ON EULER PRODUCTS AND THE CLASSIFICATION OF AUTOMORPHIC FORMS II*

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0. Introduction.

(0.1) Let f be an integer >1 and a an integer prime to f. A classical theorem of Dirichlet asserts that there are infinitely primes in the residue class of a mod f. This may be reformulated as follows. Let $\chi_1, \chi_2, \ldots, \chi_t$ be distinct characters of $(\mathbf{Z}/f)^\times$ and $c_1, c_2, \ldots, c_t$ be complex numbers. If

$$\sum_{1 \le j \le t} c_j \chi_j(p) = 0 \tag{1}$$

for all but finitely many primes p, prime to f, then $c_1 = c_2 = \cdots = c_t = 0$. A somewhat weaker statement than Dirichlet's theorem is the existence of infinitely many prime powers in the class of a. This may again be formulated in terms of Dirichlet characters, replacing (1) by the condition

$$\sum_{1 \le j \le t} c_j \chi_j(p^n) = 0, \tag{2}$$

for all integers $n \ge 1$, and all but finitely many primes p, prime to f.

We may of course view the χ_j as idele-class characters or, what amounts to the same, as automorphic representations of GL(1), regarded as a $\mathbf{Q}$-group. Then we may replace $\mathbf{Q}$ by any number field. The purpose of this paper is to further extend this result to all the linear groups. In more detail, let π be a automorphic cuspidal representation of GL(r, $\mathbf{A}$) where $\mathbf{A}$ is the ring of adeles of a global field F. We write $r = \deg \pi$. Then π decomposes into a tensor product $\pi = \otimes_v \pi_v$, over all the places v of F. Let S be a finite set of places containing all the infinite places. Suppose S is so large that π_v is unramified for $v \notin S$. For $v \notin S$ let A_v be the semi-

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simple conjugacy class, the "Langlands class," in $\mathrm{GL}(r, \mathbf{C})$ attached to π_v (c.f. (1.3) below). For $n \geq 1$ we set

$$\pi(v^n) = \operatorname{tr} A_v{}^n.$$

Suppose now that π_i, $1 \leq i \leq p$, is a family of such automorphic representations, possibly of varying degrees. Our main result is that, if the π_i are unramified outside S and

$$\sum_{1 \leq i \leq p} c_i \pi_i(v^n) = 0,$$

for all $v \notin S$ and all $n \geq 1$, then $c_i = 0$ for $1 \leq i \leq p$.

We use this result to answer a question raised by Langlands (R.P.L. II) on the classification of automorphic representations (Theorem (4.4)), proving, in conjunction with the theory of Eisenstein series, the existence of the category of "isobaric forms."

We also given an application to the strong form of Artin's conjecture. Here however our assertion will not have substance until further progress has been made.

Our proof is based on properties of the L-functions $L(s, \pi \times \pi')$ attached to pairs (π, π') of automorphic, cuspidal representations. More precisely, if π and π' are unramified outside S, we consider the partial L-functions

$$L_S(s, \pi \times \pi') = \Pi_{v \notin S} \det(1 - q_v{}^{-s} A_v \otimes A_v{}')^{-1}.$$

For $r = r' = 1$, $L_S(s, \pi \times \pi')$ is just the L-function attached to the product $\pi\pi'$. For $r = r' = 2$ the L-functions $L(s, \pi \times \pi')$ were first considered by R. Rankin and A. Selberg and to a great extent their work is the inspiration for ours (cf. [R.R.] I, II and [A.S.]). In general, these L-functions were first considered in joint work with Piatetski-Shapiro. We hope to publish a more detailed description of their properties—with applications to the determination of the poles of Eisenstein series and the formulation and proof of "converse theorems" similar to those in [J-L] and [J-P-S].

Just as in the proof of Dirichlet's Theorem our proof depends on the existence and non-existence of poles and zeros of L-functions. Essential use is made of a recent result of F. Shahidi [F.S.] on the non-vanishing of these L-functions on the edge of the critical strip.

(0.2) Upon first reading one should read Section (1), Section (2.1), the statement of Proposition (2.6), Section (3.2), the statements of Propositions (3.3), (3.6), (3.7) and finally Section (4).

(0.3) Notations are the same as in part I. We shall use standard terminology concerning automorphic forms and automorphic representations. We refer the reader to [B-J], [R.P.L. I] and in general to the two Corvallis volumes for the basic definitions and concepts. In addition we write $\mathfrak{A}_r$ for the set of equivalence classes of (irreducible) automorphic, cuspidal representations of $G_r(\mathbf{A})$ and $\mathfrak{A}$ for the disjoint union

$$\mathfrak{A} = \cup\, \mathfrak{A}_r. \tag{1}$$

We write $\deg \pi = r$ if $\pi \in \mathfrak{A}_r$. Any π in $\mathfrak{A}_r$ is uniquely expressible as an infinite (algebraic) tensor product of representations of the G_v:

$$\pi = \otimes_v \pi_v. \tag{2}$$

We call π_v the local component (at v) of π.

