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GERMS OF KLOOSTERMAN INTEGRALS FOR $GL(3)$

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Abstract. In an earlier paper we introduced the concept of Shalika germs for certain Kloosterman integrals. We compute explicitly the germs in the case of the group $GL(3)$.

1. Introduction

We let F be a local field of characteristic 0 and ψ a non trivial additive character of F. We let G be the general linear group $GL(r)$ regarded as an algebraic group over F. We often write G for $G(F) = GL(r, F)$ and $\mathcal{C}(G)$ for the space of smooth functions of compact support on $G(F)$. We use similar notations for other groups or varieties. In an earlier paper ([JY3]) we introduced the notion of Shalika germs for the Kloosterman integrals of the group $G(F)$. We also considered a quadratic extension E of F and the Kloosterman integrals relative to the symmetric space $S(r, F)$ of Hermitian matrices in $GL(r, E)$. Our purpose in this paper is to compute the Shalika germs for the group $GL(3, F)$ and show they agree, up to certain "transfer factors", with the Shalika germs for the Kloosterman integrals relative to $S(F, 3)$ (Theorem 5.1).

This can be used to give a more satisfactory proof for the global results of [JY3]. Indeed, the relative trace formula identity established there was valid only under some restrictive assumptions on the functions at hand. In more detail, let E/F be a quadratic extension of number fields (satisfying the restrictive conditions of [JY3]) and η the corresponding quadratic character. One of our goals was the following one: suppose that Π is a cuspidal automorphic representation of $GL(3, E_{\mathbb{A}})$ which is distinguished by the quasi-split unitary group H in the sense that there is a form ϕ in the space of Π, the integral of which over the group H is non zero. In [JY3] we concluded that Π is invariant under the Galois group of E/F and thus a base change by the results of [A-C]. It is now possible to show directly from the relative trace formula of [JY3]—without using the results of [A-C]—that Π is the base change of some representation π of $GL(3, F_{\mathbb{A}})$, in the sense that the L-functions

$$L(s, \pi)L(s, \pi \otimes \eta)\,,\; L(s, \Pi)$$

agree, except perhaps for a finite number of factors. Moreover, our relative trace formula suggests the following local result: suppose now that $E/F$is a quadratic

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extension of local fields and let H be a unitary group in $GL(n, E)$. If Π is a supercuspidal representation of $GL(n, E)$, it is reasonable to conjecture that the dimension of the space of linear forms invariant under H on the space of Π is at most 1. For $n = 3$ it is surely possible to derive this conjecture from our relative trace formula.

In general, the Shalika germs describe the asymptotic behavior of orbital integrals of the form:

$$\int f(n_1 g n_2)\theta(n_1 n_2) dn_1 dn_2.$$

Here N is a maximal unipotent subgroup in G and θ a generic character of N. In standard harmonic analysis, there is a theory of asymptotics of orbital integrals given by the Shalika germs and a dual theory of asymptotics of characters. The orbits of interest are the semi-simple orbits, that is, the closed orbits. The behavior at infinity of the semi-simple orbital integrals is controlled by the unipotent orbital integrals. In fact, there is an infinitesimal notion of orbital integrals (on the Lie algebra) and the Lie algebra situation is used as a model for the group situation. Likewise, for the characters, there is an infinitesimal theory, where the crucial objects are the Fourier transforms of the nilpotent orbital integrals.

The situation at hand is completely different. There is no infinitesimal version of the theory. Moreover, all the orbits, that is, the double cosets of N in G, are closed. In other words, all the orbits are elliptic. As a result, the asymptotics of orbital integrals and the theory of asymptotics of characters (Bessel distributions) are the same. Consider for instance the case of a supercuspidal representation π. Then it is reasonable to conjecture that the Bessel distribution of π is a locally integrable function equal on the open Bruhat cell to an integral of the above form, where f is a matrix coefficient of π (see [B] for the $GL(2)$ case). To prove the conjecture, the first step is to prove that the resulting function is locally integrable, and this can only be done if enough information on the germs is available (as in [B]). For non supercuspidal representations the situation is more complicated but the germs still play an important role (see [B]). In particular, the results of the present paper will be useful in proving this conjecture on Bessel distributions in the case of $GL(3)$. Finally, the germs are likely to play an important role in the proof of the "fundamental lemma" of [JY2] for $GL(n)$ and the extension of the results of [JY3] to $GL(n)$. Thus, there is every reason to study them.

We note, however, that our computation is not really explicit. We simply show that both germs can be reduced to the computation of the same (one variable) integral. For $GL(r)$ it might be possible to show similarly that the two kinds of germs agree, up to a transfer factor, without computing explicitly the germs.

The paper is arranged as follows. In section 2 we review the notion of Shalika germs adding appropriate remarks. In sections 3 and 4 we compute the germs for the Kloosterman integrals for $GL(3, F)$. In sections 5 and 6 we compute the germs for the Kloosterman integrals relative to $S(F, 3)$.

Finally, we would like to thank the referee for the patient reading of a difficult manuscript. We also thank Zenghyu Mao for making his computations on a related problem available to us. We adapted to our situation an ingenious change of variables found in his work.

2. Shalika germs

We first recall the concept of Shalika germs introduced earlier in the context of $GL(r, F)$, adding appropriate remarks. Let F be a local field, non Archimedean of characteristic 0. We denote by $\mathcal{O}_F$ the ring of integers of F, by $\wp_F$ the maximal ideal in it and by q_F the cardinality of the quotient. We let ψ_F be a non trivial additive character of F. We drop the index F if this does not create confusion. Let A be the group of diagonal matrices, $W = W(G)$ the Weyl group of A identified with the group of permutation matrices in $GL(r, F)$ and N the group of upper triangular matrices with unit diagonal. We define an algebraic group morphism from N to F by:

$$\theta_0(u) = \sum u_{i,i+1}$$

and set $\theta_\psi(u) = \psi(\theta_0(u))$. We often write θ for θ_ψ. Recall that the elements of the form wa with $w \in W$ and $a \in A(F)$ form a set of representatives for the action of $N(F) \times N(F)$ on $G(F)$ defined by:

$$s \overset{(n_1,n_2)}{\mapsto} {}^t n_1 s n_2.$$

We say that wa is relevant if $\theta_0(n_1 n_2) = 1$ when (n_1, n_2) fixes wa. If wa is relevant, then there is a standard parabolic subgroup P_w (i.e. P_w contains N) with standard Levi factor M_w (i.e. M_w contains A) such that w is the longest element of $W \cap M_w$. We then denote by A_w the center of M_w. The element a belongs to A_w. Conversely, the wa obtained this way are all relevant. See: [JY3], [F], [dG], [g.S], and [r.S], page 257. We denote by $R(G)$ the set of w of the above form. If $w \in R(G)$ and $M = M_w$, we also write $w = w_M$. In particular, w_G is the longest element of $W(G)$. For $\Phi \in \mathcal{C}(G)$ and wa relevant we consider the Kloosterman integral

$$I(wa, \Phi) = \int \Phi({}^t n_1 w a n_2)\theta(n_1 n_2) d(n_1, n_2).$$

The integral is taken over the quotient of $N(F) \times N(F)$ by the stabilizer of wa.

We now recall the notion of Shalika germs. If w, w' are in $R(G)$, we write $w \to w'$ if $A_w \supseteq A_{w'}$. This is equivalent to $M_w \subseteq M_{w'}$ or $w \in M_{w'}$. We write $w \xrightarrow{1} w'$ if $w \to w'$, $w \neq w'$ and there is no $w'' \in R(G)$ such that $w \to w'' \to w$. We can define a graph with $R(G)$ for a set of vertices: the graph is oriented and the edges are the pairs (w, w') with $w \xrightarrow{1} w'$. Note that all oriented paths from a given w to a given w' have the same length which we denote by $d(w, w')$. We write $w \xrightarrow{m} w'$ if $w \to w'$ and $d(w, w') = m$. For each $w \in R(G)$ we have $e \to w \to w_G$. For $0 \leq i \leq n$ and $g \in GL(n)$ we denote by $\delta_i(g)$ the determinant of the submatrix of g formed with the first i rows and first i columns (called $\Delta_i(g)$ in [JY3]). Thus δ_i is a map of algebraic varieties from G to F. In particular $\delta_0 = 1$ and $\delta_r(g) = \det g$. We denote by $\Delta(G)$ the set of functions of the form

$$\delta(g) = \prod_i \delta_i(g)^{n_i}$$

with $n_i \in \mathbb{Z}$, by $\Delta_0(G)$ the set of functions of the form $\delta_i, 0 \leq i \leq r$, and by $\Delta_+(G)$ the set of functions of the form $\delta_i, 1 \leq i \leq r-1$. The restriction of such a function to A is an algebraic character of A. As a matter of fact, we can identify $\Delta(G)$ with the set of algebraic characters of A. More generally, if M is a standard Levi-subgroup, then we denote by $\Delta(M)$ the set of maps which are restrictions to

M of the elements of $\Delta(G)$. We can also identify $\Delta(G)$ with $\Delta(M)$. This notation is different from the corresponding notation in [JY3].

We recall without proof the following lemma:

Lemma 2.1. *Suppose $w \in R(G)$ and $\delta \in \Delta(G)$. Suppose $w \neq w_G$ and $\delta(w_G w) \neq 0$. Then $\delta = \delta_r^m$ for some $m \in \mathbb{Z}$. Suppose $\delta(w) \neq 0$. Then $\delta(m) \neq 0$ for all $m \in M_w$.*

If $w \to w'$, we denote by $A_w^{w'}$ the set of $b \in A_w(F)$ such that $\delta(b) = \delta(w'w)$ for all $\delta \in \Delta(G)$ such that $\delta(w'w) \neq 0$. Lemma 2.1 implies that if $w \neq w_G$, then $A_w^{w_G}$ is the set of $b \in A_w$ such that $\det(b) = \det(w_G w)$. On the other hand, it is clear that $A_w^w = \{1\}$ for all w.

