0 & 0 & K_0 & K_2 & K_3 \\ 0 & 0 & 0 & K_0 & K_3 \\ 0 & 0 & 0 & 0 & K_0 \end{pmatrix} \right\}$$

where we have indicated which entries belong to the subgroups K_j. In this case

$$P_\gamma = \left\{ \begin{pmatrix} \alpha & \varepsilon & 0 & 0 & \nu \\ 0 & \beta & 0 & 0 & \mu \\ 0 & 0 & \overline{\alpha} & \overline{\varepsilon} & \overline{\nu} \\ 0 & 0 & 0 & \overline{\beta} & \overline{\mu} \\ 0 & 0 & 0 & 0 & a \end{pmatrix} : \alpha, \beta \in E^*, \varepsilon, \mu, \nu \in E \text{ and } \ a \in F^* \right\}.$$

The integral over ε will lead to vanishing of the integral when φ is in the induced space of a space of cusp forms on M_P.

The proof of Theorem 23 establishes the convergence of the integral in the previous Proposition. However, we will need to establish the convergence of the integral for functions φ which are not necessarily cuspidal, in fact, for φ a constant function. If $\varphi = 1$, then the inner integral is just the volume of the quotient $M_\eta(F)\backslash M_\eta(\mathbb{A}_F)^1$. The remaining integral is described in the next Lemma.

Lemma 27. *Let the notations be as in the previous proposition, except that λ is real and $\varphi = 1$. If $\langle \mathrm{Re}\,\lambda, \alpha^\vee \rangle >> 0$ is sufficiently large, then the following integral is finite:*

$$\int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda + \rho_P, H_{P_E}(\eta h) \rangle} \, dh.$$

Proof. It will be convenient to change notations and write the function in the integrand as $f(g, s)$ where s is real and f is defined by:

$$f\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k, s \right] = \left| \frac{\det a}{\det b} \right|^{s+m/2}.$$

Here the matrices $a, b, x, 0$ are $m \times m$ matrices and k is in $\mathbf{K}_E$. We have to show that the integral

$$\int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} f(\eta h, s) dh$$

is finite for s sufficiently large. To that end we use the familiar device of representing f by an integral. Recall the Zeta integral for the trivial representation 1_{mE} of $GL(m, \mathbb{A}_E)$:

$$\int_{GL(m,\mathbb{A}_E)} \phi(t) \mid \det t \mid^{s + \frac{m-1}{2}} dt,$$

where ϕ is a Schwartz-Bruhat function on $M(m \times m, \mathbb{A}_E)$. The integral converges for s sufficiently large and is a holomorphic multiple of the Godement-Jacquet $L-$function $L(s, 1_{mE})$. In fact, it equals $L(s, 1_{mE})$ for a suitable choice of ϕ. We can write:

$$f(g, s) = \frac{1}{L(2s + \frac{m+1}{2}, 1_{mE})} \int_{GL(m,\mathbb{A}_E)} \Phi\left[(0, t)g\right] \mid \det t \mid_E^{2s+m} dt \mid \det g \mid_E^{s+m/2},$$

where Φ is a suitable Schwartz-Bruhat function on $M(2m \times m, \mathbb{A}_E)$. We can write this integral as an integral over M_η:

$$f(g,s) = \frac{1}{L(2s+\frac{m+1}{2}, 1_{mE})} \int_{M_\eta(\mathbb{A})} \Phi\left[(0,1_m)mg\right] \mid \det m \mid_F^{2s+m} dm \mid \det g \mid_E^{s+m/2}.$$

After a change of variables we get, for $h \in H(\mathbb{A})$:

$$f(\eta h,s) = \frac{1}{L(2s+\frac{m+1}{2}, 1_{mE})} \int_{H_\eta(\mathbb{A})} \Phi\left[(0,1_m)\eta h_\eta h\right] \mid \det(h_\eta h) \mid_F^{2s+m} dh_\eta.$$

Combining the integral over $H_\eta(\mathbb{A})$ with the integral over the quotient $H_\eta(\mathbb{A})\backslash H(\mathbb{A})$ we get for our integral:

$$\frac{1}{L(2s+\frac{m+1}{2}, 1_{mE})} \int_{H(\mathbb{A})} \Phi\left[(0,1_m)\eta h\right] \mid \det(h) \mid_F^{2s+m} dh.$$

If

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\Phi\left[(0,1_m)\eta h\right] = \Phi(a-ic, b-id).$$

This is a Schwartz-Bruhat function on $M(m \times m, \mathbb{A}_F)$ and thus the integral is now a Zeta integral for the trivial representation 1_{mF} of $GL(m,\mathbb{A})$. In particular, it converges for s sufficiently large and the Lemma is proved. □

VII. Intertwining periods

Fix a Levi subgroup M of H and a cuspidal representation σ of $M_E(\mathbb{A}_E)$ trivial on A_{P_E}. For $\xi \in \Omega_2(M,M), \varphi \in \mathcal{A}_P(G)_\sigma$ and $\lambda \in (\mathfrak{A}^*_{P,\mathbb{C}})^-_\xi$, we claim that the following *intertwining period* attached to ξ is well-defined:

$$J(\xi,\varphi,\lambda) = \int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda+\rho_P, H_{P_E}(\eta h)\rangle} \left(\int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} \varphi(m\eta h)\, dm \right) dh$$

where $\eta \in G(E)$ is any element satisfying $\eta\bar{\eta}^{-1} = \xi$. Recall that $P_\eta = M_\eta N_\eta$, $H_\eta = \eta^{-1}P_\eta\eta$ and that the center of M_η is contained in the center of M_E. It follows that the inner integral defines a function of h invariant on the left under the subgroup $H_\eta(\mathbb{A})$. The function $h \longmapsto e^{\langle \lambda, H_{P_E}(\eta h)\rangle}$ is also left-invariant under $H_\eta(\mathbb{A})$, for if $x \in H_\eta(\mathbb{A})$, then

$$\langle \lambda, H_{P_E}(\eta x h)\rangle = \langle \lambda, H_{P_E}(y\eta h)\rangle = \langle \lambda, H_{P_E}(y)\rangle + \langle \lambda, H_{P_E}(\eta h)\rangle$$

where $y = \eta x \eta^{-1} \in P_\eta(\mathbb{A})$. But $\langle \lambda, H_{P_E}(y)\rangle = 0$ since $H_{P_E}(y)$ lies in the subspace $\mathfrak{A}^+_P$ of $\mathfrak{A}_P$ fixed by ξ for all $y \in P_\eta(\mathbb{A})$. Moreover, the function $h \longmapsto e^{\langle \rho_P, H_{P_E}(\eta h)\rangle}$ is the right modulus of the subgroup $H_\eta(\mathbb{A})$ (cf. the proof of Proposition 22) while the group $H(\mathbb{A})$ is unimodular. Finally it depends only on the double coset $P(F)\eta H(F)$ corresponding to ξ. We prove below that the integral defining $J(\xi,\varphi,\lambda)$ converges absolutely for λ in a suitable cone and can be analytically continued to a meromorphic function of $\lambda \in (\mathfrak{A}^*_P)^-_\xi$.

Example 2. In the special case where $P = B$ and $\xi = e$, we may take $\eta = e$. Note that λ is necessarily 0. In this case, $H_\eta = P$, $M_\eta = M$, and the Iwasawa decomposition yields

$$J(e, \varphi, \lambda) = \mathrm{vol}(M(F)\backslash M(\mathbb{A})^1) \int_{\mathbf{K}_H} \varphi(k)\ dk$$

where $\mathbf{K}_H$ is a maximal compact subgroup of $H(\mathbb{A})$.

Example 3. Let the notation be as in Theorem 23. Then we find:

$$\Pi^{G/H}(E(\varphi, \lambda)) = \int_{H_\eta(F)\backslash H(\mathbb{A})^1} e^{\langle \lambda+\rho_P, H_{P_E}(\eta h)\rangle} \varphi(\eta h)\ dh = J(\xi, \varphi, \lambda)$$

where $H_\eta = \eta^{-1} M_\eta \eta$ and $M_\eta = \{diag(g, \overline{g}) : g \in GL_m(E)\}$.

16. **Minimal involutions.** To investigate the functionals $J(\xi, \varphi, \lambda)$, we use induction beginning with the minimal involutions, defined as follows.

Definition 2. We say that $\xi \in \Omega_2(M, M)$ is *minimal* if $(\mathfrak{A}_P^*)_\xi^-$ is spanned by simple roots in Δ_P.

This definition as well as several results below apply to groups other than $GL(n)$. In our case, P corresponds to a partition $(n_1, \dots, n_r)$ of n such that $M \simeq GL(n_1) \times \cdots \times GL(n_r)$. An involution $\xi \in \Omega_2(M, M)$ permutes the factors of M and hence induces a permutation of order two on $\{1, 2, \dots, r\}$. It is easy to see that ξ is minimal if and only if the permutation is a product of disjoint transpositions of the form $(j, j + 1)$.

Let Y be the set of $j \in \{1, 2, \dots, r-1\}$ such that $n_j = n_{j+1}$. For each subset $X \subset Y$ such that $j + 1 \notin X$ if $j \in X$, there is a unique minimal involution ξ that interchanges $GL(n_j)$ and $GL(n_{j+1})$ for $j \in X$ and fixes the remaining factors. Let Q be the parabolic subgroup containing P that corresponds to the partition obtained from $(n_1, \dots, n_r)$ by replacing the pair of entries n_j, n_{j+1} by the single entry $n_j + n_{j+1}$ for $j \in X$. In other words, Δ_P^Q spans $(\mathfrak{A}_P^*)_\xi^-$. Then Q uniquely determines X and we may denote the minimal involution associated to X by ξ_Q.