Finally, if $\mathfrak{G}$ is a finite group and π a (finite-dimensional) representation of $\mathfrak{G}$, χ_π will denote the character of π. If f and g are two complex-valued functions on $\mathfrak{G}$ we write

$$(f, g) = [\mathrm{Card}\, \mathfrak{G}]^{-1} \sum_{x \in \mathfrak{G}} f(x)\overline{g(x)} \tag{3}$$

for their inner product.

1. The non-archimedean case.

(1.1) In this section, F is a non-archimedean local field. For the convenience of the reader we recall certain definitions and results.

Let r be an integer ≥ 1 and π an irreducible admissible representation of $G_r(F)$. Let $\mathcal{V}$ be the space of π. We say that π is generic if there exists a linear form $\lambda \neq 0$ on $\mathcal{V}$ such that

$$\lambda(\pi(n)v) = \theta(n)\lambda(v) \tag{1}$$

for $n \in N_r(F)$, $v \in \mathcal{V}$. The form λ is then unique up to a scalar factor. The space generated by the functions of the form

$$W(g) = \lambda(\pi(g)v) \tag{2}$$

depends only on ψ and π. We denote it by $\mathcal{W}(\pi; \psi)$.

If the central character of π is unitary then any element of $\mathcal{W}(\pi; \psi)$ is majorized by a gauge ([J-P-S] Prop. 2.3.6), that is to say a function $\xi > 0$ on G_r, invariant by the center, on the left by N, on the right by K, and given on A by

$$\xi(a) = \phi(\alpha_1(a), \alpha_2(a), \ldots, \alpha_{r-1}(a))\,|\alpha_1\alpha_2 \cdots \alpha_{r-1}(a)|^{-t},$$

with $t \geq 0$ and $\phi \geq 0$ in $\mathcal{S}(F^{r-1})$.

(1.2) Let f be a function on P_r, transforming on the left under N_r according to θ, invariant on the right under an open compact subgroup of G_r and compactly supported mod N_r. Then given π, there exists W in $\mathcal{W}(\pi; \psi)$ whose restriction to P_r coincides with f ([G-K]. (5.2)).

(1.3) Suppose that π is unramified, that is it contains the trivial representation of K. Suppose further that the largest ideal on which ψ is trivial is $\mathfrak{R}$. Then up to scalars there exists exactly one element W_0 in $\mathcal{W}(\pi; \psi)$ invariant under K on the right. Moreover $W_0(e) \neq 0$ and we normalize W_0 by requiring $W_0(e) = 1$ ([C-S], [T.S.]). We call W_0 the *essential element* of $\mathcal{W}(\pi; \psi)$.

We recall that the unramified representations of G_r (generic or not) are parameterized by the semi-simple conjugacy classes in $\mathrm{GL}(r, \mathbf{C})$. More precisely, if π is unramified then π is the unique unramified component of an induced representation of the form

$$\mathrm{Ind}(G_r, B_r; \mu_1, \mu_2, \ldots, \mu_r),$$

where the μ_i are unramified quasi-characters. The class attached to π is by definition the class of the diagonal matrix

$$A = \mathrm{diag}\,(\mu_1(\tilde{\omega}), \mu_2(\tilde{\omega}), \ldots, \mu_r(\tilde{\omega})).$$

(1.4) Let r and r' be two integers, $r > r'$. Let π (resp. π') be an irreducible admissible representation of G_r (resp. $G_{r'}$). Set for $W \in \mathcal{W}(\pi; \psi)$, $W' \in \mathcal{W}(\pi'; \psi)$,

$$\Psi(s, W, W') = \int_{N_{r'}\backslash G_{r'}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(\epsilon g)\,|\det g|^{s-(r-r')/2}\, dg, \tag{1}$$

where

$$\epsilon = \mathrm{diag}\,(1, -1, 1, -1, \ldots) \tag{2}$$

is a row vector of length r, and dg is an invariant measure on $N_{r'}\backslash G_{r'}$. It follows from the majorization of W and W' by gauges that the integral converges for $\mathrm{Re}(s)$ large.

We will also need the following. First if $W'(e) \neq 0$, then by (1.2) one can choose W so that the integrand in (1) is compactly supported. Moreover we can even assume that the resulting integral (which then exists for all s) is then identically one. Second suppose that π and π' are unramified, the largest ideal on which ψ is trivial is $\mathfrak{R}$, and that the measure dg is the quotient of the Haar measures of $G_{r'}$ and $N_{r'}$ normalized so that K and $K \cap N_{r'}$ have volume one. Then if W and W' are the essential elements of $\mathcal{W}(\pi; \psi)$ and $\mathcal{W}(\pi'; \psi)$ respectively, one has

$$\Psi(s, W, W') = \det(1 - A \otimes A' q^{-s})^{-1} \tag{3}$$

where A (resp. A') is the conjugacy class associated to π (resp. π'). In fact referring to Section 2 of [J-S], we have in the notation of that paper