It is important to keep in mind that all the notions introduced are inductive in the following sense. Suppose that $M = M_{w'}$ with $w' \in R(G)$. Then M can be written as an ordered product of linear factors $M = G_1 \times G_2 \times \cdots \times G_s$ where $G_i \simeq GL(r_i)$ and each element $m \in M$ is a diagonal matrix of square blocks:

$$m = \mathrm{diag}(g_1, g_2, \dots , g_s)$$

with $g_i \in G_i$. In particular:

$$w' = \mathrm{diag}(w'_1, w'_2, \dots, w'_s)$$

where $w'_i = w_{G_i}$. Similarly, every a in $A_{w'}$ has the form:

$$a = \mathrm{diag}(a_1, a_2, \dots, a_s)$$

with $a_i \in A_{w'_i} \subset G_i$. Thus

$$A_{w'} \simeq \prod A_{w'_i}. \tag{2.1}$$

If $w \to w'$, then $w \in M$ and

$$w = \mathrm{diag}(w_1, w_2, \dots w_r)$$

where $w_i \in R(G_i)$ (and $w_i \to w'_i$ in G_i). We have then

$$A_w \simeq \prod A_{w_i}.$$

The restriction of a $\delta \in \Delta(G)$ to M can be written as a product

$$\delta(g) = \prod_i \delta_i(g_i)$$

where $\delta_i \in \Delta(G_i)$. If $\delta(w'w) \neq 0$, then $\delta_i(w'_i w_i) \neq 0$ for each i and conversely. It follows that:

$$A_w^{w'} \simeq \prod A_{w_i}^{w'_i}. \tag{2.2}$$

We recall the following lemma, the (easy) proof of which was omitted in [JY3]:

Lemma 2.2. *Suppose that $w \to w_1 \to w'$; then*

$$A_w^{w_1} A_{w_1}^{w'} \subseteq A_w^{w'}.$$

Proof. By the inductive character of our constructions, it suffices to prove this when $w' = w_G$ and $w \neq w_G$. If $a = bc$ with $b \in A_w^{w_1}$ and $c \in A_{w_1}^{w'}$, we have to see that $\delta(w_G w) \neq 0$ implies $\delta(a) = \delta(w_G w)$. However δ is a power of the determinant by the previous lemma. Thus we may assume that $\delta = \det$. Then

$$\delta(a) = \delta(b)\delta(c) = \delta(w_1 w)\delta(w' w_1) = \delta(w' w)\delta(w_1^2) = \delta(w' w)$$

and the lemma follows. $\square$

It will be convenient to use the following notation: if f and g are functions on $A_w^{w'}$ and $A_{w'}$ respectively, then we define a new function $f * g$ on A_w by:

$$f * g(a) = \sum_{\{a=bc, b\in A_w^{w'}, c\in A_{w'}\}} f(b)g(c). \tag{2.3}$$

If f and g are functions on $A_w^{w_1}$ and $A_{w_1}^{w'}$ respectively, we define similarly a new function $f * g$ on $A_w^{w'}$ by:

$$f * g(a) = \sum_{\{a=bc, b\in A_w^{w1}, c\in A_{w_1}^{w'}\}} f(b)g(c). \tag{2.4}$$

A system of Shalika germs is a family of smooth functions $K_w^{w'}$ defined over the sets $A_w^{w'}$ for $w \to w'$ such that $K_w^w = 1$ for any w, and, for any function $f \in \mathcal{C}(G(F))$, there exist functions $\omega_w = \omega_w^f \in \mathcal{C}(A_w(F))$ with:

$$I(w., f) = \sum_{\{w': w\to w'\}} K_w^{w'} * \omega_{w'}. \tag{2.5}$$

For a given function f, the above relations determine the functions ω_w by a triangular system of linear equations. In particular $\omega_{w_G}(a)$ is just the orbital integral $I(w_G a, f)$. When we want to emphasize the dependence of the functions ω_* on the system, we will write them as $\omega_w^{K,f}$ or ω_w^K. The notion of Shalika germs depends on ψ. The choice of the invariant measures on the quotients depends on the choice of ψ and will be recalled in the case $r = 3$.

We recall the following theorem of [JY3]:

Theorem 2.3. *There exists a system of Shalika germs. If K is a system of Shalika germs, and $t_w^{w'}$ is a family of functions in $\mathcal{C}(A_w^{w'})$ such that $t_w^w = 1$ for all w, then the functions*

$$H_w^{w'} = \sum_{w\to w_1\to w'} K_w^{w_1} * t_{w_1}^{w'} \tag{2.6}$$

form another system of Shalika germs. All systems of Shalika germs are obtained in this way from a given system.

We remark that if $t_w^{w'}$ is a system of functions with the property that $t_w^{w'} = 0$ unless $w = w'$ or $w' = w_G$, then the system H defined by (2.6) verifies $H_w^{w'} = K_w^{w'}$ for $w' \neq w_G$.

It is possible to compute inductively the germs $K_w^{w'}$ in terms of the germs $K_w^{w_G}$. Indeed, suppose that for each $m < n$ we are given a system of germs for the group $GL(m, F)$; in particular we are given the functions $K_w^{w_{GL(m)}}$. Then it follows from the constructions of [JY3] that there is a system of germs on $GL(n)$ with the following property. If $w' \neq w_G$, then $M = M_{w'}$ can be written as a product of linear groups G_i as above. For $w \to w'$ write a in $A_w^{w'}$ as

$$a = \mathrm{diag}(a_1, a_2, \ldots, a_s)$$

with $a_i \in A_{w_i}^{wi'}$. Then:

$$K_w^{w'}(a) = \prod K_{w_i}^{w_i'}(a_i). \tag{2.7}$$

It will be convenient to say that a system of this form is inductive (relative to the given functions $K_w^{w_{GL(m)}}$ for $m < r$).

We want to make this assertion more precise. Let Φ be a smooth function of compact support on G such that

$$I(w_G, \Phi) = 1,$$

$$I(w_G z, \Phi) = 0 \text{ if } z \in F^\times, z^r = 1, z \neq 1.$$

We claim there is an inductive system of germs K_*^* such that

$$K_w^{w_G}(a) = I(wa, \Phi) \text{ for } a \in A_w^{w_G}.$$

Indeed, let K_*^* be an inductive system. We first observe the following. Suppose that $a \in A_w^{w_G}$ has a decomposition $a = bc$ with $b \in A_w^{w'}$, $c \in A_{w'}$ and $w \to w'$. If $w' \neq w_G$, then the element c is in fact in $A_{w'}^{w_G}$. For by definition:

$$\det(a) = \det(w_G w), \det(b) = \det(w'w);$$

hence

$$\det(c) = \det(w_G w').$$

If on the contrary $w' = w_G$, then we find that $c \in A_{w_G} \simeq F^\times$ verifies $\det c = c^r = 1$. Thus $\omega_{w_G}^{\Phi,K}(c) = 1$ if $c = 1$ and 0 otherwise. For $a \in A_w^{w_G}$ we have then

$$I(wa, \Phi) = \omega_w^{K,\Phi}(a) + \sum_{w \neq w' \neq w_G} K_w^{w'} * \omega_{w'}^{K,\Phi}(a) + K_w^{w_G}(a).$$

The "convolution" in this formula can be viewed as the "convolution" of a function on $A_w^{w'}$ and a function on $A_{w'}^{w_G}$. Define then a system of functions t_*^* as follows: $t_w^w = 1$; $t_w^{w_G}$ is the restriction of $\omega_w^{K,\Phi}$ to $A_w^{w_G}$ for $w \neq w_G$; all other elements of the family are 0. Then if H is the system of germs defined by t (see (2.6)), the above relation reads:

$$I(wa, \Phi) = H_w^{w_G}(a)$$

on $A_w^{w_G}$. Moreover $H_w^{w'} = K_w^{w'}$ for $w' \neq w_G$. Thus H_*^* is an inductive system with the required properties. Our assertion follows.

Proposition 2.4. *If m is sufficiently large, there is an inductive system of germs such that, for $w \neq w_G$, $K_w^{w_G}$ has support in the set $A_w^{w_G}(m)$ defined by*

$$| \delta(a) | \leq q^{-m}$$

for each $\delta \in \Delta_+(G)$ such that $\delta(w) \neq 0$.

Proof. Choose m so large that the character ψ is trivial on the ideal $\wp^m$ and the relations $z^r = 1$ and $z \equiv 1 \bmod \wp^m$ imply $z = 1$. Let Φ be any function with support in the set $w_G K_m$, where K_m is the principal congruence subgroup of $K = GL(r, \mathcal{O})$, such that:

$$I(w_G, \Phi) = 1.$$

Then

$$I(w_G z, \Phi) = 0 \text{ if } z \in F^\times, z^r = 1, z \neq 1.$$

For instance, we can take for Φ the characteristic function of $w_G K_m$ divided by the volume of $N \cap K_m$. Then the inductive system of germs such that $I(wa, \Phi) = K_w^{w_G}(a)$ for $a \in A_w^{w_G}$ has the required property. Indeed, if $\delta \in \Delta_+(G)$, then $| \delta(g) | \leq q^{-m}$ on $w_G K_m$. Suppose $I(wa, \Phi) \neq 0$. Then there is n_1 and n_2 such that ${}^t n_1 w a n_2 \in w_G K_m$. Suppose $\delta \in \Delta_+(G)$ and $\delta(w) \neq 0$. Then $\delta({}^t n_1 w a n_2) = \delta(wa) = \delta(w)\delta(a) = \pm\delta(a)$. Thus $| \delta(a) | \leq q^{-m}$. $\square$

We will need a refinement of the above result. Suppose $w \to w'$ with $w' \neq w_G$. Then $M_{w'} = \prod G_i$ where the G_i are linear groups. We write $w' = (w'_i)$ and $w = (w_i)$ as above. Then $A_w^{w'} = \prod_i A_{w_i}^{w'_i}$. We set

$$A_w^{w'}(m) = \prod A_{w_i}^{w'_i}(m). \tag{2.8}$$

We first prove a lemma:

Lemma 2.5. *Suppose $w \to w'$ and*

$$a = bc$$

with $a \in A_w^{w_G}, b \in A_w^{w'}, c \in A_{w'}^{w_G}$. If $a \in A_w^{w_G}(m)$, then $c \in A_{w'}^{w_G}(m)$. If $b \in A_w^{w'}(m)$ and $c \in A_{w'}^{w_G}(m)$, then $a \in A_w^{w_G}(m)$.

Proof. Let us prove the first assertion. Let $\delta \in \Delta_+(G)$. Suppose that $\delta(w') \neq 0$. Then $\delta(m) \neq 0$ for $m \in M_{w'}$. In particular $\delta(w) = \pm 1$ and $\delta(w'w) \neq 0$. Thus $\delta(b) = \pm 1$ by definition and

$$\mid \delta(c) \mid = \mid \delta(a) \mid \leq q^{-m}.$$

The first assertion of the lemma follows.

Now we prove the second assertion. Again if $\delta(w') \neq 0$, then $\delta(w) \neq 0$ and

$$\mid \delta(a) \mid = \mid \delta(c) \mid \leq q^{-m}.$$

Now suppose that $\delta(w) \neq 0$ but $\delta(w') = 0$. Write as before $M_{w'}$ as a product of linear factors G_i and correspondingly $b = (b_i)$. Then

$$\delta(b) = \prod \delta'_i(b_i)$$

where $\delta'_i \in \Delta_0(G_i)$. Moreover $\delta'_i \in \Delta_+(G_i)$ for at least one index. Thus

$$\mid \delta(b) \mid \leq q^{-m}.$$

On the other hand

$$\mid \delta(c) \mid = \prod_j |\delta_j(c)|^{r_j} ;$$

the product is over all $\delta_j \in \Delta_+(G)$ such that $\delta_j(w') \neq 0$; the exponent r_j is rational and ≥ 0. Thus $\mid \delta(c) \mid \leq 1$ and $\mid \delta(a) \mid = \mid \delta(b)\delta(c) \mid \leq q^{-m}$. The second assertion follows. $\square$

We are now ready to state our next result on inductive systems. We let m be an integer, sufficient large. We consider inductive systems of germs. Thus we have already chosen the functions $K_w^{w'}$ for $w' \neq w_G$. By induction and the previous proposition, given n we may assume that each function $K_w^{w'}$ is supported on the set $A_w^{w'}(n)$.

Proposition 2.6. *Consider a function Φ supported on the set $w_G K_m$, such that:*

$$I(w_G, \Phi) = 1,$$

$$I(w_G z, \Phi) = 0 \quad \text{if } z \in F^\times ,\ z^r = 1 ,\ z \neq 1.$$

Let $n \geq m$. Then there is an inductive system of germs such that each function $K_w^{w_G}$ for $w \neq w_G$ is supported on the set $A_w^{w_G}(n)$ and

$$K_w^{w_G}(a) = I(wa, \Phi)$$

on $A_w^{w_G}(n)$.