Let Ξ be an associate class of standard Levi subgroups. Since each $M \in \Xi$ lies in a unique standard parabolic subgroup P, there is no ambiguity if we use the index M instead of P. For example, we shall write Δ_M for Δ_P, etc. Let Φ_M be the set of roots of A_P and let Φ_M^+ be the subset of positive roots (those that occur in N_P where $M = M_P$). For $w \in \Omega(M)$ we define the length $\ell_M(w)$ to be the number of roots in Φ_M^+ sent to negative roots in $\Phi_{wMw^{-1}}$ by w. For all $\alpha \in \Delta_M$, there is an *elementary symmetry* $s_\alpha \in \Omega(M)$ uniquely characterized by the property that α is the only positive root sent to a negative root by s_α (cf. [MW], Section 1.1.7). Furthermore, for $w \in \Omega(M)$ and $\alpha \in \Delta_{wMw^{-1}}$, we have

$$\ell_{\mathrm{M}}(s_\alpha w) = \begin{cases} \ell_M(w) + 1 \text{ if } w^{-1}\alpha > 0, \\ \ell_M(w) - 1 \text{ if } w^{-1}\alpha < 0. \end{cases} \tag{60}$$

17. **Graph of involutions.** We will consider a directed graph Γ whose vertices are elements (ξ, M) where $M \in \Xi$ and $\xi \in \Omega_2(M, M)$. To define the edges, observe that for all $\alpha \in \Delta_M$, the pair $(s_\alpha \xi s_\alpha^{-1}, s_\alpha M s_\alpha^{-1})$ is also a vertex of Γ. We define an edge

$$(\xi, M) \xrightarrow{\alpha} (s_\alpha \xi s_\alpha^{-1}, s_\alpha M s_\alpha^{-1}) \tag{61}$$

provided that $\xi(\alpha) \neq \alpha$. Note that an edge may start and end at the same vertex. By Lemma 28 below, this occurs if and only if $\xi(\alpha) = -\alpha$. Furthermore, if the edge (61) has distinct vertices, then there is also an edge in the opposite direction:

$$(\xi, M) \overset{-s_\alpha \alpha}{\longleftarrow} (s_\alpha \xi s_\alpha^{-1}, s_\alpha M s_\alpha^{-1}).$$

Let Γ^0 be the subgraph whose vertices are the same at Γ, but containing those edges (61) for which $\xi(\alpha) \neq \pm\alpha$. In other words, Γ^0 is obtained from Γ by removing the loops.

Example 4. Consider the case $G = GL(3)$, $\Xi = \{M\}$ the class of the Levi subgroup of the Borel subgroup. Set $\xi_1 = (12)$, $\xi_2 = (23)$, $\xi_3 = (13)$. Then Γ is the following graph with two connected components:

$$\overset{\alpha_2}{\circlearrowleft} (\xi_2, M) \xrightarrow{\alpha_1} (\xi_3, M) \xrightarrow{\alpha_2} (\xi_1, M) \overset{\alpha_1}{\circlearrowleft} \qquad\qquad (e, M).$$

Here we have not drawn the two interior edges directed from right to left. The graph Γ^0 is obtained by removing the loops.

Example 5. Consider the case $G = GL(4)$, $\Xi = \{M\}$ the class of the Levi subgroup of the Borel subgroup. Set $\xi_1 = (12)(34)$, $\xi_2 = (13)(24)$, $\xi_3 = (14)(23)$. Then Γ contains the component

$$\begin{matrix}\overset{\alpha_1}{\circlearrowleft} \\ \overset{\alpha_3}{\circlearrowleft}\end{matrix} (\xi_1, M) \xrightarrow{\alpha_2} (\xi_2, M) \begin{matrix}\xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_3}\end{matrix} (\xi_3, M) \overset{\alpha_2}{\circlearrowleft}.$$

Again, we have not drawn the interior edges directed from right to left.

Each path

$$(\xi_1, M_1) \xrightarrow{\alpha_1} (\xi_2, M_2) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{l-1}} (\xi_\ell, M_\ell)$$

in Γ defines a word $s = s_{\alpha_{\ell-1}} \cdots s_{\alpha_1} \in \Omega(M_1, M_l)$ such that $\xi_\ell = s\xi_1 s^{-1}$ and $M_\ell = sM_1 s^{-1}$. If M_1 and M_2 are standard Levi subgroups and $\xi_i \in \Omega_2(M_i, M_i)$ for $i = 1, 2$, let $\Omega(\xi_1, \xi_2)$ be the set of all words defined by paths from (ξ_1, M_1) to (ξ_2, M_2) in Γ. Let $\Omega^0(\xi_1, \xi_2)$ be defined similarly relative to the graph Γ^0.

Lemma 28. *Assume $\xi \in \Omega_2(M, M)$ and $\alpha \in \Delta_M$. Then*

(1) *The following are equivalent:*
   (i) $s_\alpha \xi s_\alpha^{-1} = \xi$;
   (ii) $\xi(\alpha) = \pm\alpha$;
   (iii) $\ell_{s_\alpha M s_\alpha^{-1}}(s_\alpha \xi s_\alpha^{-1}) = \ell_M(\xi)$.
   Furthermore, if $\xi(\alpha) = \pm\alpha$, then $s_\alpha M s_\alpha^{-1} = M$.

(2) *If $\xi\alpha \neq \pm\alpha$, then the following are equivalent:*
   (i) $\ell_{s_\alpha M s_\alpha^{-1}}(s_\alpha \xi s_\alpha^{-1}) = \ell_M(\xi) - 2$;
   (ii) $\xi(\alpha) < 0$.

Proof. We first prove (1). Since $G = GL(n)$, M is isomorphic to a product $GL(n_1) \times \cdots \times GL(n_r)$ where $(n_1, \dots, n_r)$ is a partition of n. The elementary reflection s_α interchanges two adjacent factors $GL(n_k)$ and $GL(n_{k+1})$ for some k, and ξ interchanges certain pairs of factors of equal size. Therefore $s_\alpha \xi s_\alpha^{-1} = \xi$ if and only if either ξ fixes the factors $GL(n_k)$ and $GL(n_{k+1})$ (in which case $\xi\alpha = \alpha$) or interchanges them (in which case $\xi\alpha = -\alpha$). Therefore (i) and (ii) are equivalent. By (60), (iii) occurs if and only if either $\xi(\alpha) > 0$ and $s_\alpha\xi(\alpha) < 0$, or $\xi(\alpha) < 0$ and $s_\alpha\xi(\alpha) > 0$. But this is the case precisely when $\xi(\alpha) = \pm\alpha$. Therefore (ii) and (iii) are equivalent. To prove part (2), observe that $\ell_{s_\alpha M s_\alpha^{-1}}(s_\alpha \xi s_\alpha^{-1}) = \ell_M(\xi) - 2$ if and only if $\xi^{-1}\alpha$ and $s_\alpha \xi^{-1}\alpha$ are both negative, but if $\xi\alpha \neq \pm\alpha$, this is equivalent to the condition $\xi^{-1}\alpha < 0$. We remark that (1)(ii),(iii) and (2)(i),(ii) are equivalent for general groups but (1)(i) is sharper than (1)(ii) in general (unless $P = P_0$). □

This lemma implies, in particular, that an edge begins and ends at (ξ, M) if and only if $\xi(\alpha) = -\alpha$. Therefore Γ^0 is the graph obtained from Γ by removing the loops.

We shall use the following characterization of minimal involutions.

Proposition 29. *Let $\xi \in \Omega_2(M, M)$. Then*

1. *ξ is minimal if and only if the length function ℓ has a local minimum at ξ (i.e., $\ell_M(\xi) \leq \ell_{M'}(\xi')$ for all neighbors (ξ', M') of (ξ, M) in Γ).*
2. *ξ is conjugate to a minimal involution by an element in $\Omega(M)$.*
3. *$(\mathfrak{A}_P^*)_\xi^-$ is spanned by roots in Φ_P.*

Proof. Suppose that ξ is a local minimum for ℓ on Γ. Lemma 28 implies that $\xi(\alpha) = -\alpha$ whenever $\alpha \in \Delta_M$ and $\xi(\alpha) < 0$. Let $I = \{\alpha \in \Delta_M : \xi(\alpha) = -\alpha\}$. If $\beta \in \Phi_M^+$ is any root which is not in the span of the roots in I, then $\xi(\beta) > 0$ because $\xi(\beta)$ contains positive multiples of simple roots outside I. Therefore $(\mathfrak{A}_M^*)_\xi^-$ is contained in the span of I and hence coincides with the span of I. Therefore ξ is a minimal involution in $\Omega_2(M, M)$.

On the other hand, if ξ is minimal in $\Omega_2(M, M)$, then for any $0 < \beta \notin (\mathfrak{A}_M^*)_\xi^-$ we have $\xi(\beta) > 0$. Indeed, we may decompose β as $\beta_\xi^+ + \beta_\xi^-$ where $\xi\beta_\xi^\pm = \pm\beta_\xi^\pm$. Since $\beta_\xi^+ \neq 0$, it contains in its decomposition relative to Δ_M some positive multiple of a simple root α such that $\xi\alpha > 0$, and this multiple is also present in the decomposition of $\xi(\beta) = \beta_\xi^+ - \beta_\xi^-$. In particular, if $\beta \in \Delta_M$ and $\xi(\beta) < 0$, then $\xi\beta = -\beta$. It follows from Lemma 28 that ℓ has a local minimum at ξ.