$$\Psi(s, W, W') = \Sigma_J\, W\begin{pmatrix} \tilde{\omega}^J & 0 \\ 0 & 1 \end{pmatrix} W(\tilde{\omega}^J)\delta_r^{-1}(\tilde{\omega}^J)\,|\det \tilde{\omega}^J|^{s-(r-r')/2}, \tag{4}$$

the sum being over all r'-tuples of integers

$$J = (j_1, j_2, \ldots, j_{r'})$$

satisfying

$$j_1 \geq j_2 \geq \cdots \geq j_{r'} \geq 0.$$

Here $\delta_{r'}$ is the module of the group $B_{r'}$. As there set $\mathrm{tr}\,J = \Sigma\, j_i$, $X = q^{-s}$. Using the explicit formulae for W and W' ((2.1) of [J-S]), we find that the right side of (3) is also equal to

$$\sum_{J \in T_+(r')} \mathrm{Tr}(\rho(r, J)(A))\,\mathrm{Tr}(\rho(r', J)(A'))X^{\mathrm{Tr}(J)}. \tag{5}$$

The equality (3) then follows directly from Prop. (2.4) of [J-S]. *We will denote the Euler factor in* (3) *by* $L(s, \pi \times \pi')$.

Finally, if $r' = 1$, then π' is a (unitary) character of $F^\times$ and, up to a constant, $W'(x) = \pi'(x)$. Then the integral is simply

$$\int_{F^\times} W\begin{pmatrix} a & 0 \\ 0 & 1_{r-1} \end{pmatrix} |a|^{s-(r-1)/2} \pi'(a) d^\times a.$$

(1.5) Proposition. (i) *With the above notations, given W' there is a W such that the integral* (1.4.1) *converges and is equal to one for all s.*

(ii) *Assume π and π' are unitary. Then the integral* (1.4.1) *converges for* $\mathrm{Re}(s) \geq 1$, *normally for* $\mathrm{Re}(s)$ *in a compact set.*

(iii) *If π and π' are unitary and unramified then $L(s, \pi \times \pi')$ does not have a pole in* $\mathrm{Re}(s) \geq 1$.

Proof. The first assertion follows from (1.2). Assume π is unitary. Then by Proposition (1.3) of [J-S] the restriction of any element of $\mathcal{W}(\pi; \psi)$ to P_r is square-integrable mod N_r. The same is true for π'. The proof is then similar to the proof of (2.6)(i) below, but easier; it is left to the reader. The third assertion follows from the second one. □

2. The archimedean case.

(2.1) Let F be either **R** or **C**. We will regard $G_r(F)$ throughout as a real Lie group.

We begin by recalling the definition of a "generic representation." We then study the convergence properties of the integrals, analogous to those introduced in (1.4), associated to a pair of such representations. The basic result is Prop. (2.6) below.

We denote by $\mathfrak{L}$ and $\mathfrak{U}$ the Lie-algebra of $G_r(F)$ and the enveloping algebra of $\mathfrak{L} \otimes_{\mathbf{R}} \mathbf{C}$ respectively. Let $r \geq 1$ be an integer and π an irreducible unitary representation of $G_r(F)$ on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{H}^\infty$ be the space of smooth vectors in $\mathcal{H}$ with its canonical topology. We say that π is generic if there exist a non-zero continuous linear form λ on $\mathcal{H}^\infty$ such that

$$\lambda(\pi(n)v) = \theta(n)v \tag{1}$$

for $n \in N_r$ and $v \in \mathcal{H}^\infty$. That form is then unique up to scalars ([J.S.]) As in Section 1, the space of functions of the form

$$W(g) = \lambda(\pi(g)v), \tag{2}$$

where v is in $\mathcal{H}^\infty$, is then uniquely determined by π and ψ; we denote that space by $\mathcal{W}(\pi;\psi)$.

(2.2) We will need estimates for the elements of $\mathcal{W}(\pi;\psi)$ similar to those in (1.1). For $\phi \in C_c^\infty(G)$, set as usual $\check{\phi}(x) = \phi(x^{-1})$, $x \in G$. Then all elements v in $\mathcal{H}^\infty$ are of the form

$$v = \pi(\check{\phi})v_1$$

where ϕ is in $C_c^\infty(G)$ and v_1 is again in $\mathcal{H}^\infty$. (c.f. [D-M]). In terms of the corresponding functions W and W_1 that amounts to the relation

$$W(g) = \int_G W_1(gh)\phi(h)dh.$$

By the proof of Lemma (8.3.3) in [J-P-S] we find then that W is majorized by a gauge, the definition of a gauge for F archimedean being precisely the same as in (1.1).