Proof. We can choose an inductive system of germs K such that each function $K_w^{w_G}$ with $w \neq w_G$ is supported on $A_w^{w_G}(n)$. As before for $w \neq w_G$ we have the relation

$$I(wa, \Phi) = \omega_w^{K,\Phi}(a) + \sum_{w \neq w' \neq w_G} K_w^{w'} * \omega_{w'}^{K,\Phi}(a) + K_w^{w_G}(a)$$

on $A_w^{w_G}$. Suppose that each function $\omega_w^{K,\Phi}$ for $w \neq w_G$ vanishes on $A_w^{w_G}(n)$. Take $a \in A_w^{w_G}(n)$. Then the first term in this sum vanishes. Moreover, if $a = bc$ with $b \in A_w^{w'}$ and $c \in A_{w'}^{w_G}$, then $c \in A_{w'}^{w_G}(n)$ and $\omega_{w'}^{K,\Phi}(c) = 0$. Thus in the second term each convolution vanishes on a and the system of germs has the required property.

To obtain this result we modify the system of germs as follows. We consider a family of functions t_*^* such that $t_w^w = 1$, $t_w^{w_G}$ is supported on $A_w^{w_G}(n)$ and $\omega_w^{K,\Phi} = t_w^{w_G}$ on $A_w^{w_G}(n)$ for $w \neq w_G$; all other elements of the family are zero. Consider the system of germs defined by (2.6). Thus $H_w^{w_1} = K_w^{w_1}$ if $w_1 \neq w_G$ and

$$H_w^{w_G} = \sum_{w'} K_w^{w'} * t_{w'}^{w_G}.$$

Suppose $w \neq w_G$. By the previous lemma the function $H_w^{w_G}$ is supported on $A_w^{w_G}(n)$. The functions $\omega_w^{H,\Phi}$ is given by

$$\omega_w^{K,\Phi} = \omega_w^{H,\Phi} + t_w^{w_G}.$$

It vanishes on $A_w^{w_G}(n)$ and the system H_*^* has the required properties. □

We will need to determine how the system of germs depends on ψ. The choice of ψ determines a self-dual Haar measure on F:

$$\hat{\Phi}(y) = \int \Phi(x)\psi(-yx)dx, \int \hat{\Phi}(y)dy = \Phi(0).$$

If ψ_1 is another non trivial character, then $\psi_1(x) = \psi(sx)$ for some $s \in F^\times$. Then

$$\int \Phi(xs^{-1})\psi(x)dx = \mid s \mid^{1/2} \int \Phi(x)\psi_1(x)d_1x \tag{2.9}$$

where d_1x is the Haar measure self-dual with respect to ψ_1. Let K_*^* be a system of germs for the character ψ. The self-dual Haar measure is used to build a measure on the quotient spaces for our orbital integrals. This will be recalled below in the case $r = 3$. We set

$$S = \mathrm{diag}(s^{r-1}, s^{r-2}, \dots, s, 1).$$

For $r = 3$ a more convenient definition for S is:

$$S = \mathrm{diag}(s, 1, s^{-1}).$$

For $w \in R(W)$ we set

$$S_w = wSwS. \tag{2.10}$$

Then S_w is in A_w. Moreover for $w \to w'$

$$S_w = S_w^{w'} S_{w'} \tag{2.11}$$

where $S_w^{w'}$ is such that $\delta(S_w^{w'}) = 1$ if $\delta(w'w) \neq 0$. Given $\Phi \in \mathcal{C}(G)$ set

$$\Phi_1(x) = \Phi(SxS).$$

Denote by $I(wa, \Phi; \psi)$ the orbital integrals with respect to ψ. Then

$$I(waS_w^{-1}, \Phi_1; \psi) = \mid s \mid^{n_w} I(wa, \Phi; \psi_1)$$

where n_w is a suitable half-integer. We write

$$n_w = n_w^{w'} + n_{w'}.$$

We have then

$$\begin{aligned} I(wa, \Phi, \psi_1) &= \mid s \mid^{-n_w} I(waS_w^{-1}, \Phi_1; \psi) \\ &= \mid s \mid^{-n_w} \sum K_w^{w'} * \omega_{w'}^{K,\Phi_1}(aS_w^{-1}) \\ &= \sum K_{1w}^{w'} * \omega_{w'}^{K_1,\Phi}(a) \end{aligned}$$

where we have set

$$\begin{aligned} \omega_{w'}^{K_1,\Phi}(c) &= \mid s \mid^{-n_{w'}} \omega_{w'}^{K,\Phi_1}(cS_{w'}^{-1}), \\ K_{1,w}^{w'}(b) &= \mid s \mid^{-n_w^{w'}} K_w^{w'}(b(S_w^{w'})^{-1}). \end{aligned} \tag{2.12}$$

Thus the functions K_{1*}^{*} form a system of germs for the character ψ_1.

Finally, we set

$$J(g) = w_G\, {}^t g^{-1} w_G. \tag{2.13}$$

Thus J is an automorphism of G of order 2 which leaves N and A invariant. We have

$$\theta_\psi(J(n)) = \theta_{\psi^{-1}}(n).$$

If K is a system of germs for ψ, the functions K_{2*}^{*} defined by

$$K_{2w}^{w'}(a) = K_{Jw}^{Jw'}(Ja) \tag{2.14}$$

form a system of germs for ψ^{-1}.

3. Computation of $K_e^{w_G}$

From now on we assume $r = 3$. We want to compute the germs $K_w^{w_G}$. For our purposes, it suffices to do it when the conductor of the character ψ is the ring of integers $\mathcal{O}$. We choose an integer m_F sufficiently large. We drop the index F when this does not create a confusion. In particular, we assume that the relation $z^2 = 1$ or $z^3 = 1$ and $z \equiv 1 \bmod \wp^m$ implies $z = 1$. We also assume that the map $z \mapsto z^2$ defines an analytic bijection of $1+\wp^m$ onto $1+2\wp^m$. The inverse bijection is denoted by a square root. We define a function $\Phi \in \mathcal{C}(G)$ by the following conditions:

$$\Phi(x) = \mathrm{vol}(\wp^m)^{-3}$$

if

$$x_{31} \equiv x_{13} \equiv 1 \bmod \wp^m\, ,\, x_{22} \equiv 1 \bmod 2\wp^m,$$

$$x_{ij} \equiv 0 \bmod \wp^m \quad \text{if } i+j \neq 4.$$

If x does not satisfy the above conditions, then $\Phi(x) = 0$. Thus Φ is supported on a subset of $w_G K_m$. We have

$$I(w_G a, \Phi) = \int \Phi\left[a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ 1 & y & z \end{pmatrix}\right] \psi(x+y)\,dxdydz$$

where $dx = dy = dz$ is the self-dual Haar measure on F. Thus for $a \in F^\times$ with $a^3 = 1$ this vanishes unless $a = 1$ and is then equal to 1. Next we choose an integer

$n \geq m$ sufficiently large with respect to m and compute the inductive system of germs $K_w^{w_G}$ supported on the sets $A_w^{w_G}(n)$ which is determined by Φ, that is,

$$K_w^{w_G}(wa) = I(wa, \Phi)$$

on $A_w^{w_G}(n)$.

We first consider the germ $K_e^{w_G}$. Thus we consider a diagonal matrix

$$\alpha = \mathrm{diag}(a, b, c)$$

with $c = -1/ab$ and

$$\mid \delta_1(\alpha) \mid = \mid a \mid \leq q^{-n} \,,\ \mid \delta_2(\alpha) \mid = \mid ab \mid \leq q^{-n}.$$

and compute $K_e^{w_G}(\alpha)$ when $\mid b \mid \leq 1$.

Proposition 3.1. *With the previous notations, suppose* $\mid a \mid \leq q^{-n}$, $\mid b \mid \leq 1$. *Then:*

$$K_e^{w_G}(\alpha) = \mid b \mid^{1/2} \mid ab \mid^{-1} \gamma(1, \psi)\gamma(-b, \psi)(2, b) \int \psi\left(2x - \frac{2x}{b\sqrt{\mu}}\right)(x, b)dx$$

where we have set $\mu := b + ax^2$ *and the range of the integral is* $\mu \equiv 1 \bmod 2\wp^m$.

As usual $(.,.)$ denotes the quadratic residue symbol and the constant γ is the Weil constant. We recall that it is defined by the formula

$$\int \hat{\Phi}(x)\psi(\frac{ax^2}{2})dx = \mid a \mid^{-1/2} \gamma(a, \psi) \int \Phi(x)\psi(-\frac{x^2}{2a})dx. \tag{3.1}$$

We will not try to evaluate the integral of the proposition further because we will show that the germ for the quadratic extension is given by the **same** formula—up to a transfer factor. Regarding the computation of the integral, we remark that the phase function has critical points for $\mu = 1$. If $b \neq 1$, the critical points are non singular (the second derivative is not 0 at the critical point). We can then use the method of stationary phase to evaluate the integral for b fixed and $\mid a \mid$ small. However, if $b = 1$, the only critical point is at $\mu = 1$ and it is a singular point, so we cannot evaluate the integral by the method of stationary phase in this case.

The orbital integral is defined by

$$I(\alpha, \Phi) = \int \Phi({}^t n_2 \alpha n_1)\theta(n_2 n_1)dn_2 dn_1; \tag{3.2}$$

for $i = 1, 2$, we have set:

$$n_i = \begin{pmatrix} 1 & x_i & z_i \\ 0 & 1 & y_i \\ 0 & 0 & 1 \end{pmatrix}, \ dn_i = dx_i dy_i dz_i,$$

where the measures on the right are equal to the self-dual Haar measure on F; $\theta(n_i) = \psi(x_i + y_i)$.

To begin the computation we use a change of variables suggested by the work of Z. Mao. Let

$$S = \begin{pmatrix} a & ax_1 \\ ax_2 & \mu \end{pmatrix}, \ \mu = b + ax_1x_2.$$

Consider the matrix

$$T = \begin{pmatrix} S & S\begin{pmatrix} z_1 \\ y_1 \end{pmatrix} \\ \begin{pmatrix} z_2 & y_2 \end{pmatrix} S & \begin{pmatrix} z_2 & y_2 \end{pmatrix} S \begin{pmatrix} z_1 \\ y_1 \end{pmatrix} + c \end{pmatrix}.$$

Then $K_e^{w_G}$ is given by the integral

$$\mathrm{vol}(\wp^m)^{-3}\int \psi(x_1+x_2+y_1+y_2)dx_1dx_2dy_1dy_2dz_1dz_2$$

over the range $\Phi(T)\neq 0$. The conditions on the matrix S are

$$ax_1\equiv ax_2\equiv 0 \bmod \wp^m\,,\ \mu\equiv 1 \bmod 2\wp^m.$$

We can write

$$S=\begin{pmatrix}1 & \frac{ax_1}{\mu}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{ab}{\mu} & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{ax_2}{\mu} & 1\end{pmatrix}.$$

We introduce new variables v_1,u_1,v_2,u_2 by:

$$\begin{pmatrix}v_1\\ u_1\end{pmatrix}=\begin{pmatrix}1 & 0\\ \frac{ax_2}{\mu} & 1\end{pmatrix}\begin{pmatrix}z_1\\ y_1\end{pmatrix},\ \begin{pmatrix}v_2 & u_2\end{pmatrix}=\begin{pmatrix}z_2 & y_2\end{pmatrix}\begin{pmatrix}1 & \frac{ax_1}{\mu}\\ 0 & 1\end{pmatrix}.$$