Part (2) follows immediately from (1) by the description of the edges in Γ. Part (3) follows from (2). □

Part (1) of the previous proposition implies

Corollary 30. *Let $\xi \in \Omega_2(M, M)$. If (ξ, M) is not minimal, then there exists $\alpha \in \Delta_M$ such that the element $\xi' = s_\alpha \xi s_\alpha^{-1}$ satisfies $\ell(\xi') = \ell(\xi) - 2$.*

18. **The functional equations.** Fix a parabolic subgroup $P = MN$ and $\xi \in \Omega_2(M, M)$. Let Φ_P^+ be the set of positive roots of A_P in N_P and set

$$\Phi_P^\xi = \{\beta \in \Phi_P^+ : \xi\beta < 0\}.$$

We shall set

$$D_{P,\xi} = \{\lambda \in (\mathfrak{A}_P^*)_\xi^- : \langle \lambda, \beta^\vee \rangle >> 0 \ \text{ for } \beta \in \Phi_P^\xi\}$$

where $>>$ denotes *sufficiently large*, but we shall not specify the implied constants. However, observe that $D_{P,\xi}$ is non-empty because it contains $P_\xi^-\lambda$ if λ is sufficiently positive in $\mathfrak{A}_P^{*+}$. Indeed, if $\beta \in \Phi_P^+$, then $P_\xi^-\beta^\vee = \frac{1}{2}(\beta^\vee - \xi\beta^\vee)$ is a sum of positive roots and hence

$$\left\langle P_\xi^-\lambda, \beta^\vee \right\rangle = \left\langle \lambda, P_\xi^-\beta^\vee \right\rangle >> 0.$$

Our main result in this section is the following

Theorem 31. *Let $\xi \in \Omega_2(M, M)$. Then*

(1) *$J(\xi, \varphi, \lambda)$ is defined by an absolutely convergent integral for* $\mathrm{Re}\,\lambda \in D_{P,\xi}$.
(2) *$J(\xi, \varphi, \lambda)$ has a meromorphic continuation to $(\mathfrak{A}_P^*)_\xi^- \otimes \mathbb{C}$.*
(3) *If $s \in \Omega(\xi_1, \xi_2)$, then*

$$J(\xi_1, \varphi, \lambda) = J(\xi_2, M(s, \lambda)\varphi, s\lambda). \tag{62}$$

In the case of minimal involutions, parts (1) and (2) of this Theorem as well as certain cases of (3) follow from the previous section. Indeed,

Lemma 32. *If $\xi = \xi_Q$ is minimal and $Q = Q_\xi$, then $(\mathfrak{A}_P^*)_\xi^- = (\mathfrak{A}_P^Q)^*$ and $D_{P,\xi}$ is the half-space*

$$D_{P,\xi} = \{\lambda \in (\mathfrak{A}_P^Q)^* : \langle \mathrm{Re}\,\lambda, \alpha^\vee \rangle >> 0 \text{ for all } \alpha \in \Delta_P^Q\}.$$

In this case,

(1) *The integral defining $J(\xi_Q, \varphi, \lambda)$ is absolutely convergent if* $\mathrm{Re}\,\lambda \in D_{P,\xi}$.
(2) *$J(\xi_Q, \varphi, \lambda)$ has a meromorphic continuation to $(\mathfrak{A}_P^Q)^* \otimes \mathbb{C}$.*
(3) *$J(\xi_Q, \varphi, \lambda) = J(\xi_Q, M(s_\alpha, \lambda)\varphi, s_\alpha\lambda)$ for all $\alpha \in \Delta_P^Q$.*

Proof. We can take $\eta = \eta_Q \in M_Q(E)$. Then $H(\mathbb{A}) = N_Q(\mathbb{A})M_Q(\mathbb{A})\mathbf{K}_H$ with $dh = e^{-\langle \rho_Q, H_E(m)\rangle} dndmdk$ and $H_\eta = H_{\eta,Q}N_Q$ where $H_{\eta,Q} = H_\eta \cap M_Q$. Therefore, since $\rho_P = \rho_Q + \rho_P^Q$

$$J(\xi_Q, \phi, \lambda)$$

$$= \int_{H_{n,Q}(\mathbb{A})N_Q(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda + \rho_P, H_{P_E}(\eta_Q h)\rangle} \left(\int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} \phi(l\eta_Q h) dl \right) dh$$

$$= \int_{H_{n,Q}(\mathbb{A})\backslash M_Q(\mathbb{A})} e^{\langle \lambda + \rho_P^Q, H_{P_E}(\eta_Q m)\rangle} \left(\int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} \phi^{\mathbf{K}_F}(l\eta_Q h) dl \right) dm$$

$$= J^{M_Q}(\xi_Q, \phi^{\mathbf{K}_F}, \lambda_P^Q),$$

where we set

$$\phi^{\mathbf{K}_F}(g) = \int_{\mathbf{K}_F} \phi(gk) dk.$$

By Proposition 26 (applied to a product of linear groups, namely M_{Q_E}) this integral is absolutely convergent in the indicated domain and equal to the period $\Pi^{M_{Q_E}/M_Q}\left(E(\phi^{\mathbf{K}_H}, \lambda_P^Q)\right)$ of the Eisenstein series on M_{Q_E} for the parabolic subgroup $P \cap M_{Q_E}$. This shows that the integral extends to a meromorphic function. The stated functional equation is inherited from the functional equation of the Eisenstein series. □

The next proposition provides functional equations that can be verified by direct calculation.

Proposition 33. *Let $\xi \in \Omega_2(M, M)$ and let $\alpha \in \Delta_M$ be a root such that $\xi(\alpha) < 0$ but $\xi(\alpha) \neq -\alpha$. Then for $\lambda \in D_{P,\xi}$ we have an equality*

$$J(\xi, \varphi, \lambda) = J(s_\alpha \xi s_\alpha^{-1}, M(s_\alpha, \lambda)\varphi, s_\alpha \lambda) \tag{63}$$

where both sides are defined by absolutely convergent integrals.

Proof. We first establish the integral formula (66) below. Let w_α be a representative for s_α and set:

$$\xi' = w_\alpha \xi w_\alpha^{-1}, \quad \eta' = w_\alpha \eta, \quad M' = w_\alpha M w_\alpha^{-1}.$$

Then M' is the standard Levi-subgroup of a (standard) parabolic subgroup $P' = M'N'$. We have $s_\alpha \alpha = -\alpha'$ with $\alpha' \in \Delta_{P'}$. We also introduce the parabolic subgroup R such that $R \supset P$ and $\Delta_P^R = \{\alpha\}$. Note that $R \supset P'$ and $\Delta_{P'}^R = \{\alpha'\}$. The standard Levi-decomposition of R reads $R = M_R N_R$ and w_α belongs to M_R and hence normalizes N_R. On the other hand, we have $N = N_\alpha N_R$ and $N' = N_{\alpha'} N_R$. Here we denote by N_α the abelian subgroup exponential of the eigenspace of the Lie algebra corresponding to α. Likewise for $N_{\alpha'}$.

Define an involution θ by $\theta(x) = (\xi')^{-1}\overline{x}'\xi'$. The subgroup $N'_{\eta'}$ is the set of fixed points of θ in N'. We claim that

$$w_\alpha N_\eta w_\alpha^{-1} = N'_{\eta'} \cap N_R. \tag{64}$$

To prove this, observe first that N_η is contained in N_R. Indeed, N_η is the space of $u \in N$ such that $\xi^{-1} u \xi = \overline{u}$ and therefore is contained in $N \cap \xi^{-1} N \xi$. However, $N \cap \xi^{-1} N \xi$ is contained in N_R, for if not, then N_α would intersect $N \cap \xi^{-1} N \xi$ non-trivially and then $\xi N_\alpha \xi^{-1} \subset N$, contradicting our assumption that $\xi(\alpha) < 0$. Since w_α normalizes N_R, we also conclude that $w_\alpha N_\eta w_\alpha^{-1}$ is contained in N_R. If an element x satisfies $\xi^{-1} x \xi = \overline{x}$, then the element $x' = w_\alpha x w_\alpha^{-1}$ is fixed by θ. Therefore $w_\alpha N_\eta w_\alpha^{-1}$ is contained in $N_R \cap N'_{\eta'}$. Similarly we find that $w_\alpha^{-1}(N_R \cap N'_{\eta'}) w_\alpha$ is contained in N_η and the equality (64) follows.

The involution θ maps $N_{\alpha'}$ to N_R. Indeed, if $u \in N_{\alpha'}$, then

$$\theta(u) = w_\alpha \xi^{-1} w_\alpha^{-1} \overline{u} w_\alpha \xi w_\alpha^{-1}.$$

The element $w_\alpha^{-1} \overline{u} w_\alpha$ is in $N_{-\alpha}$ and thus $\xi^{-1} w_\alpha^{-1} \overline{u} w_\alpha \xi$ lies in some N_β with $\beta > 0$, $\beta \neq \alpha$. It follows that $\beta' = s_\alpha \beta$ is a positive root in $\Delta_{P'}$ different from α' and hence $\theta(u)$ lies in $N_{\beta'} \subset N_R$.

We now claim that given any element $u \in N_{\alpha'}$, there exists $s(u) \in N_R$, unique modulo $N'_{\eta'} \cap N_R$, such that $u\theta(u)s(u) \in N'_{\eta'}$. Furthermore, the map

$$\begin{array}{ccc} N_{\alpha'} & \longrightarrow & N'_{\eta'}/N'_{\eta'} \cap N_R \\ u & \mapsto & u\theta(u)s(u) \ \left(N'_{\eta'} \cap N_R\right) \end{array} \tag{65}$$

defines an algebraic group isomorphism (locally, globally), whose inverse is the projection on the M_R part. To prove the claim, observe that the element $t = u^{-1}\theta(u)^{-1}u\theta(u)$ belongs to N_R and satisfies $\theta(t)t = 1$. Hence $t = \theta(s)s^{-1}$ for some $s \in N_R$ which is unique modulo $N'_{\eta'} \cap N_R$. Since $u\theta(u)s$ is fixed by θ, we may set $s(u) = s$.