(2.3) On the other hand let τ_r be the irreducible unitary representation of P_r induced by the character θ_r of N_r. By [J-S] Prop. (3.8), there is an $A \neq 0$ in $\mathrm{Hom}_{P_r}(\pi|P_r, \tau_r)$. On the other hand one can identify the space $\mathcal{H}_r$ of τ_r with the Hilbert space of functions on G_{r-1} transforming on the left under N_{r-1} like θ_{r-1} and square-integrable mod N_{r-1}. For ϕ in this space, $h \in G_{r-1}(F)$, we have obviously

$$\left(\tau_r\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}\phi\right)(g) = \phi(gh), \qquad g \in G_{r-1}(F).$$

Thus, if $v \in \mathcal{H}^\infty$, then Av, being a differentiable vector for τ_r is actually a smooth function. Thus $v \mapsto \mathrm{Av}(e)$ is a linear form satisfying (2.1.1). We may then assume $\lambda(v) = \mathrm{Av}(e)$ and we deduce immediately the following consequence:

LEMMA. (i) *There exists a positive constant c such that for all* $v \in \mathcal{H}^\infty$ *one has*

$$\int_{N_{r-1}\backslash G_{r-1}} |W|^2 \begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} dg \leq c\,||v||^2,$$

where W *is the element of* $\mathcal{W}(\pi;\psi)$ *defined by* (2.1.2).

(ii) *If v runs through a dense subset of $\mathcal{H}$ contained in $\mathcal{H}^\infty$, then the corresponding functions*

$$g \mapsto W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

run through a dense subset of $\mathcal{K}_r$.

(2.4) We will also need an elementary result whose proof we give for the convenience of the reader. Let U be a unipotent subgroup of G_r; as an algebraic variety U is isomorphic to a vector space over F. Regarding that vector space as a real vector space, we may then speak of polynomial functions, Schwartz functions, and so forth on U.

PROPOSITION. *Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Let ϕ be a Schwartz function on U. Then for all $v \in \mathcal{H}^\infty$ the vector*

$$\pi(\phi)v = \int_U \pi(u)v\phi(u)du$$

is again in $\mathcal{H}^\infty$ and the map $v \mapsto \pi(\phi)v$ of $\mathcal{H}^\infty$ to itself is continuous.

Here the integral is the usual Bochner integral for Hilbert space-valued functions.

Proof. Clearly the integral is absolutely convergent. Let $\langle\ ,\ \rangle$ denote the given inner product on $\mathcal{H}$. To prove the first assertion it suffices to show that for all $w \in \mathcal{H}$, $g \in G$, $x \in \mathfrak{g}(G)$, the function f on $\mathbf{R}$ defined by

$$f(t) = \langle \pi(\exp tXg)\pi(\phi)v, w \rangle \tag{1}$$

is smooth at $t = 0$. We have

$$\pi(g)\pi(\phi)v = \pi(\phi')\pi(g)v$$

where ϕ' is a Schwartz function on the unipotent group gUg^{-1}. Since the unipotent group U is arbitrary we may assume $g = e$. Next we have

$$\frac{d}{dt}\pi(\exp tX)v = \pi(\exp tX)\pi(X)v.$$

Thus if

$$g(t, u) = \langle \pi(\exp tX)\pi(u)v, w\rangle$$

we get immediately

$$\frac{\partial g}{\partial t}(t, u) = \langle \pi(\exp tX)\pi(X)\pi(u)v, w\rangle. \tag{2}$$

Next write $\operatorname{ad}(u^{-1})X = \Sigma P_i(u)X_i$ where the P_i are polynomials. Then

$$\pi(X)\pi(u) = \Sigma P_i(u)\pi(u)\pi(X_i). \tag{3}$$

Substituting in (2) we find that $u \mapsto (\partial g/\partial t)(t, u)\phi(u)$ is integrable on U uniformly in t. Thus the double integral

$$\int_0^x \int \frac{\partial g}{\partial t}(t; u)\phi(u)du dt$$

is absolutely convergent and represents a differentiable function of x. Interchanging the order of integration we get for this integral $f(x) - f(0)$. Thus f is differentiable; moreover

$$f'(t) = \int \langle \pi(\exp tX)\pi(X)\pi(u)v, w\rangle\, \phi(u)du. \tag{4}$$

Using (3) again, we find that $f'(t)$ is a finite sum of functions of the same type as f. Thus f is C^∞ everywhere and in particular at zero.

Finally evaluating (4) at $t = 0$ we get

$$\pi(D)\pi(\phi)v = \int \pi(D)\pi(u)v\phi(u)du, \tag{5}$$

at first for $D \in \mathfrak{U}$ of degree ≤ 1, but then by an easy induction for all D. The second assertion follows. We leave the details to the reader.

Now suppose λ is any continuous linear form on $\mathcal{H}^\infty$. Then

$$\lambda(v) = \Sigma\, \lambda_D(\pi(D)v),\ v \in \mathcal{H}^\infty,$$

(finite sum) where the λ_D are continuous linear forms on $\mathcal{H}$. It follows from (5) applied to $\mathrm{ad}(g^{-1})D$ that

$$\lambda(\pi(g)\pi(\phi)v) = \int \lambda(\pi(gu)v)\phi(u)du.$$

In particular if π is irreducible and generic, W an element in $\mathcal{W}(\pi; \psi)$, ϕ a Schwartz function on U, then the integral

$$\int_U W(gu)\phi(u)du$$

converges and the function of g it defines is again in $\mathcal{W}(\pi; \psi)$.