In terms of these new variables the integral becomes:

$$\mathrm{vol}(\wp^m)^{-3}\int \psi\left(x_1+x_2+u_1+u_2-\frac{ax_2v_1}{\mu}-\frac{ax_1v_2}{\mu}\right)dx_1dx_2dv_1dv_2du_1du_2$$

and the domain of integration is defined by the following congruences mod $\wp^m$, except for μ:

$$ax_1\equiv ax_2\equiv 0\,,\ \mu\equiv 1 \bmod 2\wp^m,$$

$$abv_1\equiv abv_2\equiv 1\,,\ u_1\equiv u_2\equiv 0\,,\ v_2v_1\frac{ab}{\mu}+\mu u_1u_2+c\equiv 0.$$

After integrating over u_1,u_2 we find:

$$\mathrm{vol}(\wp^m)^{-1}\int \psi\left(x_1+x_2-\frac{ax_2v_1}{\mu}-\frac{ax_1v_2}{\mu}\right)dx_1dx_2dv_1dv_2$$

integrated over:

$$ax_1\equiv ax_2\equiv 0 \bmod \wp^m\,,\ \mu\equiv 1 \bmod 2\wp^m,$$

$$abv_1\equiv abv_2\equiv 1 \bmod \wp^m\,,\ abv_2abv_1\equiv \mu \bmod ab\wp^m.$$

We note that $\sqrt{\mu}\equiv 1 \bmod \wp^m$. Thus we can change v_1 and v_2 to $v_1\sqrt{\mu}/ab$ and $v_2\sqrt{\mu}/ab$ respectively to get:

$$\mathrm{vol}(\wp^m)^{-1}\mid ab\mid^{-2}\int \psi\left(x_1+x_2-\frac{x_2v_1}{b\sqrt{\mu}}-\frac{x_1v_2}{b\sqrt{\mu}}\right)dx_1dx_2dv_1dv_2$$

over

$$ax_1\equiv ax_2\equiv 0 \bmod \wp^m\,,\ \mu\equiv 1 \bmod 2\wp^m,$$

$$v_1\equiv v_2\equiv 1 \bmod \wp^m\,,\ v_2v_1\equiv 1+ \bmod ab\wp^m.$$

Next, we set

$$v_2=\frac{1+\xi}{v_1}$$

and integrate over $\xi\in ab\wp^m$. We get

$$\mid ab\mid^{-1}\int \psi\left(x_1+x_2-\frac{x_2v}{b\sqrt{\mu}}-\frac{x_1}{vb\sqrt{\mu}}\right)dx_1dx_2dv$$

over

$$ax_1 \equiv ax_2 \equiv 0 \bmod \wp^m \,,\; \mu \equiv 1 \bmod 2\wp^m, v \equiv 1 \bmod \wp^m.$$

Finally we change x_1 to $x_1 v$ and x_2 to x_2/v to get:

$$| ab |^{-1} \int \psi \left(-\frac{x_2}{b\sqrt{\mu}} - \frac{x_1}{b\sqrt{\mu}} \right) T(x_1, x_2) dx_1 dx_2 \tag{3.3}$$

over

$$ax_1 \equiv ax_2 \equiv 0 \bmod \wp^m \,,\; \mu \equiv 1 \bmod 2\wp^m,$$

where we have set $\mu = b + ax_1 x_2$ and

$$T(x_1, x_2) = \int_{1+\wp^m} \psi \left(x_1 v + \frac{x_2}{v} \right) dv. \tag{3.4}$$

Lemma 3.2. $T(x_1, x_2) = T(x_2, x_1)$. *Suppose* $k \geq 0$. *If* $| x_2 | \leq q^{2m+k}$, *then* $T(x_1, x_2) = 0$ *unless* $| x_2 - x_1 | \leq q^{m+k}$ *and then* $| x_1 | \leq q^{2m+k}$.

Proof. The first assertion is clear. For the second assertion, assume that $| x_2 | \leq q^{2m+k}$. If $T(x_1, x_2) \neq 0$, then there is $v \in 1 + \wp^m$ such that

$$\int_{1+\wp^{m+k}} \psi(x_1 v v_0 + \frac{x_2}{v v_0}) dv_0 \neq 0.$$

We can write $v_0 = 1 + u_0$ and the above integral is then equal to

$$\psi(x_1 v + \frac{x_2}{v}) \int_{\wp^{m+k}} \psi[(x_1 v - \frac{x_2}{v}) u_0] du_0.$$

This integral is 0 unless

$$| x_1 v - \frac{x_2}{v} | \leq q^{m+k}.$$

This relation implies that $| x_1 | \leq q^{2m+k}$ and then $| x_1 - x_2 | \leq q^{m+k}$ as claimed. □

.

The lemma allows us to write the integral for $K_e^{w_G}$ as the sum of two integrals I and II with the same integrand and the same conditions on the variables, except that for I we demand that

$$| x_1 | \leq q^{2m+k} \,,\; | x_1 - x_2 | \leq q^{m+k},$$

and for II we demand that

$$| x_1 | = | x_2 | > q^{2m+k}.$$

We fix an integer k even such that $q^{-k} \leq | 2 |^2$.

Recall that $| a | \leq q^{-n}$. Taking n sufficiently large with respect to m we see that on the domain of integration for I we have $| ax_1 x_2 | \leq | 2 | q^{-2m-k}$. Thus b is a unit; in fact $b \equiv 1 \bmod 2\wp^m$. Moreover $\sqrt{\mu} \equiv \sqrt{b} \bmod \wp^{2m+k}$. In particular on the domain of I we have:

$$\psi \left[-\frac{x_i}{b\sqrt{\mu}} \right] = \psi \left[-\frac{x_i}{b\sqrt{b}} \right].$$

The domain is defined by:

$$| x_1 | \leq q^{2m+k} \,,\; | x_1 - x_2 | \leq q^{m+k} \,,\; b \equiv 1 \bmod 2\wp^m.$$

After a change of variables, we find

$$I = \mid a \mid^{-1} \int\int \psi\left[x(v+\frac{1}{v}) - \frac{2x}{b\sqrt{b}}\right]\left(\int \psi\left[y(v-\frac{1}{b\sqrt{b}})\right]dy\right)dvdx. \tag{3.5}$$

The integral is over:

$$\mid y \mid \leq q^{m+k}\,,\ \mid x \mid \leq q^{2m+k}\,,\ v \in 1+\wp^m.$$

The integral over y is 0 unless

$$\mid v - \frac{1}{b\sqrt{b}} \mid \leq q^{-m-k}$$

and is then equal to q^{m+k} This inequality amounts to

$$v = \frac{1}{b\sqrt{b}}(1+u)$$

with $\mid u \mid \leq q^{-m-k}$. Thus

$$I = \mid a \mid^{-1} q^{m+k} \int\int \psi\left[x\left(\frac{1+u}{b\sqrt{b}} + \frac{b\sqrt{b}}{1+u} - \frac{2}{b\sqrt{b}}\right)\right]dxdu$$

over

$$\mid x \mid \leq q^{2m+k}\,,\ \mid u \mid \leq q^{-m-k}.$$

Over the range of integration we have $\mid xu^2 \mid \leq 1$ and also

$$\mid x(\frac{1}{b\sqrt{b}} - b\sqrt{b})u \mid \leq 1.$$

Thus the integrand does not depend on u and after integrating over u we obtain

$$I = \mid a \mid^{-1} \int \psi\left[x(b\sqrt{b} - \frac{1}{b\sqrt{b}})\right]dx$$

over $\mid x \mid \leq q^{2m+k}$. We claim that this integral is also equal to

$$I = \mid a \mid^{-1} \int \psi\left[2x(1 - \frac{1}{b\sqrt{b}})\right]dx$$

over the same range. Indeed, we can write $b\sqrt{b} = 1/(1+t)$ with $\mid t \mid \leq q^{-m}$. Then

$$b\sqrt{b} - \frac{1}{b\sqrt{b}} = \frac{1}{1+t} - (1+t) = -2t + t^2 - t^3 + \cdots = -2tu$$

where

$$u = 1 - \frac{t}{2} + \frac{t^2}{2} + \cdots.$$

We may (in fact we already) assume that $q^{-m} <\mid 2 \mid$; thus u is a unit. On the other hand

$$2(1 - \frac{1}{b\sqrt{b}}) = -2t.$$

Changing x to xu^{-1} we obtain our assertion. Thus finally:

$$I = \mid a \mid^{-1} \int \psi\left[2x\left(1 - \frac{1}{b\sqrt{\mu}}\right)\right]dx \tag{3.6}$$

where $\mu = b + ax^2$ and the domain of integration is defined by $\mu \equiv 1 \bmod 2\wp^m$ and $\mid x \mid \leq q^{2m+k}$.

We pass to the computation of II (see (17)):

$$II = \mid ab \mid^{-1} \int T(x_1, x_2)\psi \left[-\frac{x_1 + x_2}{b\sqrt{\mu}} \right] dx_1 dx_2 \tag{3.7}$$

taken over

$$ax_i \equiv 0 \bmod \wp^m \,,\ \mu := b + ax_1x_2 \equiv 1 \bmod 2\wp^m,$$

$$\mid x_1 \mid = \mid x_2 \mid > q^{2m+k}.$$

We need two lemmas.

Lemma 3.3. *Over the range of II if $T(x_1, x_2) \neq 0$, then $x_1 = x_2u^2$ with $u \in 1 + \wp^m$.*

Proof. Let us write $\mid x_1 \mid = \mid x_2 \mid = q^{2m+k+h}$ with $h > 0$. If $T(x_1, x_2) \neq 0$, then there is $v \in 1 + \wp^m$ such that

$$\int_{1+\wp^{m+h+k/2}} \psi \left[x_1vv_0 + \frac{x_2}{vv_0} \right] dv_0 \neq 0.$$

Up to a constant factor this integral is equal to

$$\int \psi \left((x_1v - \frac{x_2}{v})u_0 \right) du_0$$

over $\wp^{m+h+k/2}$. This integral vanishes unless

$$\mid x_1v - \frac{x_2}{v} \mid \leq q^{m+h+k/2}$$

or

$$\frac{x_2}{x_1v^2} \in 1 + \wp^{m+k/2}.$$

Since $q^{-k/2} \leq \mid 2 \mid$, this element is the square of an element in $1+\wp^m$ and the lemma follows. □

.