The isomorphism (65) can be used to compute the Haar measure on $N'_{\eta'}$ (locally or globally) as:

$$\int_{N'_{\eta'}} f(z)dz = \int_{N_{\alpha'}} \left(\int_{N'_{\eta'} \cap N_R} f(u\theta(u)s(u)n)dn \right) du. \tag{66}$$

To continue the proof, we start with the intertwining period

$$J(s_\alpha \xi s_\alpha^{-1}, M(s_\alpha, \lambda)\phi, s_\alpha \lambda),$$

which is equal to

$$\int_{H_{\eta'}(\mathbb{A})\backslash H(\mathbb{A})} \int_{M_{\eta'}(F)\backslash M_{\eta'}(\mathbb{A})^1} \int_{N_{\alpha'}(\mathbb{A})} e^{\left\langle \lambda+\rho_P, H_{P_E}(w_\alpha^{-1} nm'\eta' h)\right\rangle} \phi(w_\alpha^{-1} nm'\eta' h) dn dm' dh.$$

We will check below that this triple integral is absolutely convergent. The integrand, viewed as a function of $n \in N'(\mathbb{A})$, is left-invariant under $N_R(\mathbb{A})$. It follows that the integral over $N_{\alpha'}(\mathbb{A})$ can be rewritten as an integral over the quotient

$$w_\alpha N_\eta w_\alpha^{-1}(\mathbb{A})\backslash N'_{\eta'}(\mathbb{A}).$$

Now we have $P_{\eta'} = M_{\eta'} N'_{\eta'}$ and $M_{\eta'} = w_\alpha M_\eta w_\alpha^{-1}$. Also $N'_{\eta'} \supset w_\alpha N_\eta w_\alpha^{-1}$. Thus $P_{\eta'} = w_\alpha P_\eta w_\alpha^{-1} N'_{\eta'}$ with $N'_{\eta'}$ normal and the intersection of the two subgroups is $w_\alpha N_\eta w_\alpha^{-1}$. Conjugating by $(\eta')^{-1}$, we obtain

$$H_{\eta'} = H_\eta \left(\eta'^{-1} N'_{\eta'} \eta' \right).$$

The second group is normal and the intersection of the two groups is $\eta^{-1} N_\eta \eta$. Using the fact that m' normalizes both $N'_{\eta'}(\mathbb{A})$ and $w_\alpha N_\eta w_\alpha^{-1}(\mathbb{A})$ and conjugation by m' does not change the Haar measures, we see that

$$J(s_\alpha \xi s_\alpha^{-1}, M(s_\alpha, \lambda)\phi, s_\alpha \lambda)$$

can be written as

$$\int\int\int \; e^{\langle \lambda+\rho, H_{P_E}(w_\alpha^{-1} m'\eta' uh)\rangle} \; \phi(w_\alpha^{-1} m'\eta' uh) \; dm' du dh.$$

The integral in the h variable is taken over $H_{\eta'}(\mathbb{A})\backslash H(\mathbb{A})$ and the integral in the u variable is taken over $\eta^{-1} N_\eta \eta(\mathbb{A}) \backslash (\eta')^{-1} N'_{\eta'} \eta'(\mathbb{A})$. We may now combine the integrations over u and h into an integration over $H(\mathbb{A})$ modulo $H_\eta(\mathbb{A})$ to obtain

$$\int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} \int_{M_{\eta'}(F)\backslash M_{\eta'}(\mathbb{A})^1} e^{\langle \lambda+\rho_P, H_{P_E}(w_\alpha^{-1} m'\eta' h)\rangle} \; \phi(w_\alpha^{-1} m'\eta' h) \; dm' dh.$$

Finally, using that $w_\alpha^{-1} m'\eta' = m\eta$ with $m \in M_\eta$, we obtain equation (63) as desired.

It remains to verify that the triple integral defining $J(\xi', M(s_\alpha, \lambda)\phi, s_\alpha\lambda)$ is absolutely convergent. To this end, we take the variable λ to be real and ϕ to be the constant 1, and define a "scalar" intertwining period:

$$j(\xi, \lambda) = \mathrm{vol}(M_\eta(F)\backslash M_\eta^1(\mathbb{A})) \int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda+\rho_P, H_{P_E}(\eta h)\rangle} dh$$

(with positive integrand). Computing formally as above, we find a formula:

$$j(\xi, \phi) = j(s_\alpha \xi s_\alpha^{-1}, s_\alpha \lambda) m(s_\alpha, \lambda)$$

where

$$m(s_\alpha,\lambda)=\int_{N_{\alpha'}(\mathbb{A})} e^{\langle\lambda+\rho_P,H_{P_E}(w_\alpha^{-1}n)\rangle}dn$$

is a scalar intertwining operator. The integral defining $m(s_\alpha,\lambda)$ is known to converge absolutely in a domain

$$C=\{\lambda:\ \langle\lambda,\alpha^\vee\rangle >> 0\}\,.$$

By Lemma 27 above, we know that if ξ is minimal, then $j(\xi,\lambda)$ converges in some domain $D_{P,\xi}$. By induction on the length of ξ we may assume that $j(\xi',\lambda')$ converges in $D_{P',\xi'}$. Now

$$D_{P,\xi}\subset s_\alpha^{-1}D_{P',\xi}\cap C.$$

Thus for $\lambda\in D_{P\ ,\xi}$ we find that both $j(s_\alpha\xi s_\alpha^{-1},s_\alpha\lambda)$ and $m(s_\alpha,\lambda)$ are defined by absolutely convergent integrals and our assertion follows: the integral $j(\xi,\lambda)$ is absolutely convergent for $\lambda\in D_{P\ ,\xi}$. The same is therefore true of $J(\xi,\phi,\lambda)$, and the proof is complete. □

We now prove parts (1) and (2) of Theorem 31 by induction on $\ell_M(\xi)$. If ξ is not minimal, there exists a root $\alpha\in\Delta_M$ such that $\xi(\alpha)<0$ and $\xi'=s_\alpha\xi s_\alpha^{-1}$ has length $\ell(\xi)-2$ by Corollary 30. Let P' be the parabolic subgroup with $\mathfrak{A}^*_{P'}=s_\alpha A^*_P$. Then

$$D_{P,\xi}=s_\alpha^{-1}\left(D_{P',\xi'}\right)\cap\left\{\lambda\in(\mathfrak{A}^*_P)^-_\xi:\langle\lambda,\alpha^\vee\rangle\,,\langle\lambda,-\xi\alpha^\vee\rangle>>0\right\}$$

since $\Phi^\xi_P=s_\alpha\Phi^{\xi'}_{P'}\cup\{\alpha,-\xi(\alpha)\}$. We may assume by induction that the integral defining $J(\xi',\varphi,\lambda)$ converges absolutely in $D_{\xi'}$ and has a meromorphic continuation. Then $J(\xi,\varphi,\lambda)$ converges absolutely in D_ξ and has a meromorphic continuation by Proposition 33.

It remains to prove part (3) of Theorem 31. The functional equations corresponding to $s\in\Omega^0(\xi,\xi')$ are a consequence of Proposition 33 and the functional equations $M(s_\alpha^{-1},s_\alpha\lambda)M(s_\alpha,\lambda)=I$. It remains to examine the case of a loop. So assume that $\xi=s_\alpha\xi s_{\alpha^{-1}}$ where α is a root such that $\xi\alpha=-\alpha$. By Lemma 29, (2), there exists a minimal involution ξ' such that $\xi'=w\xi w^{-1}$ with $w\in\Omega^0(\xi,\xi')$, i.e., w is equal to a word associated to a path in Γ^0 (start with any $w\in\Omega(\xi,\xi')$ and remove loops in the path attached to w). Since $s_\alpha\xi s_\alpha^{-1}=\xi$, the element $s=ws_\alpha w^{-1}$ lies in $\Omega(\xi',\xi')$. In fact, since s acts trivially on $(\mathfrak{A}^*_{P'})^+_{\xi'}$ and ξ' is minimal, s must be a product of commuting elementary symmetries s_γ for roots γ contained in $(\mathfrak{A}^*_{P'})^-_{\xi'}$. We deduce that

$$\begin{aligned}J(\xi,M(s_\alpha,\lambda)\varphi,s_\alpha\lambda)&=J(\xi',M(w,s_\alpha\lambda)M(s_\alpha,\lambda)\varphi,ws_\alpha\lambda)\\&=J(\xi',M(ws_\alpha,\lambda)\varphi,ws_\alpha\lambda)\end{aligned}$$

by the functional equation $M(ws_\alpha,\lambda)=M(w,s_\alpha\lambda)M(s_\alpha,\lambda)$. On the other hand, by Lemma 32

$$\begin{aligned}J(\xi',M(ws_\alpha,\lambda)\varphi,ws_\alpha\lambda)&=J(\xi',M(sws_\alpha,\lambda)\varphi,sws_\alpha\lambda)\\&=J(\xi',M(w,\lambda)\varphi,w\lambda).\end{aligned}$$

Finally $J(\xi',M(w,\lambda)\varphi,w\lambda)=J(\xi,\varphi,\lambda)$, again since $w\in\Omega^0(\xi,\xi')$. □

19. **A description of** $\Omega(\xi,\xi')$**.** Since the sets $\Omega(\xi,\xi')$ determine the functional equations of the intertwining periods, it is of interest to have a description of them independent of the graphs. We do this in the next proposition. For $\xi \in \Omega_2(M,M)$, set

$$C_\xi = \{x \in \mathfrak{A}_M^* : \langle x, \beta^\vee\rangle > 0 \text{ for } \beta \in \Phi_M^+ \cap (\mathfrak{A}_M^*)_\xi^+\}.$$

Proposition 34. *Let M and M' be standard Levi subgroups, $\xi \in \Omega_2(M,M)$, and $\xi' \in \Omega_2(M',M')$. Suppose that $w\xi w^{-1} = \xi'$. Then the following are equivalent:*

(a) *$w \in \Omega(\xi,\xi')$.*
(b) *$w \in \Omega(M,M')$ and $wC_\xi = C_{\xi'}$.*
(c) *$w \in \Omega(M,M')$ and $w^{-1}\beta > 0$ for all $\beta \in \Phi_{M'}^+ \cap (\mathfrak{A}_{M'}^*)_{\xi'}^+$.*

Proof. The equivalence of (b) and (c) follows immediately from the definitions. To prove the equivalence of (a) and (c), assume first that $w = s_\alpha$ is an elementary symmetry. Then $w^{-1}\beta > 0$ for all $\beta \in \Phi_{M'}^+ \cap (\mathfrak{A}_{M'}^*)_{\xi'}^+$ if and only if $\alpha \notin (\mathfrak{A}_M^*)_\xi^+$. But $\alpha \notin (\mathfrak{A}_M^*)_\xi^+$ if and only if $\xi \xrightarrow{\alpha} \xi'$. Notice that when (c) holds, w^{-1} maps $\Phi_{M'}^+ \cap (\mathfrak{A}_{M'}^*)_{\xi'}^+$ bijectively to $\Phi_M^+ \cap (\mathfrak{A}_M^*)_\xi^+$.