(2.5) Suppose that r and r' are two integers with $1 \leq r' \leq r$, π (resp. π') an irreducible (unitary) generic representation of G_r (resp. $G_{r'}$) on the Hilbert space $\mathcal{H}$ (resp. $\mathcal{H}'$). As in (1.4), for $W \in \mathcal{W}(\pi; \psi)$ and W' in $\mathcal{W}(\pi'; \psi)$, set

$$\Psi(s, W, W') = \int_{N_{r'}\backslash G_{r'}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(\epsilon g)|\det g|^{s-(r-r')/2}dg \qquad (1)$$

with ϵ and dg as before. As there the integral converges for $\mathrm{Re}(s)$ large.

(2.6) Proposition. (i) *With the preceding notation, one can choose W and W' such that the integral* (2.5.1) *converges normally for* $\mathrm{Re}(s) \geq 1$. *Accordingly it defines a continuous function of s, holomorphic for* $\mathrm{Re}(s) > 1$.

(ii) *If s_0 is given with* $\mathrm{Re}(s_0) \geq 1$, *one can choose W and W' with the additional property that* $\Psi(s_0, W, W') \neq 0$.

Proof. Let p and m be two integers with $1 \leq p \leq m$. We will use the following integration formula:

$$\int_{N_m\backslash G_m} \phi(g)dg = \int_{Q_{m,p}\backslash S_m} dh \int_{N_p\backslash G_p} \phi\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} h\right] |\det x|^{-(m-p)}dx. \qquad (1)$$

where S_m is the group $\mathrm{SL}(m)$, $Q_{m,p}$ is the subgroup of all matrices of the form

$$q = \begin{pmatrix} g & u \\ 0 & n \end{pmatrix}, \qquad g \in S_p, n \in N_{m-p},$$

and the measures are the appropriate invariant measures on the respective quotients.

For example, by Lemma (2.3), one has for all $W \in \mathcal{W}(\pi; \psi)$ the relation

$$\int_{Q_{r-1,r'}\backslash S_{r-1}} dh \int_{N_{r'}\backslash G_{r'}} |W|^2 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right] |\det g|^{-(r-r'-1)} dg$$
$$= \int_{N_{r-1}\backslash G_{r-1}} \left| W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right|^2 dg < +\infty.$$

There is of course at least one element of $\mathcal{W}(\pi; \psi)$ whose restriction to G_{r-1} is $\neq 0$, i.e. the above integral is positive. Thus there exists at least one $h \in S_{r-1}$ for which the integral

$$\int_{N_{r'}\backslash G_{r'}} |W|^2 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'} \end{pmatrix} h \right] |\det g|^{1-(r-r')} dg$$

is both finite and positive. Then by replacing W by a translate we obtain the existence of an element W_0 of $\mathcal{W}(\pi; \psi)$ for which

$$0 < \int_{N_{r'}\backslash G_{r'}} |W_0|^2 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'} \end{pmatrix} \right] |\det g|^{1-(r-r')} dg < \infty. \tag{2}$$

For this W_0 we also have, for $s \geq 0$,

$$\int_{N_{r'}\backslash G_{r'}} |W_0|^2 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'} \end{pmatrix} \right] |\det g|^{s+1-(r-r')} dg < +\infty. \tag{3}$$

To see this break up the integral into an integral over $|\det g| \leq 1$ and one over $|\det g| \geq 1$. In the first integral we may decrease s to zero and use (2). In the second we may take s large, integrate over all of $N_{r'}\backslash G_{r'}$ and replace W by a gauge to obtain the necessary convergence.

Next let Φ be a Schwartz function on $F^{r'}$ given any W in $\mathcal{W}(\pi;\psi)$, set

$$W_1(g) = \int_{F^{r'}} W_0\left[g\begin{pmatrix} 1_{r'} & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{r-r'-1}\end{pmatrix}\right]\Phi(u)du \tag{4}$$

(u being a column of height r'). By Prop. (2.4) and the remark following that Proposition, W_1 is again in $\mathcal{W}(\pi;\psi)$. Moreover, for $g \in G_{r'}$, $g^{-1}N_r g$ contains the group we are integrating over in (4). Changing variables we get

$$W_1\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} = W_0\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\phi(\eta g) \tag{5}$$

where ϕ is the Fourier transform of Φ defined by

$$\phi(x) = \int \Phi(u)\psi(x\cdot u)du, \tag{6}$$

and η is the row-vector of length r' given by

$$\eta = (0, 0, \ldots, 0, 1). \tag{7}$$

Since as Φ varies, ϕ represents an arbitrary Schwartz function in r' variables, we may choose Φ so that ϕ is smooth and compactly supported with support contained in the open orbit