Lemma 3.4. *Suppose $\mid t \mid > q^{2m+k}$. Then the integral*

$$S(t) := \int_{1+\wp^m} \psi \left[t\left(v + \frac{1}{v} \right) \right] dv$$

is equal to

$$\mid 2t \mid^{-1/2} \psi(2t)\gamma(2t, \psi).$$

Proof. If we write $v = 1 + s$ with $s \in \wp^m$, then

$$v + \frac{1}{v} = 2 + \frac{s^2}{1+s} = 2 + u^2$$

where

$$u = \frac{s}{\sqrt{1+s}}.$$

In view of our assumption on m the map $s \mapsto u$ is an analytic bijection of $\wp^m$ onto itself. Thus we can rewrite the integral as

$$\psi(2t) \int \Phi(u)\psi \left(\frac{2tu^2}{2} \right) du$$

where Φ is the characteristic function of $\wp^m$. By (3.1) this is equal to

$$\psi(2t) \mid 2t \mid^{-1/2} \gamma(2t,\psi) \int \psi\left(-\frac{u^2}{4t}\right) \hat{\Phi}(u) du.$$

On the support of the new integrand $\mid u^2/4t \mid \leq 1$. Thus the integral on the right is the integral of the Fourier transform of Φ and is equal to $\Phi(0) = 1$. $\square$

We now compute II. We remark that the condition $\mu \equiv 1$ implies $\mid ax_1x_2 \mid \leq 1$ which, together with $\mid x_1 \mid = \mid x_2 \mid$, implies the condition $ax_1 \equiv ax_2 \equiv 0 \bmod \wp^m$. Thus:

$$II = \mid ab \mid^{-1} \int T(x_1,x_2)\psi\left(-\frac{x_1+x_2}{b\sqrt{\mu}}\right) dx_1 dx_2$$

taken over

$$\mu := b + ax_1x_2 \equiv 1 \bmod 2\wp^m \,,\, \mid x_1 \mid = \mid x_2 \mid > q^{2m+k}.$$

By Lemma (3.2), the integral does not change if we impose the further restriction that x_1/x_2 be the square of an element of $1+\wp^m$. We can then change variables as follows:

$$x_1 = xu^2 \,,\, x_2 = x$$

with $u \in 1+\wp^m$. Then $dx_1dx_2 = \mid 2x \mid dxdu$. The integral takes the form:

$$II = \mid ab \mid^{-1} \mid 2 \mid$$

$$\times \int \left(\int \psi\left(xv + xu^2v^{-1}\right) dv \right) \left(\int \psi\left(-\frac{x+xu^2}{b\sqrt{\mu}}\right) du \right) \mid x \mid dx.$$

Here $\mu = b + ax^2u^2$ and the range of integration is $\mu \equiv 1 \bmod 2\wp^m$ and $\mid x \mid > q^{2m+k}$, $u, v \in 1+\wp^m$. We can further change x to xu^{-1} and v to vu to arrive at:

$$II = \mid ab \mid^{-1} \mid 2 \mid \int S(x) S(-\frac{x}{b\sqrt{\mu}}) \mid x \mid dx$$

over

$$\mid x \mid > q^{2m+k} \,,\, \mu := b + ax^2 \equiv 1 \bmod 2\wp^m.$$

Recall that $\mid b \mid \leq 1$. Thus we can apply Lemma (3.3) to each one of the functions S to get:

$$II = \mid ab \mid^{-1} \mid b \mid^{1/2} \int \psi\left(2x - 2\frac{x}{b\sqrt{\mu}}\right) \gamma(2x,\psi)\gamma(-\frac{2x}{b\sqrt{\mu}},\psi) dx$$

taken over

$$\mu := b + ax^2 \equiv 1 \bmod 2\wp^m \,,\, \mid x \mid > q^{2m+k}.$$

If we take m sufficiently large, then $\sqrt{\mu}$ is a square and so disappears from the γ factor. Now we recall the formula

$$\gamma(\alpha,\psi)\gamma(\beta,\psi) = \gamma(1,\psi)\gamma(\alpha\beta,\psi)(\alpha,\beta).$$

We see that the product of the γ factors in the integrand is equal to

$$\gamma(1,\psi)\gamma(-b,\psi)(2,b)(b,x).$$

Thus we find

$$II = \gamma(1,\psi)\gamma(-b,\psi)(2,b) \mid ab \mid^{-1} \mid b \mid^{1/2} \int \psi\left(2x - 2\frac{x}{b\sqrt{\mu}}\right)(b,x)dx \tag{3.8}$$

taken over

$$\mu := b + ax^2 \equiv 1 \bmod 2\wp^m\,,\ \mid x \mid > q^{2m+k}.$$

At this point we remark that if $b \equiv 1 \bmod 2\wp^m$, then b is a square and we have

$$\gamma(1,\psi)\gamma(-b,\psi)(2,b) = \gamma(1,\psi)\gamma(-1,\psi) = 1,$$

$$\mid b \mid = 1\,,\ (b,x) = 1.$$

Thus we can rewrite I in the same form as II but over the domain:

$$\mu := b + ax^2 \equiv 1 \bmod 2\wp^m\,,\ \mid x \mid \leq q^{2m}.$$

If we add I and II, we get the result announced in Proposition (3.1).

4. Computation of $K^{w_G}_{w_1}$

We continue with the notations of the previous section. Apart from w_G the remaining elements of $R(G)$ are w_1, w_2 where:

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now we compute $K^{w_G}_{w_1}$. Let $\alpha \in A^{w_G}_{w_1}(n)$. Thus

$$\alpha = \mathrm{diag}(a, a, a^{-2}) = a\ \mathrm{diag}(1, 1, a^{-3})$$

with $\mid a \mid^2 \leq q^{-n}$.

Proposition 4.1. *For* $\mid a \mid^2 \leq q^{-n}$

$$K^{w_G}_{w_1}(\alpha) = \mid a \mid^{-2} \mid 3 \mid^{-1/2}\ \psi\left(\frac{3}{a}\right)\gamma(2a,\psi)\gamma(6a,\psi).$$

Proof. As before:

$$K^{w_G}_{w_1}(\alpha) = \int \Phi\left({}^t n_2 \alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} n_1\right)\theta(n_1 n_2)\psi(x)dx dn_1 dn_2 \tag{4.1}$$

where

$$n_i = \begin{pmatrix} 1 & 0 & z_i \\ 0 & 1 & y_i \\ 0 & 0 & 1 \end{pmatrix},$$

$dn_i = dy_i dz_i$ and $\theta(n_i) = \psi(y_i)$. As usual the measures are equal to the self-dual Haar measure. After changing z_i to $z_i - xy_i$ this becomes

$$\int \Phi\left[a\begin{pmatrix} 0 & 1 & y_1 \\ 1 & x & z_1 \\ y_2 & z_2 & a^{-3} + y_2 z_1 + z_2 y_1 - x y_1 y_2 \end{pmatrix}\right]$$

$$\times \psi(x + y_1 + y_2)dx dy_1 dy_2 dz_1 dz_2$$

where the support of Φ is defined by:

$$ay_i \equiv 1 \bmod \wp^m\,,\ ax \equiv 1 \bmod 2\wp^m\,,\ az_i \equiv 0 \bmod \wp^m,$$

$$a^{-2} + ay_2z_1 + ay_1z_2 - axy_1y_2 \equiv 0 \bmod \wp^m.$$

Changing z_1, z_2 to $z_1(ay_2)^{-1}, z_2(ay_1)^{-1}$ (and noting that $| ay_i |= 1$ on the domain of integration) we obtain for new domain of integration:

$$ay_i \equiv 1 \bmod \wp^m \,,\, ax \equiv 1 \bmod 2\wp^m \,,\, az_i \equiv 0 \bmod \wp^m,$$

$$a^{-2} + z_1 + z_2 - axy_1y_2 \equiv 0 \bmod \wp^m.$$

Next we change x to xa^{-1} and change all other variables similarly. We get:

$$\mathrm{vol}(\wp^m)^{-3} \mid a \mid^{-5} \int \psi \left[\frac{x + y_1 + y_2}{a} \right] dxdy_1dy_2dz_1dz_2$$

over

$$z_i \equiv 0 \bmod \wp^m \,,\, x \equiv 1 \bmod 2\wp^m \,,\, y_i \equiv 1 \bmod \wp^m,$$

$$z_1 + z_2 + a^{-1} - \frac{xy_1y_2}{a} \equiv 0 \bmod a\wp^m.$$

The integral over z_1, z_2 is 0 unless

$$a^{-1} - \frac{xy_1y_2}{a} \equiv 0 \bmod \wp^m.$$

If we impose this condition, we can change z_1 to

$$z_1 - a^{-1} - \frac{xy_1y_2}{a}$$

and integrate z_1, z_2 over the range:

$$z_1 \equiv z_2 \equiv 0 \mod \wp^m, z_1 + z_2 \equiv 0 \mod a\wp^m.$$

We get:

$$\mid a \mid^{-4} \mathrm{vol}(\wp^m)^{-1} \int \psi \left[\frac{x + y_1 + y_2}{a} \right] dxdy_1dy_2$$

over

$$x \equiv 1 \bmod 2\wp^m \,,\, y_i \equiv 1 \bmod \wp^m,$$

$$xy_1y_2 \equiv 1 \bmod a\wp^m.$$

We set

$$y_2 = \frac{1 + u}{xy_1}$$

with $u \in a\wp^m$ and integrate over u. We get:

$$\mid a \mid^{-3} \int \psi \left[\frac{Q(x, y)}{a} \right] dxdy$$

over

$$x \equiv 1 \bmod 2\wp^m \,,\, y \equiv 1 \bmod \wp^m$$

where we have set

$$Q(x, y) = x + y + x^{-1}y^{-1}.$$

If we set $x = 1 + u, y = 1 + v$, then Q has only one point critical point namely the point $u = 0, v = 0$ and it is a regular point since the Taylor expansion of Q up to order 2 at this point reads:

$$Q = 3 + u^2 + v^2 + uv + \cdots .$$

By the principle of stationary phase, if n is sufficiently large with respect to m, the integral depends only on the quadratic part of the Taylor expansion of Q and is thus equal to:

$$\mid a \mid^{-3} \ \psi(\frac{3}{a}) \int \psi \left(\frac{u^2+v^2+uv}{a} \right) dudv.$$

The integral is taken over a small enough neighborhood of 0. If we set

$$u_1 = u + \frac{v}{2}\,,\ v_1 = v,$$

the integral becomes:

$$\mid a \mid^{-3} \ \psi(\frac{3}{a}) \int \psi \left(\frac{2u_1^2}{2a} \right) du_1 \int \psi \left(\frac{3v_1^2}{4a} \right) dv_1.$$

By (3.1), if n is sufficiently large we obtain Proposition (4.1).

5. Germs over the quadratic extension

We consider a quadratic extension E/F and denote by $\eta_{E/F}$ or simply η the quadratic character of $F^\times$ attached to E. We write $E = F(\sqrt{\tau})$ where $\mid \tau \mid_F = 1$ or $\mid \tau \mid_F = q_F^{-1}$. We fix an additive character $\psi = \psi_F$ of F and set $\psi_E(z) = \psi_F(z + \overline{z})$. We denote by dx, $x \in F$, the self-dual Haar measure on F and by dz, $z \in E$, the self-dual Haar measure on E. If we write $z = z_1 + z_2\sqrt{\tau}$ with $z_i \in F$, then $dz = \mid 2 \mid_F \mid \tau \mid_F^{1/2} dz_1 dz_2$. We denote by $S(r, F)$ the set of invertible Hermitian matrices in $GL(r, E)$.