Now assume (a) holds. Then we may write w as a product $w = s_{\alpha_k}\cdots s_{\alpha_1}$ of elementary symmetries arising from a path in Γ. Set $w_j = s_{\alpha_{j-1}}\cdots s_{\alpha_1}$, $\xi_j = w_j\xi w_j^{-1}$and $M_j = w_jMw_j^{-1}$. By induction, w_k^{-1} maps $\Phi_{M_k}^+ \cap (\mathfrak{A}_{M_k}^*)_{\xi_k}^+$ bijectively to $\Phi_M^+ \cap (\mathfrak{A}_M^*)_\xi^+$ and $s_{\alpha_k}^{-1}$ maps $\Phi_{M'}^+ \cap (\mathfrak{A}_{M'}^*)_{\xi'}^+$ bijectively to $\Phi_{M_k}^+ \cap (\mathfrak{A}_{M_k}^*)_{\xi_k}^+$. Hence w^{-1} maps $\Phi_{M'}^+ \cap (\mathfrak{A}_{M'}^*)_{\xi'}^+$ bijectively to $\Phi_M^+ \cap (\mathfrak{A}_M^*)_\xi^+$ and (c) holds.

Finally, assume (c) holds and let $w = s_{\alpha_k}\cdots s_{\alpha_1}$ be a reduced decomposition of w. Set $w_j = s_{\alpha_{j-1}}\cdots s_{\alpha_1}$. Since the decomposition is reduced, $w_j^{-1}\alpha_j > 0$ and $ww_j^{-1}\alpha_j < 0$ for $j = 1, \ldots, k$ ([MW], p. 14). In particular, the root $\beta = -s_{\alpha_k}\alpha_k$ lies in $\Phi_{M'}^+$ and $w^{-1}\beta = -w_k^{-1}\alpha_k < 0$. Our hypothesis implies that $\beta \notin (\mathfrak{A}_{M'}^*)_{\xi'}^+$. Therefore $\xi_k \xrightarrow{\alpha_k} \xi'$ is an edge in Γ. It also follows that w_k satisfies condition (c) with M_k in place of M' and, by induction, $w_k \in \Omega(\xi, w_k\xi w_k^{-1})$. Hence $w = s_{\alpha_k}w_k$ belongs to $\Omega(\xi,\xi')$ as required. ☐

Corollary 35. *Let $w \in \Omega(\xi,\xi')$. Then every reduced decomposition $w = s_{\alpha_k}\cdots s_{\alpha_1}$ is obtained from a word in the graph Γ.*

Remark 4. Assume that $M = M_0$ and let $\{\beta_1, \ldots, \beta_r\}$ be the positive roots contained in $(\mathfrak{A}_M^*)_\xi^-$. It is straightforward to check that $\Omega(\xi,\xi) \cong (\mathbb{Z}/2)^r \rtimes S_r$. The group $\Omega(\xi,\xi)$ acts by signed permutations on $(\mathfrak{A}_0^*)_\xi^-$, i.e., S_r permutes the β_j and the j^{th} factor of $(\mathbb{Z}/2)^r$ acts by sending β_j to $-\beta_j$. The subgroup Ω_ξ^+ of Ω generated by reflections about roots contained in $(\mathfrak{A}_0^*)_\xi^+$ is isomorphic to S_{n-2r} and

$$\{w \in \Omega : w(\mathfrak{A}_0^*)_\xi^- = (\mathfrak{A}_0^*)_\xi^-\} = \Omega(\xi,\xi) \times \Omega_\xi^+.$$

It is sufficient to check these facts for $\xi = (1,2)\cdots(2r-1,2r)$.

20. **Unramified computations.** In this section we consider a local analogue of the intertwining period in the unramified case. Assume that $n = 2m$ and let $P = MN$ be the standard parabolic subgroup of type (m,m). We use the notation of Section VI. Thus ξ is the non-trivial element in $\Omega(M,M)$ and $\eta \in GL_n(E)$ is such that $\eta\overline{\eta}^{-1} = \xi$. The subgroup M_η consists of the matrices $diag(h,\overline{h})$ where $h \in GL_m(E)$.

Assume that $\sigma = \sigma_1 \otimes \sigma_2$ is a cuspidal representation of $M_E(\mathbb{A}_E)$ trivial on A_M such that $\sigma_2 \simeq \overline{\sigma}_1^*$. There is a unique (up to scalars) non-zero linear form L' on the space of σ which is invariant under $M_\eta(\mathbb{A})$, namely

$$L'(\varphi) = \int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} \varphi(m)dm.$$

For $j = 1, 2$, we choose an identification of σ_j with a restricted tensor product $\otimes' \sigma_{jv}$ where the product is over places v of F. This identification presupposes the choice of K_v-fixed vectors x_{jv} in the space of σ_{jv} for almost all v. Thus, σ_{jv} is a representation of $GL_m(E_v)$ where $E_v = E \otimes F_v$. We set $\sigma_v = \sigma_{1v} \otimes \sigma_{2v}$. Since $\sigma_{2v} \simeq \overline{\sigma}_{1v}^*$, there exists a non-zero linear map

$$L_v' : \sigma_{1v} \otimes \sigma_{2v} \longrightarrow \mathbb{C}$$

invariant under M_η. It is also unique up to scalar multiples. We may assume that $L_v'(x_{1v} \otimes x_{2v}) = 1$ for almost all v. Then for a suitable normalization of L', we have

$$L'(\varphi) = \prod_v L_v'(\varphi_v)$$

whenever φ corresponds to a pure tensor $\otimes \varphi_v$.

Set

$$\pi_v = \mathrm{Ind}_{P_v}^{G_v} \sigma_v, \quad \pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \sigma$$

and let $V(\pi_v)$ and $V(\pi)$ denote the space of π_v and π, respectively. If $\varphi_v \in V(\pi_v)$, then $\varphi_v(g)$ lies in the space of $\sigma_{1v} \otimes \sigma_{2v}$ for all $g \in G_v$ and hence we may define a function $L_v(\varphi_v)(g)$ whose value is the image of $\varphi_v(g)$ under L_v'. This function is left-invariant under $M_\eta(E_v)$. Similarly, for φ belonging to the space $V(\pi)$ of π, we may set $L(\varphi)(g)$ equal to the image of $\varphi(g)$ under L'. Then

$$L(\varphi)(g) = \prod_v L_v(\varphi_v)(g_v).$$

In the global theory we used unnormalized induction. However, it will be more convenient to us to use normalized induction for the local computation. Thus, we now have

$$J(\xi, \varphi, \lambda) = \int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda, H_{P_E}(\eta h)\rangle} \ L(\varphi)(\eta h)\ dh.$$

(The ρ_P is already absorbed in φ because we use normalized induction.) Define the following *local intertwining period:*

$$J_v(\xi, \varphi_v, \lambda) = \int_{H_\eta(F_v)\backslash H(F_v)} e^{\langle \lambda, H_{P_E}(\eta h_v)\rangle} \ L_v(\varphi_v)(\eta h_v)\ dh_v$$

for $\lambda \in (\mathfrak{A}_P^*)_\xi^-$. Then we have the factorization (for a suitable normalization of measures):

$$J(\xi, \varphi, \lambda) = \prod_v J_v(\xi, \varphi_v, \lambda).$$

In this sense, the global intertwining period is equal to the product of the local intertwining periods. We shall now compute the local factor $J_v(\xi, \varphi_v, \lambda)$ in the unramified case.