$$\eta G_{r'} \simeq Q_{r',r'-1}\backslash S_{r'}.$$

Next with ϕ and W_0 chosen in the above fashion, we will show that for *all* W' in $\mathcal{W}(\pi';\psi)$ the integral $\Psi(s, W_1, W')$ converges (absolutely) for $\mathrm{Re}(s) \geq 1$, normally for $\mathrm{Re}(s)$ in a compact subset of $\mathrm{Re}(s) \geq 1$. By (5) we are reduced to proving the convergence for $s \geq 1$ of the integral

$$\int_{N_{r'}\backslash G_{r'}} |W_0| \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |W'|(\epsilon g)|\phi(\eta g)||\det g|^{s-(r-r')/2} dg, \tag{8}$$

uniformly for $b > s > a \geq 1$. Breaking the integral up as before over $|\det g| \leq 1$ and $|\det g| \geq 1$ we need only prove the convergence for $s \geq 1$. Then, by Cauchy-Schwartz, the integral is majorized by the square-root of the product of the integral

$$\int_{N_{r'}\backslash G_{r'}} |W_0|^2 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |\det g|^{s-(r-r')} dg \tag{9}$$

and the integral

$$\int_{N_{r'}\backslash G_{r'}} |W'|^2(\epsilon g)|\phi|^2(\eta g)|\det g|^s dg. \tag{10}$$

By (3) the integral (9) is finite.

On the other hand there is a Schwartz function ϕ_1 on $F^{r'}$ such that $|\phi(x)| \leq \phi_1$. Then $\phi_2 = \phi_1{}^2$ is again a Schwartz function and (10) is majorized by the integral

$$\int |W'|^2(\epsilon g)\phi_2(g)|\det g|^s dg,$$

whose convergence has been proved in [J-S] Proposition (3.17). We have therefore proved the first part of Proposition (2.6).

We now prove the second part. We *fix* W_0 satisfying (2), and s_0 with $\mathrm{Re}(s_0) \geq 1$. We will prove that there is a ϕ as above with support in $\eta G_{r'}$ and a $W' \in \mathcal{W}(\pi'; \psi)$ such that

$$\Psi(s_0, W_1, W') \neq 0.$$

Suppose in fact that this integral vanishes for all such ϕ and W'. Using once more the formula (1) we get:

$$\int_{Q_{r',r'-1}\backslash S_{r'}} \phi(\eta h) dh \int_{N_{r'-1}\backslash G_{r'-1}} W_0 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'+1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1_{r-r'} \end{pmatrix} \right]$$

$$\times W'\left[\epsilon \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h\right] |\det g|^{s_0-1-(r-r')/2} dg = 0. \tag{14}$$

Fix W'. The inner integral represents a locally integrable function f on $Q_{r',r'-1}\backslash S_{r'}$. That function is orthogonal to all smooth functions of compact support on $Q_{r',r'-1}\backslash S_{r'}$. Since the latter space is countable at infinity it follows that f vanishes almost everywhere. Thus given W', *there exists a set $M(W')$ of measure zero in $Q_{r',r'-1}\backslash S_{r'}$ such that for all $h \notin M(W')$, the integral*

$$\int_{N_{r'-1}\backslash G_{r'-1}} W_0\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'+1}\end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1_{r-r'}\end{pmatrix}\right]$$

$$\times W'\left[\epsilon\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h\right] |\det g|^{s_0-1-1/2(r-r')}\,dg \tag{15}$$

is convergent and equal to zero.

Next apply the integration formula (1) this time to (3) (with $s = 2(\mathrm{Re}(s_0) - 1)$). We get

$$\int_{Q_{r',r'-1}\backslash S_{r'}} dh \int_{N_{r'-1}\backslash G_{r'-1}} |W_0|^2\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'+1}\end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1_{r-r'}\end{pmatrix}\right]$$

$$\times |\det g|^{2(\mathrm{Re}\, s_0-1)-(r-r')}\,dg < +\infty. \tag{16}$$

There exists then a set M_0 of measure zero in $Q_{r',r'-1}\backslash S_{r'}$ such that for $h \notin M_0$ the function

$$g \mapsto W_0\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'+1}\end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1_{r-r'}\end{pmatrix}\right] |\det g|^{s_0-1-(r-r')/2} \tag{17}$$

on $N_{r'-1}\backslash G_{r'-1}$ is square-integrable. Thus for h $\notin M_0$, each such function is in the space $\mathcal{K}_{r'}$.

On the other hand, let $\mathcal{H}'$ denote the space of π' and fix X a *countable* dense subset of $\mathcal{H}'$ contained in $(\mathcal{H}')^\infty = \mathcal{W}(\pi'; \psi)$. For any $h \in G_{r'}$, the set $\pi'(h)X$ has similar properties. Let Y denote the image of X in

$\mathcal{W}(\pi'; \psi)$. Applying Lemma (2.3), we find in particular, that, for a given $h \in S_{r'}$, the set of functions

$$g \mapsto \overline{W'}\left[\epsilon \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h\right], \qquad W' \in Y,$$

is dense in $\mathcal{K}_{r'}$.