The group $N(E)$ operates on $S(r, F)$ by:

$$s \overset{n}{\mapsto} {}^t\overline{n} s n.$$

We can use this action to define the relevant orbits of $N(E)$ on $S(r, F)$. As before, the elements of the form wa with $w \in R(G)$ and $a \in A_w(F)$ form a set of representatives for the relevant orbits. We can then define orbital integrals by:

$$J(wa, \Phi) = \int \Phi({}^t\overline{n} wan) \theta_\psi(n\overline{n}) dn,$$

the integral over the quotient of $N(E)$ by the stabilizer of wa. The choice of the invariant measures depends on ψ and will be recalled in the case of $r = 3$. The product $n\overline{n}$ is in $N(F)$ modulo an element of the derived group of $N(E)$ so that $\theta_\psi(n\overline{n})$ is well defined. We can define the notion of a system of Shalika germs L_*^* for these orbital integrals. In particular, our results on the support of the Shalika germs and the dependence of the germs on the choice of the character ψ_F apply to the present situation. Our goal is the following theorem:

Theorem 5.1. *There exist systems of germs L_*^* and K_*^* such that:*

$$L_e^{w_G}(a, b, -1/ab) = \eta_{E/F}(b) K_e^{w_G}(a, b, -1/ab), \tag{5.1}$$

$$L_{w_1}^{w_G}(a, a, a^{-2}) = \eta_{E/F}(a) c(E/F, \psi) K_{w_1}^{w_G}(a, a, a^{-2}), \tag{5.2}$$

$$L_{w_2}^{w_G}(a^2, a^{-1}, a^{-1}) = \eta_{E/F}(-a) c(E/F, \psi) K_{w_2}^{w_G}(a^2, a^{-1}, a^{-1}), \tag{5.3}$$

$$L_e^{w_1}(a, a^{-1}, 1) = \eta_{E/F}(a) c(E/F, \psi) K_e^{w_1}(a, a^{-1}, 1), \tag{5.4}$$

$$L_e^{w_2}(1, a, a^{-1}) = \eta_{E/F}(-a) c(E/F, \psi) K_e^{w_2}(1, a, a^{-1}), \tag{5.5}$$

where the constant c is defined in terms of the Weil constant by

$$c(E/F,\psi)=\gamma(\tau,\psi)\gamma(1,\psi)^{-1}\eta_{E/F}(2). \tag{5.6}$$

If $\psi_1(x)=\psi(sx)$, then $c(E/F,\psi_1)=c(E/F,\psi)\eta_{E/F}(s)$. In particular,

$$c(E/F,\psi^{-1})=\eta_{E/F}(-1)c(E/F,\psi).$$

If K_*^* is a system of germs for the character ψ and $\psi_1(x)=\psi(sx)$, then the following formulas define a system of germs for the character ψ_1:

$$K_{1e}^{w_G}(a,b,-1/ab) = K_e^{w_G}(as^{-2},b,-s^2/ab)\mid s\mid^{-5/2}, \tag{5.7}$$
$$K_{1w_1}^{w_G}(a,a,a^{-2}) = K_{w_1}^{w_G}(as^{-1},as^{-1},s^2a^{-2})\mid s\mid^{-1/2}, \tag{5.8}$$
$$K_{1w_2}^{w_G}(a^2,a^{-1},a^{-1}) = K_{w_2}^{w_G}(s^{-2}a^2,sa^{-1},sa^{-1})\mid s\mid^{-1/2}, \tag{5.9}$$
$$K_{1e}^{w_1}(a,a^{-1},1) = K_e^{w_1}(as^{-1},sa^{-1},1)\mid s\mid^{-2}, \tag{5.10}$$
$$K_{1e}^{w_2}(1,a,a^{-1}) = K_e^{w_2}(1,s^{-1}a,sa^{-1})\mid s\mid^{-2}\,. \tag{5.11}$$

Similar results apply to the germs L_*^*. Thus it suffices to prove the theorem for one character ψ. In particular, we may assume the conductor of ψ_F is $\mathcal{O}_F$. Identities (5.4) and (5.5) have been proved in [JY3] (Propositions (2.3) and (3.1)).

We now consider the system of germs constructed in the previous sections. It depends on the choice of the two integers m_F and n as well as the character ψ_F with conductor $\mathcal{O}_F$. Recall that we first choose the integer m_F sufficiently large. We then choose the integer n sufficiently large with respect to m_F and then the functions of the germs have support in $A_*^*(n)$. In this section we choose an integer $m=m_E$ as follows. If E/F is unramified, we take $m_E=m_F$. If E/F is ramified, we take $m_E=2m_F$. Thus in all cases $\wp_E^m\cap F=\wp_F^{m_F}$. How large the integer m_F (or m_E) needs be depends on the quadratic extension. We let $U(m)$ be the group of $z\in 1+\wp_E^m$ such that $z\overline{z}=1$. If m is sufficiently large, the elements of $U(m)$ can be written in the form:

$$z=\sqrt{1+v^2\tau}+v\sqrt{\tau}\,,\ v\in F\,,\ v\sqrt{\tau}\in\wp_E^m. \tag{5.12}$$

Then, if dv denotes the self-dual Haar measure on F,

$$dz=dv \tag{5.13}$$

is a Haar measure on $U(m)$. We denote by $A(m)$ the set of elements of $E^\times$ of the form

$$tz\,,\ t\in(1+\wp_E^m)\cap F\,,\ z\in U(m). \tag{5.14}$$

The set $A(m)$ is a subgroup of $1+\wp_E^m$ and contains $1+\wp_E^{2m}$. As a matter of fact $A(m)$ is the set of elements of the form $x+y\sqrt{\tau}$ with $x\in F\cap(1+\wp_E^m)$ and $y\in F$, $y\sqrt{\tau}\in\wp_E^m$. We denote by $\Psi\in\mathcal{C}(S(3,F))$ the function defined by the conditions

$$\Psi(x)=\operatorname{vol}(\wp_E^m)^{-1}\operatorname{vol}(\wp_E^m\cap F)^{-1}$$

if

$$x_{22}\equiv 1 \bmod 2\wp_E^m\ ,\ x_{13},x_{31}\in A(m),$$

$$x_{ij}\equiv 0 \bmod \wp_E^m\,,\ \text{ if } i+j\neq 4,$$

and $\Psi(x) = 0$ otherwise. Now:

$$J(w_G a, \Psi) = \int \Psi \left[a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ 1 & \overline{x} & z \end{pmatrix} \right] \psi(x + \overline{x}) dx dz$$

where dx is the self-dual Haar measure on E and dz the self-dual Haar measure on F. As before if $a \in F^\times$ with $a^3 = 1$ and m is sufficiently large, then $J(w_G a, \Psi) = 1$ if $a = 1$ and $J(w_G a, \Psi) = 0$ otherwise. If n is sufficiently large (with respect to m), there is an inductive system of germs L_*^* supported on the sets $A_*^*(n)$ such that on $A_w^{w_G}(n)$:

$$L_w^{w_G}(a) = J(wa, \Psi).$$

The automorphism J (see (2.13)) leaves the function Φ of the previous section and the function Ψ invariant and thus transforms the systems K and L defined by m_F, n, ψ into the systems defined by m_F, n, ψ^{-1}. Since $J(w_1) = w_2$, assertion (5.2) implies (5.3). Similarly, it suffices to prove assertion (5.1) for $\mid b \mid \leq 1$ since $J(a, b, -1/ab) = (-ab, b^{-1}, a^{-1})$. Thus Theorem (5.1) will be a consequence of Propositions (3.1), (4.1) and Propositions (5.1) and (6.1) below:

Proposition 5.2. *Suppose*

$$\alpha = \mathrm{diag}(a, b, -1/ab)$$

with

$$\mid a \mid_F \leq q_F^{-n} , \mid b \mid_F \leq 1.$$

Then:

$$L_e^{w_G}(\alpha) =$$

$$\mid ab \mid_F^{-1} \mid b \mid_F^{1/2} \gamma(-b, \psi) \gamma(1, \psi)(2, b) \eta_{E/F}(b) \int \psi \left(2t - \frac{2t}{b\sqrt{\mu}} \right) (t, b) dt,$$

the integral over the set defined by $t \in F$ and:

$$\mu := b + at^2 \equiv 1 \bmod 2\wp_F^{m_F}.$$

Proof. As before $L_e^{w_G}(\alpha) = J(\alpha, \Psi)$. We introduce the matrices:

$$S = \begin{pmatrix} a & ax \\ a\overline{x} & \mu \end{pmatrix}, \mu = b + ax\overline{x},$$

$$T = \begin{pmatrix} S & S \begin{pmatrix} z \\ y \end{pmatrix} \\ \begin{pmatrix} \overline{z} & \overline{y} \end{pmatrix} S & \begin{pmatrix} \overline{z} & \overline{y} \end{pmatrix} S \begin{pmatrix} z \\ y \end{pmatrix} + c \end{pmatrix}.$$

Then $L_e^{w_G}$ is given by the integral

$$\mathrm{vol}(\wp_E^m)^{-1} \mathrm{vol}(\wp_E^m \cap F)^{-1} \int \psi(x + \overline{x} + y + \overline{y}) dx dy dz$$

over the range $\Psi(T) \neq 0$. The conditions on the matrix S are

$$ax \equiv 0 \bmod \wp_E^m , \mu \equiv 1 \bmod 2\wp_E^m.$$

We can write

$$S = \begin{pmatrix} 1 & \frac{ax}{\mu} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{ab}{\mu} & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{a\overline{x}}{\mu} & 1 \end{pmatrix}.$$

We introduce new variables v, u by:

$$\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{a\overline{x}}{\mu} & 1 \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}.$$

Then the integral can be written as:

$$\mathrm{vol}(\wp_E^m)^{-1}\,\mathrm{vol}(\wp_E^m \cap F)^{-1} \int \psi\left(x + \overline{x} + u + \overline{u} - \frac{a\overline{x}v}{\mu} - \frac{ax\overline{v}}{\mu}\right) dxdvdu$$

taken over

$$ax \equiv 0 \bmod \wp_E^m\,,\ \mu := b + ax\overline{x} \equiv 1 \bmod 2\wp_E^m,$$

$$\frac{abv}{\mu} + axu \in A(m)\,,\ \mu u \equiv 0 \bmod \wp_E^m,$$

$$\frac{abv\overline{v}}{\mu} + \mu u\overline{u} - \frac{1}{ab} \equiv 0 \bmod \wp_E^m.$$

This can be simplified:

$$ax \equiv 0 \bmod \wp_E^m\,,\ \mu := b + ax\overline{x} \equiv 1 \bmod 2\wp_E^m,$$

$$u \equiv 0 \bmod \wp_E^m\,,\ abv \in A(m)\,,\ (abv)(ab\overline{v}) \equiv \mu \bmod ab\wp_E^m.$$

We integrate over u and change v to v/ab to get

$$\mathrm{vol}(\wp_E^m \cap F)^{-1} \mid ab \mid_F^{-2} \int \psi\left(x + \overline{x} - \frac{\overline{x}v + x\overline{v}}{b\mu}\right) dxdv$$

taken over

$$ax \equiv 0 \bmod \wp_E^m\,,\ \mu := b + ax\overline{x} \equiv 1 \bmod 2\wp_E^m,$$

$$v \in A(m)\,,\ v\overline{v} \equiv \mu \bmod ab\wp_E^m.$$

If m is sufficiently large, μ has a square root in $F \cap (1 + \wp_E^m)$. We change v to $v\sqrt{\mu}$ and remark that $\sqrt{\mu}$ is in $A(m)$. Thus the integral takes the form:

$$\mathrm{vol}(\wp_E^m \cap F)^{-1} \mid ab \mid_F^{-2} \int \psi\left(x + \overline{x} - \frac{\overline{x}v + x\overline{v}}{b\sqrt{\mu}}\right) dxdv$$

taken over

$$ax \equiv 0 \bmod \wp_E^m\,,\ \mu := b + ax\overline{x} \equiv 1 \bmod 2\wp_E^m,$$

$$v \in A(m)\,,\ v\overline{v} \equiv 1 \bmod ab\wp_E^m.$$

Now $v\overline{v}$ is in $F \cap (1 + ab\wp_E^m)$. Thus it is the square of an element $1 + t$ with $t \in F \cap 2^{-1}ab\wp_E^m$. If n is sufficiently large, the element t is also in $\wp_E^m$. Then

$$v = u(1 + t)$$

with $u \in U(m)$ and $t \in F \cap 2^{-1}ab\wp_E^m$. Recall (see (5.12)) that if m is sufficiently large, we can write

$$u = \sqrt{1 + s^2\tau} + s\sqrt{\tau}$$

where $s\sqrt{\tau} \in \wp_E^m$ and $s \in F$ and then $du = ds$ is a Haar measure on $U(m)$. We have $dv =\mid 2 \mid_F \mid \tau \mid_F^{1/2} dudt$. Moreover,

$$\left|\frac{xut}{b}\right|_E \leq \left|\frac{ax}{2}\right|_E \leq \mid 2 \mid_E^{-1} q_E^{-m} \leq 1.$$