For the rest of this section, assume that v is a finite place such that σ_{jv} is unramified for $j = 1, 2$. We shall show below that the integral converges absolutely for $\mathrm{Re}\,\lambda >> 0$ and compute its value assuming that φ_v is a K_v-fixed vector in $V(\pi_v)$. To this end, we recall the definition of the Asai L-function of an unramified representation of $GL_m(E)$ ([AS], [F1]). We view $GL(m)_E$ as a group over F. Its L-group is then

$$ {}^L GL(m)_E = GL_m(\mathbb{C}) \times GL_m(C) \rtimes Gal(E/F) $$

where the non-trivial element $\sigma_{E/F} \in Gal(E/F)$ acts on the connected component by interchanging the factors. Let $V = \mathbb{C}^m$ and let T be the automorphism of $V \otimes V$ sending $x \otimes y$ to $y \otimes x$. We identify $GL(V \otimes V)$ with $GL_{m^2}(\mathbb{C})$ and define

$$ \rho_A : \ {}^L GL(m, E) \longrightarrow \ GL_{m^2}(\mathbb{C}), $$

where $\rho_A(g \times h \times 1) = g \otimes h$ and $\rho_A(1 \times 1 \times \sigma_{E/F}) = T$. Write $\omega_{E/F}$ for the character of ${}^L GL(m)_E$ obtained by pulling back the non-trivial character of $Gal(E/F)$. Then $\rho_A \otimes \omega_{E/F}$ is also defined. If σ_v is an unramified representation, we may define the local Langlands L-factors $L(\lambda, \sigma_v, \rho_A)$ and $L(\lambda, \sigma_v, \rho_A \otimes \omega_{E/F})$ attached to ρ_A and $\rho_A \otimes \omega_{E/F}$ respectively. Note that the direct sum $\rho_A \oplus \rho_A \otimes \omega_{E/F}$ is isomorphic to the representation of ${}^L GL(m)_E$ induced from the tensor product representation of $GL_m(\mathbb{C}) \times GL_m(C)$. Since formation of L -functions is invariant under induction, we obtain

$$ L(\sigma_v, s, \rho_A)\, L(\sigma_v, s, \rho_A \otimes \omega_{E/F}) = L(s, \sigma_v \times \overline{\sigma_v}) $$

where $L(s, \sigma_v \times \overline{\sigma_v})$ is the Rankin-Selberg convolution of σ_v and $\overline{\sigma_v}$ ([JS]).

Actually, we shall need to use the *contragredient* representation ρ_A^*. Let us write down the local factors explicitly. Recall that σ_v is a representation of $GL_m(E_v)$ where $E_v = E \otimes F_v$. If v splits in E, then $GL_m(E_v) = GL_m(E_{w_1}) \times GL_m(E_{w_2})$ and $\sigma_v = \sigma_{w_1} \otimes \sigma_{w_2}$ where w_1 and w_2 are the place of E dividing v. For $j = 1, 2$, σ_{w_j} corresponds to a Langlands class $g(\sigma_{w_j}) \in GL_m(\mathbb{C})$ and the Langlands class of σ_v in ${}^L GL(m, E)$ is $g(\sigma_v) = g(\sigma_{w_1}) \times g(\sigma_{w_2}) \times 1$. In this case,

$$ L(\lambda, \sigma_v, \rho_A^*) = L(\lambda, \sigma_v, \rho_A^* \otimes \omega_{E/F}) = \det(1 - q_v^{-\lambda} g(\sigma_{w_1})^* \otimes g(\sigma_{w_2})^*) $$

where q_v is the order of the residue field of F_v and $g^* = {}^t g^{-1}$. This is the local factor in the Rankin-Selberg product $\sigma_{w_1}^* \times \sigma_{w_2}^*$. If v remains prime in E, let w be the unique place of E dividing v. Then E_v is a field and $\sigma_v = \sigma_w$ corresponds to a Langlands class $g(\sigma_w) \in GL_m(\mathbb{C})$. It also corresponds to a Langlands class in ${}^L GL(m, E)$, namely $g(\sigma_w) \times 1 \times \sigma_{E/F}$, and we have

$$ L(\lambda, \sigma_v, \rho_A^*) = \det(1 - q_v^{-\lambda} (g(\sigma_w)^* \otimes 1) T). $$

If $g(\sigma_w)$ has eigenvalues $q_w^{\lambda_1}, ..., q_w^{\lambda_n}$ (where $q_w = q_v^2$), then

$$ \begin{aligned} L(\lambda, \sigma_v, \rho_A^*) &= \prod_{1 \le i < j \le n} (1 - q_w^{-\lambda_i - \lambda_j} q_w^{-\lambda})^{-1} \prod_{i=1}^{n} (1 - q_w^{-\lambda_i} q_v^{-\lambda})^{-1}, \\ L(\lambda, \sigma_v, \rho_A^* \otimes \omega_{E/F}) &= \prod_{1 \le i < j \le n} (1 - q_w^{-\lambda_i - \lambda_j} q_w^{-\lambda})^{-1} \prod_{i=1}^{n} (1 + q_w^{-\lambda_i} q_v^{-\lambda})^{-1}. \end{aligned} $$

We now begin our computation of $J_v(\xi, \varphi_v, \lambda)$ for φ_v fixed by K_v. Suppose that σ_{1v} is the unramified constituent of $\mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)} \chi$ where B_m is the standard upper-triangular Borel subgroup of $GL(m)$ and $\chi = (\chi_1, ..., \chi_m)$ is an m-tuple of unramified characters of E_v^*. Then σ_{2v} is the unramified constituent of $\mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)} \chi^{-1}$. Let χ^* be the character of the upper-triangular Borel subgroup $B(E_v)$ of $G(E_v)$ defined by the n -tuple $(\chi_1, ..., \chi_m, \chi_1^{-1}, ..., \chi_m^{-1})$. We identify $\mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)} \chi \otimes \mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)} \chi^{-1}$ with $\mathrm{Ind}_{B(E_v)}^{M(E_v)} \chi^*$. For ψ in the space of $\mathrm{Ind}_{B(E_v)}^{M(E_v)} \chi^*$, set

$$L'_v(\psi) = \int_{B_\eta(F_v)\backslash M_\eta(F_v)} \psi(m)\, dm.$$

Here we use that the modulus function of $B_\eta(F_v)$ is equal to the restriction of the character $e^{\langle \rho_B, H_{B_E}(m)\rangle}$. This linear functional is $M_\eta(F_v)$-invariant and, up to multiples, it is the unique such functional. We may identify π_v with the unramified constituent of the induced representation $\Sigma_v = \mathrm{Ind}_{B(E_v)}^{G(E_v)} \chi^*$ and on the induced space of Σ_v, the functional L_v can be written

$$L_v(\varphi)(g) = \int_{B_\eta(F_v)\backslash M_\eta(F_v)} \varphi(mg)\, dm$$

where dm is the semi-invariant measure on $B_\eta\backslash M_\eta(F_v)$. The local intertwining period can be written

$$\begin{aligned} J_v(\xi,\varphi,\lambda) &= \int_{H_\eta(F_v)\backslash H(F_v)} e^{\langle \lambda, H_{P_E}(\eta h)\rangle}\ L_v(\varphi)(\eta h)\, dh \\ &= \int_{H_\eta(F_v)\backslash H(F_v)} \int_{B_\eta\backslash M_\eta(F_v)} e^{\langle \lambda, H_{P_E}(\eta h)\rangle}\ \varphi(m\eta h)\, dm\, dh \\ &= \int_{B'_\eta(F_v)\backslash H(F_v)} e^{\langle \lambda, H_{P_E}(\eta h)\rangle}\ \varphi(\eta h)\, dh \end{aligned}$$

where $B'_\eta(F_v) = \eta^{-1}B_\eta(F_v)\eta = H(F_v) \cap \eta^{-1}B_E(E_v)\eta$.

The *essential vector* is the unique function φ_v in the space Σ_v which is right-invariant under $GL_n(\mathcal{O}_v)$ (where $\mathcal{O}_v$ is the ring of integers in E_v) and satisfies $\varphi_v(e) = 1$. We identify $\mathfrak{A}^*_{P,\mathbb{C}}$ with $\mathbb{C}$ by sending ϖ to 1, where $\hat{\Delta}_P = \{\varpi\}$.

Theorem 36. *Assume that $v \notin S$ and let φ_v be the essential vector. For a suitable normalization of measures we have*

$$J_v(\xi,\varphi_v,\lambda) = \frac{L(\lambda, \sigma_{1v}, \rho_A^*)}{L(\lambda+1, \sigma_{1v}, \rho_A^* \otimes \omega_{E/F})}.$$

We first prove this theorem in the case that v splits in E. Then we may identify $G(E_v)$ with $GL_n(E_{w_1}) \times GL_n(E_{w_2})$ where w_1, w_2 are the places of E dividing v. We have $\sigma_{jv} = \sigma_{jw_1} \otimes \sigma_{jw_2}$ where $\sigma_{1w_1} \simeq \sigma^*_{2w_2}$ and $\sigma_{1w_2} \simeq \sigma^*_{2w_1}$. Conjugation acts by $(x,y) \longrightarrow (y,x)$ and $H(F_v)$ is imbedded diagonally. We may also take $\eta = (1,\xi)$. Then $H_\eta(F_v)$ is the Levi factor of the parabolic subgroup $P = MN$ type (m,m) in $H(F_v)$ and $B'_\eta(F_v) = B(F_v) \cap H_\eta(F_v)$. Using the Iwasawa decomposition

$H(F_v) = B'_\eta(F_v)N(F_v)\mathbf{K}_F$ we obtain

$$\begin{aligned} J_v(\xi,\varphi,\lambda) &= \int_{B'_\eta(F_v)\backslash H(F_v)} e^{\langle \lambda, H_P(\eta h)\rangle}\ \varphi(\eta h)\ dh \\ &= \int_{B'_\eta(F_v)\backslash H(F_v)} e^{\langle \lambda, H_P(h)+H_P(\xi h)\rangle}\ \varphi_1(h)\varphi_2(\xi h)\ dh \\ &= \int_{N(F_v)} e^{\langle \lambda, H_P(\xi n)\rangle}\ \varphi_2(\xi n)\ dn. \end{aligned}$$

In other words, $J_v(\xi,\varphi_v,\lambda)$ coincides with the standard intertwining operator applied to the essential vector in $\mathrm{Ind}_{P_{w_2}}^{G_{w_2}}(\sigma_{1w_2}\otimes\sigma_{2w_2})$. By the Gindikin-Karpelevic formula, the integral is equal to

$$\frac{L(\lambda,\sigma^*_{1w_2}\otimes\sigma_{2w_2})}{L(\lambda+1,\sigma^*_{1w_2}\otimes\sigma_{2w_2})} = \frac{L(\lambda,\sigma^*_{1w_2}\otimes\sigma^*_{1w_1})}{L(\lambda+1,\sigma^*_{1w_2}\otimes\sigma^*_{1w_1})}$$

and this is equal to $L(\lambda,\sigma^*_{1v},\rho_A)/L(\lambda+1,\sigma^*_{1v},\rho_A\otimes\omega_{E/F})$ as claimed.