Now let $M = \bigcup M(W') \cup M_0$, the countable union being over $W' \in Y$. We conclude that for a given $h \notin M$, the inner product (15) of the function defined by (17) with the elements of a dense subset of $\mathcal{K}_{r'}$ is zero. Let α be the canonical projection

$$\alpha: S_{r'} \to Q_{r',r'-1} \backslash S_{r'}.$$

The same statement is true for $h \notin \alpha^{-1}(M)$. Thus by the continuity of W_0, we have

$$W_0\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'+1} \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1_{r-r'} \end{pmatrix}\right] = 0, \tag{18}$$

at first, for $g \in G_{r'-1}$, $h \notin \alpha^{-1}(M)$, but then again by continuity for all g in $G_{r'-1}$ and $h \in S_{r'}$. Taking then $h = e$ in (18), we conclude that

$$W_0 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = 0$$

for $g \in G_{r'}$ in obvious contradiction to (2). □

3. Partial L-functions.

(3.1) Let F be a global field and $F_{\mathbf{A}} = \mathbf{A}$ the associated ring of adeles. If π is an irreducible unitary representation of $G_r(\mathbf{A})$ on a Hilbert space $\mathcal{K}$, then π is uniquely representable as a topological tensor product

$$\pi = \hat{\bigotimes_v} \pi_v, \qquad \mathcal{K} = \hat{\bigotimes_v} \mathcal{K}_v,$$

where π_v is an irreducible unitary representation of $G_r(F_v)$ on $\mathcal{H}_v$. As above let $\mathcal{H}_v^\infty$ denote the space of smooth vectors in $\mathcal{H}_v$. (If v is finite, this is the space of K_v-finite vectors in $\mathcal{H}$.) Let then

$$\mathcal{H}^\infty = \bigotimes_v \mathcal{H}_v^\infty$$

be the algebraic tensor product of the $\mathcal{H}_v^\infty$, that is, the linear span of the pure tensors:

$$w = \bigotimes_v w_v, \tag{1}$$

w_v being for almost all v a fixed K_v-invariant vector e_v in $\mathcal{H}_v$ (or $\mathcal{H}_v^\infty$). Fix once for all a non-trivial character

$$\psi = \Pi_v \psi_v$$

of $F_{\mathbf{A}}$ trivial on F. If each π_v is generic then the local spaces $\mathcal{W}(\pi_v; \psi_v)$ have been defined. Let

$$\mathcal{W}(\pi; \psi) = \bigotimes_v \mathcal{W}(\pi_v; \psi_v).$$

be the algebraic tensor product. We have for each v an isomorphism of $\mathcal{H}_v^\infty$ with $\mathcal{W}(\pi_v; \psi_v)$, given by (1.1.2) if v is finite and similarly (2.1.2) if v is infinite. If π_v is unramified we normalize this isomorphism so that e_v corresponds to the essential element of $\mathcal{W}(\pi_v; \psi_v)$. Taking tensor products we get then an isomorphism

$$\mathcal{H}^\infty \simeq \mathcal{W}(\pi; \psi). \tag{2}$$

It is clear from (2.2) that any pure tensor $w \in \mathcal{H}^\infty$ is of the form

$$w = \pi(\phi)w_1 \tag{3}$$

where $\phi \in C_c^\infty(G_r(\mathbf{A}))$ and w_1 is again in $\mathcal{H}^\infty$. A similar statement applies to $\mathcal{W}(\pi; \psi)$.

Now suppose ω is a character (unitary) of $F^\times \backslash F_{\mathbf{A}}^\times$. Let ${}^0L_2(\omega)$ denote the space of cusp forms ([B-J]) on $G_r(\mathbf{A})$ transforming under $Z_r(\mathbf{A})$

according to ω. Suppose that π is a component of ${}^0L_2(\omega)$ and let $\mathfrak{A}(\pi)$ denote the corresponding irreducible subspace. Let also $\mathfrak{A}^\infty(\pi)$ denote the image of $\mathfrak{H}^\infty$ in $\mathfrak{A}(\pi)$. By (3) any function ϕ in $\mathfrak{A}^\infty(\pi)$ is a smooth function of $G_r(\mathbf{A})$. Moreover by the general theory, if F is a number field, ϕ is "rapidly-decreasing."

We will need to consider the "Fourier series" of an element of $\mathfrak{A}^\infty(\pi)$. For $0 \leq m \leq r - 1$, let $N_{r,m}$ be the unipotent radical of the parabolic of type $(m + 1, 1, \ldots, 1)$, that is the subgroup of N_r consisting of all matrices (n_{ij}) for which $n_{ij} = 0$ for $1 \leq i \leq m + 1$, $1 \leq j \leq m + 1$, $i \neq j$. We set

$$V_{m,\phi}(g) = \int_{N_{r,m}(F)\backslash N_{r,m}(\mathbf{A})} \phi(ug)\overline{\theta}_r(u)du \tag{3}$$

so that $V_{r-1,\phi} = \phi$. Note that

$$V_{m,\phi}(\gamma g) = V_{m,\phi}(g), \qquad \gamma \in G_m(F). \tag{4}$$

Moreover $N_{r,0} = N_r$ and $V_{0,\phi} \in \mathfrak{W}(\pi; \psi)$ (c.f. [S], Section 4). Thus in particular $V_{0,\phi}$ is majorized by a gauge ([J-P-S], Props. (2.3.6), (2.4.1) and Section 12).