Thus the integrand does not depend on t. After integrating over t, the integral takes the form:

$$\mid \tau \mid_F^{1/2} \mid ab \mid_F^{-1} \int \psi\left(x + \overline{x} - \frac{\overline{x}u + x\overline{u}}{b\sqrt{\mu}}\right) dxdu$$

taken over

$$ax \equiv 0 \bmod \wp_E^m\,,\ \mu := b + ax\overline{x} \equiv 1 \bmod 2\wp_E^m\,,\ u \in U(m).$$

At this point, it is convenient to change u to its inverse and then x to xu to get:

$$L_e^{w_G}(\alpha) =\mid \tau \mid_F^{1/2} \mid ab \mid^{-1} \int \psi\left(-\frac{\overline{x} + x}{b\sqrt{\mu}}\right) T(x)dx \tag{5.15}$$

taken over

$$ax \equiv 0 \bmod \wp_E^m\,,\ \mu := b + ax\overline{x} \equiv 1 \bmod 2\wp_E^m,$$

where we have set

$$T(x) = \int_{U(m)} \psi(xu + \overline{xu})du. \tag{5.16}$$

As before we write the integral as the sum of two integrals I and II with the same integrand but I is over the set of x with $\mid x \mid_E \leq q_E^{2m+k}$ and II over the set of x such that $\mid x \mid_E > q_E^{2m+k}$. The integer $k \geq 0$ is even if $\mid \tau \mid = 1$, odd if $\mid \tau \mid = q_F^{-1}$. Moreover, it satisfies additional conditions which will be specified below.

Consider the integral I. In view of our assumption on $\mid a \mid$, if n is sufficiently large with respect to m and k, the condition $\mid x \mid_E \leq q_E^{2m+k}$ implies $ax \equiv 0 \bmod \wp_E^m$ and $ax\overline{x} \equiv 0 \bmod 2\wp_E^{2m+k}$. Thus $I = 0$ unless $b \equiv 1 \bmod 2\wp_E^m$ and then

$$I =\mid ab \mid^{-1} \mid \tau \mid_F^{1/2} \int \psi\left(-\frac{\overline{x} + x}{b\sqrt{b}}\right) T(x)dx \tag{5.17}$$

taken over $\mid x \mid_E \leq q_E^{2m+k}$. We need a lemma:

Lemma 5.3. *Let k be even if $\mid \tau \mid_F = 1$ and odd otherwise. Suppose further $q_E^{-k} \leq \mid 2 \mid_E$. Then if $\mid x \mid_E \leq q_E^{2m+k}$, $T(x) = 0$ unless $\mid x - \overline{x} \mid_E \leq q_E^{m+k}$.*

Proof. As before if $T(x) \neq 0$, then there is $u \in U(m)$ such that the following integral is non zero:

$$\int_{U(m+k)} \psi(xuu_0 + \overline{xuu_0})du_0.$$

If we set $xu = z_1 + z_2\sqrt{\tau}$, the integral reads

$$\int \psi(2z_1\sqrt{1 + s_0^2\tau} + 2z_2 s_0 \tau)ds_0$$

where $s_0 \in F$ and $\mid s_0\sqrt{\tau} \mid_E \leq q_E^{-m-k}$. We get then $\mid s_0^2\tau \mid_E \leq q_E^{-2m-2k}$. On the other hand $\mid 2z_1 \mid_E \leq q_E^{2m+k}$. In view of the assumption on k, we get $\mid z_1 s_0^2 \tau \mid_E \leq 1$ and the integral reads:

$$\psi(2z_1) \int \psi(2z_2 s_0 \tau) ds_0.$$

We claim that this integral vanishes unless $\mid 2z_2\sqrt{\tau} \mid_E \leq q_E^{m+k}$. Indeed, if the extension is unramified, then $\mid \tau \mid_F = 1$ and $q_E = q_F^2$. The range of s_0 is then $\mid s_0 \mid_F \leq q_F^{-m-k}$ and the integral vanishes unless $\mid 2z_2 \mid_F \leq q_F^{m+k}$ which is equivalent to $\mid 2z_2\sqrt{\tau} \mid_E \leq q_E^{m+k}$. Now suppose the extension is ramified. Then $q_E = q_F = q$ and m is even. Suppose first $\mid \tau \mid_F = 1$. Recall k is then even. The range of s_0 is defined by $\mid s_0 \mid_E \leq q^{-m-k}$ or $\mid s_0 \mid_F \leq q^{-(m+k)/2}$. Then the integral vanishes unless $\mid 2z_2 \mid_F \leq q^{(m+k)/2}$ which is equivalent to $\mid 2z_2\sqrt{\tau} \mid_E \leq q^{m+k}$. Finally, assume $\mid \tau \mid_F = q^{-1}$. Recall k is then odd. Then $\mid \sqrt{\tau} \mid_E = q^{-1}$. The range of s_0 is then defined by $\mid s_0 \mid_E \leq q^{1-m-k}$ or $\mid s_0 \mid_F \leq q^{(1-m-k)/2}$. Thus the integral vanishes unless $\mid 2z_2\tau \mid_F \leq q^{(m+k-1)/2}$ or $\mid 2z_2 \mid_F \leq q^{(m+k+1)/2}$, that is, $\mid 2z_2\sqrt{\tau} \mid_E \leq q^{m+k}$.

If we write $x = x_1 + x_2\sqrt{\tau}$ and $u = \sqrt{1+s^2\tau} + s\sqrt{\tau}$, we have

$$2z_2\sqrt{\tau} = 2x_2\sqrt{\tau}\sqrt{1+s^2\tau} + 2x_1 s\sqrt{\tau}.$$

Since $\mid 2x_1 \mid_E \leq q_E^{2m+k}$, we find $\mid 2x_1 s\sqrt{\tau} \mid_E \leq q_E^{m+k}$ which implies that

$$\mid 2x_2\sqrt{\tau} \mid_E \leq q_E^{m+k}$$

as claimed. □

From now on we assume that k satisfies the conditions of the lemma. If we write $x = x_1 + x_2\sqrt{\tau}$ in the integral I, then by the previous lemma the integral does not change if we restrict x_2 to the range $\mid 2x_2\sqrt{\tau} \mid_E \leq q_E^{m+k}$. By the assumption on k, this inequality implies $\mid x_2\sqrt{\tau} \mid_E \leq q_E^m$. Then the condition $\mid x \mid_E \leq q_E^{2m+k}$ is equivalent to $\mid x_1 \mid_E \leq q_E^{2m+k}$. Thus we can write

$$I = \mid ab \mid_F^{-1} \mid 2\tau \mid_F \int \psi\left[2x_1\sqrt{1+s^2\tau} + 2x_2 s\tau - \frac{2x_1}{b\sqrt{b}}\right] dx_1 dx_2 ds,$$

the integral over

$$\mid x_1 \mid_E \leq q_E^{2m+k} , \mid 2x_2\sqrt{\tau} \mid_E \leq q_E^{m+k} , \mid s\sqrt{\tau} \mid_E \leq q_E^{-m}.$$

As in the proof of the previous lemma, if we integrate over x_2 first, the resulting integral vanishes unless $\mid s\sqrt{\tau} \mid_E \leq q_E^{-m-k}$. Thus the integral does not change if we take for the range of s the set $\mid s\sqrt{\tau} \mid_E \leq q_E^{-m-k}$. Then $\psi(2x_1\sqrt{1+s^2\tau}) = \psi(2x_1)$. Thus the integral I contains as a factor the integral

$$\int ds \int \psi(2x_2 s\tau) dx_2 = \mid 2 \mid_F^{-1} \mid \tau \mid_F^{-1} .$$

Thus we find that $I = 0$ unless $b \equiv 1 \bmod 2\wp_E^m$ and then

$$I = \mid ab \mid_F^{-1} \int \psi\left[2x - \frac{2x}{b\sqrt{b}}\right] dx$$

taken over $x \in F$ with $\mid x \mid_E \leq q_E^{2m+k}$. This can also be written

$$I = \mid ab \mid_F^{-1} \int \psi\left[2x - \frac{2x}{b\sqrt{\mu}}\right] dx \tag{5.18}$$

where $\mu = b + ax^2$ and the range is defined by $\mu \equiv 1 \bmod 2\wp_E^m$ and $x \in F$, $| x |_E \leq q_E^{2m+k}$.

We pass to the computation of II. As before (see (38)):

$$II = | ab |_F^{-1} | \tau |_F^{1/2} \int \psi\left(-\frac{\overline{x}+x}{b\sqrt{\mu}}\right) T(x) dx \tag{5.19}$$

taken over

$$ax \equiv 0 \bmod \wp_E^m \, , \; | x |_E > q_E^{2m+k} \, , \; \mu := b + ax\overline{x} \equiv 1 \bmod 2\wp_E^m .$$

As before, if n is sufficiently large in comparison with m, the first condition is a consequence of the other conditions. At this point we need another lemma:

Lemma 5.4. *Let k' be an integer satisfying the conditions of the previous lemma. Set $k = 3k'$. Then if m is sufficiently large and $| x |_E > q_E^{2m+k}$, the integral $T(x)$ vanishes unless $x \in F^\times U(m)$.*

Proof. We stress that, in this lemma, how large m needs be depends on the quadratic extension but not on the integer k. We can write

$$| x |_E = q_E^{2m+2h+k} \, , \text{ or } \; | x |_E = q_E^{2m+2h-1+k} ,$$

where $h > 0$. In any case

$$| x |_E \leq q_E^{2m+2h+k} .$$

By the previous lemma $T(x) = 0$ unless x has the form:

$$x = x_1 + x_2\sqrt{\tau}$$

with $| 2x_2\sqrt{\tau} |_E \leq q_E^{m+h+2k'}$. Then

$$| x_1 |_E = q_E^{2m+2h+k} \, , \text{ or } \; | x_1 |_E = q_E^{2m+2h-1+k} ,$$

and we can write

$$x = x_1\left(1 + \frac{x_2\sqrt{\tau}}{x_1}\right),$$

$$\left| \frac{x_2\sqrt{\tau}}{x_1} \right|_E \leq | 2 |_E^{-1} q_E^{-m-k'-h+1} \leq | 2 |_E^{-1} q_E^{-m-k'} \leq q_E^{-m} .$$

Thus $x \in F^\times A(m) = F^\times U(m)$. $\square$

From now on we assume that k satisfies the conditions of Lemma (5.2). Then it satisfies the conditions of Lemma (5.1) as well. Thus in the integral II we can set $x = tv$ with $v \in U(m)$ and $t \in F^\times$. Then $dx = | 2 |_F | \tau |_F^{1/2} | t |_F \, dt dv$. The measure dv has been defined earlier. We find:

$$II = | ab |_F^{-1} | 2\tau |_F \int \left(\int \psi(t(u+\overline{u})) du \right) \left(\int \psi\left[-\frac{t(v+\overline{v})}{b\sqrt{\mu}} \right] dv \right) | t |_F \, dt$$

over $u, v \in U(m)$ and $t \in F^\times$ with $| t |_E > q_E^{2m+k}$ and $\mu := b + at^2 \equiv 1 \bmod 2\wp_E^m$. We apply again the method of stationary phase in the following form:

Lemma 5.5. *If m is sufficiently large and if k is large enough in comparison with m, then, for $t \in F^\times$ with $| t |_E > q_E^{2m+k}$, the integral $T(t)$ (see (39)) is given by:*

$$T(t) = | 2\tau t |_F^{-1/2} \psi(2t)\gamma(2t\tau, \psi).$$

Proof. The integral has the form

$$T(t) = \int \psi(2t\sqrt{1+s^2\tau})ds$$

for $\mid s\sqrt{\tau} \mid_E \leq q_E^{-m}$. If m is sufficiently large, we can set

$$u = \frac{s}{\sqrt{\frac{\sqrt{1+s^2\tau}+1}{2}}}.$$

Then

$$T(t) = \psi(2t) \int \psi(tu^2\tau)du.$$

If k is sufficiently large, the integral has the required value. □

Thus we now assume that the integer k satisfies the conditions of the three previous lemmas. We can then use the last lemma to compute the inner integrals of II:

$$II = \mid ab \mid_F^{-1} \mid b \mid_F^{1/2} \int \psi\left(2t - \frac{2t}{b\sqrt{\mu}}\right)\gamma(2t\tau,\psi)\gamma(-2tb\sqrt{\mu}\tau,\psi)dt.$$

If m is sufficiently large, $\sqrt{\mu}$ is a square and so the second γ factor does not depend on μ. The product of the γ factors is

$$\gamma(-b,\psi)\gamma(1,\psi)(2,b)(t,b)\eta_{E/F}(b).$$

Thus we find

$$II = \mid ab \mid_F^{-1} \mid b \mid_F^{1/2} \gamma(-b,\psi)\gamma(1,\psi)(2,b)\eta_{E/F}(b) \int \psi\left(2t - \frac{2t}{b\sqrt{\mu}}\right)(t,b)dt \tag{5.20}$$

over $t \in F$ with

$$\mid t \mid_E > q_E^{2m+k},\ \mu := b + at^2 \equiv 1 \bmod 2\wp_E^m.$$

The last congruence can be written $\bmod 2\wp_F^{m_F}$. Finally, just as before, we can combine this integral with (5.18) to obtain Proposition (5.1) and assertion (5.1) of Theorem (5.1). □

6. Computation of $L_{w_1}^{w_G}$

We pass to the computation of $L_{w_1}^{w_G}$.

Proposition 6.1. *Let*

$$\alpha = \mathrm{diag}(a,a,a^{-2}) = a\,\mathrm{diag}(1,1,a^{-3})$$

with $\mid a \mid^2 \leq q^{-n}$. *Then:*

$$L_{w_1}^{w_G}(\alpha) = \eta(a)c(E/F,\psi) \mid a \mid_F^{-2} \mid 3 \mid_F^{-1/2} \psi(\frac{3}{a})\gamma(2a,\psi)\gamma(6a,\psi).$$

Comparing with the corresponding formula for K (Proposition (4.1)) we see that Proposition (6.1) implies assertion (5.2) of Theorem (5.1).

It remains to prove the proposition. As before:

$$L_{w_1}^{w_G}(\alpha) = \int \Psi\left({}^t\overline{n}\alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} n\right)\theta(n\overline{n})\psi(x)dxdn$$

where

$$n = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

$dn = dydz$ and $\theta(n\overline{n}) = \psi(y + \overline{y})$. As usual the measures dy and dz are equal to the self-dual Haar measure on E and dx is the self-dual Haar measure on F. Then, as before,

$$L_{w_1}^{w_G}(\alpha) = \int \Psi \left[a \begin{pmatrix} 0 & 1 & y \\ 1 & x & z \\ \overline{y} & \overline{z} & a^{-3} + y\overline{z} + z\overline{y} - xy\overline{y} \end{pmatrix} \right] \psi(x + y + \overline{y}) dxdydz.$$

The range of the integral is

$$ay \in A(m)\,,\ ax \equiv 1 \bmod 2\wp_E^m\,,\ az \equiv 0 \bmod \wp_E^m,$$

$$\frac{1}{a^2} + a\overline{y}z + ay\overline{z} - axy\overline{y} \equiv 0 \bmod \wp_E^m.$$

Changing z to $z/a\overline{y}$ the last condition becomes:

$$\frac{1}{a^2} + z + \overline{z} - axy\overline{y} \equiv 0 \bmod \wp_E^m$$

while the other conditions do not change since $\mid ay \mid = 1$. We can change x to xa^{-1} and change all other variables similarly to obtain:

$$L_{w_1}^{w_G}(\alpha) = \mid a \mid_F^{-5} \operatorname{vol}(\wp_E^m)^{-1} \operatorname{vol}(\wp_E^m \cap F)^{-1} \int \psi \left[\frac{x + y + \overline{y}}{a} \right] dxdydz$$

over

$$y \in A(m)\,,\ x \equiv 1 \bmod 2\wp_E^m\,,\ z \equiv 0 \bmod \wp_E^m,$$

$$z + \overline{z} + \frac{1 - xy\overline{y}}{a} \equiv 0 \bmod a\wp_E^m.$$

If n is sufficiently large, we have

$$a\wp_E^m \cap F \subset \operatorname{Tr}(\wp_E^m).$$

Thus the integral over z is 0 unless

$$\frac{1 - xy\overline{y}}{a} \in \operatorname{Tr}(\wp_E^m).$$

If it is so, we can write this element in the form $t + \overline{t}$ with $t \in \wp_E^m$ and change z to $z - t$. Integrating with respect to z first we have to compute the volume of the set defined by the conditions:

$$z + \overline{z} \in a\wp_E^m \cap F\,,\ z \in \wp_E^m.$$

We write

$$z = \frac{z_1}{2} + z_2\sqrt{\tau}.$$

Then $dz = dz_1 dz_2 \mid \tau \mid_F^{1/2}$. The first condition reads $z_1 \in a\wp_E^m \cap F$. It implies (if n is large enough) $z_1/2 \in \wp_E^m$. Then the second condition is equivalent to $z_2\sqrt{\tau} \in \wp_E^m$. If $\mid \tau \mid = 1$, this is in turn equivalent to $z_2 \in \wp_E^m \cap F = \wp_F^{m_F}$. If $\mid \tau \mid = q_F^{-1}$, then the extension is ramified. The condition on z_2 amounts to $\mid z_2 \mid_E \leq q^{-m+1}$ which is equivalent to $\mid z_2 \mid_F \leq q^{-m/2}$ or $z_2 \in \wp_E^m \cap F$. Thus the volume in question is

$$\mid a \mid_F \mid \tau \mid_F^{1/2} \mathrm{vol}(\wp_E^m \cap F)^2.$$

The integral is therefore equal to

$$\mid \tau \mid_F^{1/2} \mid a \mid_F^{-4} \mathrm{vol}(\wp_E^m \cap F)\ \mathrm{vol}(\wp_E^m)^{-1}$$
$$\times \int \psi \left[\frac{x + y + \overline{y}}{a} \right] dxdy$$

over

$$x \in 1 + 2\wp_E^m \cap F\,,\ y \in A(m)\,,\ xy\overline{y} \in 1 + a\,\mathrm{Tr}(\wp_E^m).$$

We write

$$y = ts\,,\ t \in 1 + \wp_E^m \cap F\,,\ s = \sqrt{1 + v^2\tau} + v\sqrt{\tau}\,,\ v\sqrt{\tau} \in \wp_E^m,$$

$$dy = \mid 2 \mid_F \mid \tau \mid_F^{1/2} dtdv,$$

and the integral becomes:

$$L_{w_1}^{w_G}(\alpha) = \mid 2\tau \mid_F \mid a \mid_F^{-4} \mathrm{vol}(\wp_E^m \cap F)\ \mathrm{vol}(\wp_E^m)^{-1}$$
$$\times \int \psi \left[\frac{x + 2t\sqrt{1 + v^2\tau}}{a} \right] dxdtdv.$$

If we change x to xt^{-2}, then t^{-2} is in $1 + 2\wp_E^m$ and so the conditions on x read:

$$x \in 1 + 2\wp_E^m\,,\ x \in 1 + a\,\mathrm{Tr}(\wp_E^m).$$

If n is sufficiently large, the second condition implies the first. Moreover $\psi(t^{-2}x/a) = \psi(t^{-2}/a)$. Thus we can integrate over x and get

$$\mid 2\tau \mid_F \mid a \mid_F^{-3} \mathrm{vol}(\wp_E^m \cap F)\ \mathrm{vol}(\wp_E^m)^{-1}\,\mathrm{vol}(\mathrm{Tr}(\wp_E^m))$$
$$\times \int \psi \left[\frac{t^{-2} + 2t\sqrt{1 + v^2\tau}}{a} \right] dtdv.$$

Recall that the self-dual Haar measure on E is given by $dz = \mid 2 \mid_F \mid \tau \mid_F^{1/2} dz_1 dz_2$ if $z = z_1 + z_2\sqrt{\tau}$. In other words, let E_0 be the F-vector space of elements of E with trace 0. Then $\mid \tau \mid_F^{1/2} dz_2$ is a measure on E_0. We have an exact sequence

$$0 \to E_0 \to E \xrightarrow{\mathrm{Tr}} F \to 0$$

and the image by the trace of the quotient measure of the self-dual Haar measure on E by the measure on E_0 is the self-dual Haar measure on F. We have then

$$\mathrm{vol}(\wp_E^m) = \mathrm{vol}(\mathrm{Tr}(\wp_E^m))\,\mathrm{vol}(\wp_E^m \cap E_0).$$

However $z_2\sqrt{\tau} \in \wp_E^m$ is equivalent to $z_2 \in \wp_E^m \cap F$, since m is even if τ is not a unit. Thus we get

$$\mathrm{vol}(\wp_E^m) = \mid \tau \mid_F^{1/2} \mathrm{vol}(\mathrm{Tr}(\wp_E^m))\,\mathrm{vol}(\wp_E^m \cap F)$$

As a consequence we can simplify the factors in our integral:

$$L_{w_1}^{w_G} = \mid 2 \mid_F \mid \tau \mid_F^{1/2} \mid a \mid_F^{-3} \int \psi \left[\frac{Q(t, v)}{a} \right] dtdv$$

where we have set:

$$Q(t,v) = t^{-2} + 2t\sqrt{1+v^2\tau}.$$

We write once more $t = 1+u$. Then the function Q has only one critical point at $u=0, v=0$. Its Taylor expansion, up to quadratic terms, at this point is:

$$Q(u,v) = 3 + 3u^2 + v^2\tau + \cdots .$$

Thus if n is sufficiently large, the integral is equal to

$$\mid 2 \mid_F \mid \tau \mid_F^{1/2} \mid a \mid_F^{-3} \psi(\frac{3}{a}) \int \psi\left[\frac{3u^2}{a}\right] du \int \psi\left[\frac{v^2\tau}{a}\right] dv$$

or

$$\mid a \mid_F^{-2} \mid 3 \mid_F^{-1/2} \psi(\frac{3}{a})\gamma(6a,\psi)\gamma(2a\tau,\psi).$$

We have

$$\gamma(2a\tau,\psi) = \gamma(2a,\psi)\eta(a)c(E/F,\psi)$$

and we obtain the result announced in Proposition (6.1). This concludes the proof of the proposition and Theorem (5.1). ☐

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