We now suppose that v remains prime in E, and, for simplicity, we drop v from the notation. We first consider the case $H = GL_2(F)$ and $G = GL_2(E)$ where E/F is an unramified extension of p-adic fields. We also assume that $p \neq 2$. We write $|\cdot|$ and $||\cdot||$ for the absolute values on F and E, respectively, q for the order of the residue field of F, and $q_E = q^2$ for the order of the residue field of E. Fix $i \in E^*$ such that $Tr_{E/F}(i) = 0$ and set

$$\eta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

Then

$$\eta\bar{\eta}^{-1} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2i} & \frac{1}{2i} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the subgroup $T = H\cap\eta^{-1}B(E)\eta$ is a torus isomorphic to E^*. We shall compute

$$\int_{T\backslash H} \varphi(\eta h)\ dh$$

where

$$\varphi(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix}k) = ||\frac{a}{b}||^{\frac{\lambda+1}{2}}.$$

Thus φ is the essential vector in the unramified representation $GL_2(E)$ with Langlands class

$$\begin{pmatrix} q_E^{\frac{\lambda}{2}} & 0 \\ 0 & q_E^{-\frac{\lambda}{2}} \end{pmatrix} \in GL_2(\mathbb{C}).$$

To fix the quotient measure on $T\backslash H$, we observe that every $h \in H$ can be written uniquely in the form

$$h = t\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

with $t \in T$. In these coordinates, a right-invariant measure on $T\backslash H$ is given by $\frac{da}{|a|}db$. Assume that the measures da and db assign measure 1 to the ring of integers

$\mathcal{O}_F$. With this normalization, we have the following result. It is proved in [JL] using the method of Lemma 27 above. Here we give a direct computational proof.

Proposition 37. *We have*

$$\int_{T\backslash H} \varphi(\eta h)dh = ||i||^{\frac{1}{2}} \ \frac{1+q^{-\lambda-1}}{1-q^{-\lambda}}.$$

Proof. Since

$$\eta \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+i \\ a & b-i \end{pmatrix},$$

we have

$$\begin{aligned} \varphi(\begin{pmatrix} a & b+i \\ a & b-i \end{pmatrix}) &= ||2ai||^{\frac{\lambda+1}{2}} (\max\{||a||,\ ||b-i||\})^{-\lambda-1} \\ &= ||i||^{\frac{\lambda+1}{2}} ||a||^{\frac{\lambda+1}{2}} \max\{||a||,\ ||i-b||\}^{-\lambda-1} \end{aligned}$$

and hence

$$\int_{T\backslash H} \varphi(\eta h)\ dh = ||i||^{\frac{\lambda+1}{2}} \int_{F^*} f(a) ||a||^{\frac{\lambda+1}{2}} \frac{da}{|a|}$$

where

$$f(a) = \int_F \max\{||a||,\ ||i-b||\}^{-\lambda-1}\ db.$$

We now compute $f(a)$. Set

$$X = 1 + \int_{|b|>1} |b|^{-2\lambda-1} \frac{db}{|b|}\ .$$

It is easily checked that

$$X = \frac{1-q^{-2\lambda-2}}{1-q^{-2\lambda-1}}$$

with the normalization $\mathrm{vol}(\mathcal{O}_F) = 1$.

Lemma 38. $f(a) = |a|^{-2\lambda-1}X$ *if* $||a|| \geq ||i||$ *and* $f(a) = ||i||^{-\lambda-\frac{1}{2}}X$ *if* $||a|| < ||i||$.

Proof. If $||a|| \geq ||i||$, then

$$\begin{aligned} f(a) &= \int_{|b|\leq|a|} ||a||^{-\lambda-1}\ db + \int_{|b|>|a|} ||b||^{-\lambda-1}\ db \\ &= |a|^{-2\lambda-2}\mathrm{vol}(\{b : |b| \leq |a|\}) + |a|^{-2\lambda-1} \int_{|b|>1} |b|^{-2\lambda-1} \frac{db}{|b|} \\ &= |a|^{-2\lambda-1}X. \end{aligned}$$

If $||a|| < ||i||$, then

$$\begin{aligned} f(a) &= \int_F ||i-b||^{-\lambda-1}\ db \\ &= ||i||^{-\lambda-1}\mathrm{vol}(\{b : ||b|| \leq ||i||\}) + \int_{||b||>||i||} |b|^{-2\lambda-1} \frac{db}{|b|} \\ &= ||i||^{-\lambda-\frac{1}{2}} + ||i||^{-\lambda-\frac{1}{2}} \int_{|b|>1} |b|^{-2\lambda-1} \frac{db}{|b|} \\ &= ||i||^{-\lambda-\frac{1}{2}}X. \qquad \square \end{aligned}$$

Now we have

$$\int_{F^*} f(a)||a||^{\frac{\lambda+1}{2}}\frac{da}{|a|} = \sum_{n=-\infty}^{\infty} f(p^n)q^{-n(\lambda+1)}$$
$$= q^{r(2\lambda+1)}X\sum_{n>r} q^{-n(\lambda+1)} + X\sum_{n\le r} q^{n(2\lambda+1)}q^{-n(\lambda+1)}$$

and it is easily checked that this equals

$$q^{r\lambda}\left(\frac{1+q^{-\lambda-1}}{1-q^{-\lambda}}\right)$$

and the result follows. □

The next proposition is the inert case of Theorem 36.

Proposition 39. *Let E/F be an unramified quadratic extension of p-adic fields and let σ_1 be an unramified representation of $GL_m(E)$. For the unique normalized K-invariant $\varphi \in I_P(\sigma_1 \times \overline{\sigma}_1^*)$ we have*

$$J(\xi,\varphi,\lambda) = \frac{L(\lambda,\sigma_1,\rho_A^*)}{L(\lambda+1,\sigma_1,\rho_A^*\otimes\omega_{E/F})}.$$

Proof. In the above notation, suppose that $\sigma_1 = \mathrm{Ind}_{B_m(E)}^{GL_m(E)}\,\chi$ where $\chi=(\chi_1,...,\chi_m)$ with $\chi_i = |\cdot|_E^{\lambda_i}$. As before, $\sigma_{2v} = \mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)}\,\chi^{-1}$ and we view φ as an element of $\mathrm{Ind}_{B_{nv}}^{G_{nv}}\,(\chi_1,\dots,\chi_n,\chi_1^{-1},\dots,\chi_n^{-1})$. To compute

$$J_v(\xi,\varphi,\lambda) = \int_{B'_\eta(F_v)\backslash H(F_v)} e^{\langle\lambda,H_{B_E}(\eta h)\rangle}\ \varphi(\eta h)\ dh,$$

we shall regard it as a local intertwining period for an Eisenstein series induced from the Borel subgroup and reduce to the case $n=2$ by making use of the functional equations. Let $\xi' = (1,2)\cdots(2n-1,2n)$. Then ξ' is a minimal involution and $\xi = w^{-1}\xi' w$ where w is defined by $w(i) = 2i-1$ and $w(i+n) = 2i$ for $i = 1,\dots,n$. Then w has the following reduced decomposition:

$$w = (s_{2n-2})(s_{2n-4}s_{2n-3})\cdots(s_4\cdots s_n s_{n+1})(s_2\cdots s_{n-1}s_n).$$

We observe that an analogue of Proposition 33 holds in the local case. It is proved in the same way, by re-writing the absolutely convergent integral. Therefore

$$J_v(\xi,\varphi,\lambda) = J_v(\xi', M(w,\lambda)\varphi, w\lambda).$$

As in the proof of Lemma 32, the right hand side can be written as a local intertwining period with respect to the group $GL_2\times\cdots\times GL_2$ (m times) and the induction from $(\chi_1,\chi_1^{-1},\dots,\chi_m,\chi_m^{-1})$. By Proposition 37 we have

$$J(\xi',\varphi,\lambda) = \prod_{i=1}^{n}(1-q_E^{-\lambda_i}q_F^{-\lambda})^{-1}(1+q_E^{-\lambda_i}q_F^{-(\lambda+1)}).$$

By the formula of Gindikin and Karpelevic, $M(w,\lambda)\varphi = c(\lambda)\varphi$ where

$$c(\lambda) = \prod_{1\le i<j\le n}\frac{(1-q_E^{-\lambda_i-\lambda_j}q_E^{-\lambda})^{-1}}{(1-q_E^{-\lambda_i-\lambda_j}q_E^{-(\lambda+1)})^{-1}}.$$

This completes the proof. □

VIII. Periods of truncated Eisenstein series

The next theorem gives our formula for $\Pi^{G/H}(\Lambda_m^T E(\varphi,\lambda))$. Let P be a parabolic subgroup, let $(n_1,\dots,n_r)$ be the corresponding partition of n, and fix a cuspidal representation σ of $M_{P_E}(\mathbb{A}_E)^1$. Let $\mathcal{G}(P,\sigma)$ be the set of pairs (Q,s) consisting of a parabolic subgroup Q and an element $s\in\Omega(M_P,M_{P'})$ with $P'\le Q$ such that $s^{-1}\alpha>0$ for all $\alpha\in\Delta_{P'}^Q$ and there exists a set $X\subset\{1,2,\dots,r-1\}$ of indices satisfying conditions (1), (2), (3) below. Let $(n'_1,\dots,n'_r)$ be the partition corresponding to $M'=M_{P'}$, and set $s\sigma=\sigma'=\sigma'_1\otimes\sigma'_2\otimes\cdots\otimes\sigma'_r$. Then

(1) For all $j\in X$ we have $j+1\notin X$, $n'_j=n'_{j+1}$ and $(\sigma'_{j+1})^*\simeq\overline{\sigma'_j}$.
(2) For all j such that $j,j-1\notin X$, σ'_j is distinguished with respect to $GL_{n'_j}(\mathbb{A}_F)$.
(3) Q corresponds to the partition obtained from $(n'_1,\dots,n'_r)$ by replacing the pair of entries n'_j, n'_{j+1} by the single entry $n'_j+n'_{j+1}$ for all $j\in X$.

Let $\mathcal{G}_P(\sigma)$ be the set of all parabolic subgroups Q that appear in some pair $(Q,s)\in\mathcal{G}(P,\sigma)$.

The Levi subgroup M' of P' is isomorphic to $GL_{n'_1}\times\cdots\times GL_{n'_r}$. Let ξ_Q be the unique element in $\Omega(M',M')$ such that ξ interchanges the j^{th} and $(j+1)^{th}$ factors of M' for $j\in X$ and fixes all other factors. We write $\lambda=\lambda_Q+\lambda_{P'}^Q$ for the decomposition of an element $\lambda\in\mathfrak{A}_{P'}^*$ relative to the direct sum $\mathfrak{A}_{P'}^*=\mathfrak{A}_Q^*\oplus(\mathfrak{A}_{P'}^Q)^*$. Let

$$v_Q=2^{d(H)-d(Q)}\operatorname{vol}\{\sum_{\alpha\in\Delta_Q}a_\alpha\alpha^\vee:0\le a_\alpha\le 1\}.$$

Theorem 40. *Let $\varphi\in\mathcal{A}_{P_E}(G)_\sigma$ where σ is a cuspidal representation of $M_E(\mathbb{A}_E)$, and let $E(g,\varphi,\lambda)$ be the associated Eisenstein series. Then as a meromorphic function of λ,*

$$\int_{H(F)\backslash H(\mathbb{A})^1}\Lambda_m^T E(h,\varphi,\lambda)dh$$

is equal to

$$\sum_{(Q,s)\in\mathcal{G}(P,\sigma)}v_Q\;\frac{e^{\langle(s\lambda)_Q,T\rangle}}{\prod_{\alpha\in\Delta_Q}\left\langle(s\lambda)_Q,\alpha^\vee\right\rangle}J(\xi_Q,M(s,\lambda)\varphi,(s\lambda)_{P'}^Q).$$

Proof. Since σ is cuspidal, $E_{Q_E}(g,\varphi,\lambda)$ vanishes unless Q contains an associate of P and we obtain

$$\int_{H(F)\backslash H(\mathbb{A})^1}\Lambda_m^T E(g,\varphi,\lambda)\;dg$$

$$=\sum_Q(-1)^{d(Q)-d(H)}\int^*_{Q(F)\backslash H(\mathbb{A})^1}E_{Q_E}(g,\varphi,\lambda)\;\hat\tau_Q(H_{Q_E}(g)-T)\;dg$$

where Q ranges over such parabolics. Let $\Omega(P,Q)$ be the union over all $P'\subset Q$ of the subset of elements $s\in\Omega(M_P,M_{P'})$ such that $s^{-1}\alpha>0$ for all $\alpha\in\Delta_{P'}^Q$. Then

$$E_{Q_E}(g,\varphi,\lambda)=\sum_{s\in\Omega(P,Q)}E^{Q_E}(g,M(s,\lambda)\varphi,s\lambda)$$

where $E^{Q_E}(g, M(s,\lambda)\varphi, \lambda)$ is the Eisenstein series on Q_E induced from the function $M(s,\lambda)\varphi$. We must therefore compute the integrals

$$\int^{*}_{Q(F)\backslash H(\mathbb{A})^1} E^{Q_E}(g, M(s,\lambda)\varphi, s\lambda)\ \hat{\tau}_Q(H_{Q_E}(g) - T)\ dg \tag{67}$$

for $s \in \Omega(P,Q)$. By definition, (67) is equal to

$$\Pi^{M_{Q_E}/M_Q}\left(E^{Q_E}\left(\cdot\ , (M(s,\lambda)\phi)^{\mathbf{K}_F}, (s\lambda)^Q_{P'}\right)\right)\int^{*}_{\mathfrak{A}_Q} e^{\langle (s\lambda)_Q, 2X\rangle}\hat{\tau}_Q(2X-T)dX \tag{68}$$

where we have set:

$$(M(s,\lambda)\phi)^{\mathbf{K}_F}(m) = \int_{\mathbf{K}_F} M(s,\lambda)\phi(mk)dk$$

and E^{Q_E} is an Eisenstein series for the parabolic subgroup $P_E \cap M_{Q_E}$ of the group M_{Q_E}. By Theorem 23 (applied to a product of linear groups), the first factor in (67) vanishes unless (Q,s) belongs to $\mathcal{G}(P,\sigma)$. If $(Q,s) \in \mathcal{G}(P,\sigma)$, then it is equal to the following intertwining period integral for the group M_Q:

$$J^{M_Q}\left(\xi_Q, (M(s,\lambda)\phi)^{\mathbf{K}_F}, (s\lambda)^Q_{P'}\right).$$

As in the proof of Lemma 32, this is the same as the following intertwining period for the group G:

$$J(\xi_Q, M(s,\lambda)\phi, (s\lambda)^Q_{P'}).$$

On the other hand,

$$\int^{*}_{\mathfrak{A}_Q} e^{\langle (s\lambda)_Q, 2X\rangle}\hat{\tau}_Q(2X-T)dX = v_Q\frac{e^{\langle (s\lambda)_Q, T\rangle}}{\prod_{\alpha\in\Delta_Q}\langle (s\lambda)_Q, \alpha^\vee\rangle}.$$

The required formula follows. ☐

As an example, consider the case $P = B$, the Borel subgroup. Then the integral (67) vanishes unless Q is a parabolic of type $(m_1, \dots, m_k)$ with $m_j = 1$ or 2. Thus M_Q is isomorphic to a product of copies of GL_1 and GL_2. Given such a Q, let ξ_Q be a representative for the longest element in the Weyl group of M_Q, and let $\eta = \eta_Q \in Q(E)$ be such that $\eta\bar{\eta}^{-1} = \xi_Q$. Then we have

Proposition 41. *Let σ be a character of $T_E(E)\backslash T_E(\mathbb{A}_E)^1$ and let $E(g,\varphi,\lambda)$ be an Eisenstein series where $\varphi \in \mathcal{A}_{B_E}(G)_\sigma$. Then*

$$\int_{H(F)\backslash H(\mathbb{A})^1} \Lambda^T_m E(h,\varphi,\lambda)\ dh$$

is equal to the sum of the terms

$$v_Q\frac{e^{\langle (w\lambda)_Q, T\rangle}}{\prod_{\alpha\in\Delta_Q}\left\langle (w\lambda)_Q, \alpha^\vee\right\rangle}J(\xi_Q, M(w,\lambda)\varphi, (w\lambda)^Q_0)$$

where Q ranges over the parabolic subgroups of type $(n_1, \dots, n_r)$ with $n_j = 1$ or 2 and w ranges over the elements of the Weyl group Ω such that $w\sigma$ is trivial on $T_{\eta_Q}(\mathbb{A})^1$, and $w^{-1}\alpha > 0$ for $\alpha \in \Delta_0^Q$.

In this proposition, the intertwining integral $J(\xi_Q, M(w,\lambda)\varphi, w\lambda_0^Q)$ reduces to

$$\mathrm{vol}(H_{\eta_Q}(F)A_{M_Q}\backslash H_{\eta_Q}(\mathbb{A}))\int_{H_{\eta_Q}(\mathbb{A})\backslash H(\mathbb{A})} e^{\left\langle (w\lambda)_0^Q+\rho_B, H(\eta_Q h)\right\rangle}\; M(w,\lambda)\varphi(\eta_Q h)\; dh.$$

Example 6. Consider the case $G = GL(2)$, and $E(g, \varphi, \lambda)$ is an Eisenstein series induced from a character σ of $B_E(F)\backslash B_E(\mathbb{A}_E)$ trivial on A_0. Here we identify $\mathfrak{A}_{0,\mathbb{C}}^*$ with $\mathbb{C}$ by sending the fundamental weight to 1 and thus we view λ as a complex parameter. Let w be the non-trivial element in the Weyl group. There are three possible pairs (Q, w) : (B, e), (B, w), and (G, e). If $Q = B$, then $\xi = \eta = e$ and $B_\eta = H_\eta = B$. The pairs (B, e) and (B, w) occur if and only if σ is trivial on $B(\mathbb{A}_F)$. Identifying σ with a pair of Hecke characters (σ_1, σ_2) of $\mathbb{A}_E^*$, the condition is that σ_1 and σ_2 are trivial on $\mathbb{A}_F^*$. If so, the contribution of the pairs (B, e) and (B, w) is equal to $\frac{1}{2}\,\mathrm{vol}(T(F)A_0\backslash T(\mathbb{A}))$ times

$$\frac{e^{\lambda T}}{\lambda}\int_{\mathbf{K}_F}\varphi(k)\; dk + \frac{e^{-\lambda T}}{-\lambda}\int_{\mathbf{K}_F} M(w,\lambda)\varphi(k)\; dk.$$

If $Q = G$, then $\xi = w$ and B_η is the torus

$$B_\eta = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} : \alpha \in E^* \right\}.$$

The pair (G, e) contributes if and only if $\sigma_2(\alpha) = \sigma_1(\overline{\alpha})^{-1}$. If so, the contribution is

$$\mathrm{vol}(B_\eta(F)A_0\backslash B_\eta(\mathbb{A}))\int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda+1, H(\eta h)\rangle}\; \varphi(\eta h)\; dh.$$

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