Now we have at once

$$\int_{\mathbf{A}^m/F^m} V_{m,\phi}\left[\begin{pmatrix} 1_m & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{r-m-1} \end{pmatrix} g\right]$$

$$\overline{\theta}_{m+1}\left(\begin{bmatrix} 1_m & u \\ 0 & 1 \end{bmatrix}\right) du = V_{m-1,\phi}(g). \tag{5}$$

On the other hand, if U is the unipotent radical of a standard parabolic of type $(m, r - m)$, the constant term in the Fourier expansion of the function

$$u \mapsto V_{m,\phi}\left[\begin{pmatrix} 1_m & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{r-m-1} \end{pmatrix} g\right] \tag{6}$$

on the group $\mathbf{A}^m/F^m$ factors through an integral of the form

$$\int_{U_F \backslash U_{\mathbf{A}}} \phi(ug)du = 0$$

and is therefore zero. Next any non-trivial character of $\mathbf{A}^m/F^m$ has the form

$$u \mapsto \psi(\eta\gamma u),$$

where as before

$$\eta = (0, 0, \ldots, 0, 1)$$

is a row vector of length m, $\gamma \in P_m(F)\backslash G_m(F)$, and u is regarded as a column of height m. Thus if in (5) we replace u by γu and use (4) we get

$$V_{m,\phi}(g) = \Sigma_{P_m(F)\backslash G_m(F)} V_{m-1,\phi} \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1_{r-m} \end{pmatrix} g \right]. \tag{6}$$

Then by recursion starting with m we get, at first formally,

$$V_{m,\phi}(g) = \Sigma_{N_m(F)\backslash G_m(F)} V_{0,\phi} \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1_{r-m} \end{pmatrix} g \right]. \tag{7}$$

Since, as we have noted, $V_{0,\phi}$ is majorized by a gauge, by Proposition (1.2.2) of [J-P-S], the series in (7) is absolutely convergent. Thus the passage from (6) to (7) by combining the various Fourier series is justified.

(3.2) Let again r and r' be integers. Let π and π' be admissible irreducible representations of $G_r(\mathbf{A})$. Then, if S is a sufficiently large finite set of places containing the places at infinity, the local components π_v and $\pi_v{}'$ are unramified. Let A_v and $A_v{}'$ be the associated conjugacy classes, respectively in $\mathrm{GL}_r(\mathbf{C})$ and $\mathrm{GL}_{r'}(\mathbf{C})$ (c.f. (1.3)). We set

$$L(s, \pi_v \times \pi_v{}') = \det(1 - q_v{}^{-s} A_v \otimes A_v{}')^{-1} \tag{1}$$

as before and then set

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi_v'). \tag{2}$$

If now π and π' are unitary, then as is well known the eigenvalues λ of A_v (resp. A_v') are in absolute value $\leq q_v^{(r-1)/2}$ (resp. $q_v^{(r'-1)/2}$) (c.f. [R.G.-J] (6.12)). In fact we even have $\lambda < q_v^{1/2}$ (c.f. [J-S] (2.5)). Thus in that case the Euler product in (2) is absolutely convergent for $\mathrm{Re}(s)$ large.

Now suppose in addition that π and π' are automorphic cuspidal representations. Then by Theorem (5.3) of [J-S] the product (2) is actually absolutely convergent for $\mathrm{Re}(s) > 1$. Thus $L_S(s, \pi \times \pi')$ is defined in the half-plane $\mathrm{Re}(s) > 1$. It follows from (5.3.2) and the uniform convergence of the series (5.3.5) of [J-S] in compact subsets of $\mathrm{Re}(s) > 1$, that $L_S(s, \pi \times \pi')$ is also holomorphic and $\neq 0$ in $\mathrm{Re}(s) > 1$.

(3.3) PROPOSITION. *Suppose $r \neq r'$ and the unitary representations π and π' are automorphic and cuspidal. Then the function $L_S(s, \pi \times \pi')$ has a continuous extension to the closed half plane* $\mathrm{Re}(s) \geq 1$.

Proof. 1st part: We note that by (1.5)(iii) we may enlarge S at will. We may assume then that for $v \notin S$, $\mathfrak{R}_v$ is the largest ideal on which ψ_v is trivial.

That being so, let for $v \notin S$, W_v (resp. W_v') be the essential element of $\mathcal{W}(\pi_v; \psi_v)$. (resp. $\mathcal{W}(\pi_v'; \psi_v)$). For v finite in S, choose W_v and W_v' so that each (local) integral $\Psi(s, W_v, W_v')$ converges for all s and is equal to one (c.f. (1.5)). For v infinite choose W_v and W_v' so that the integral $\Psi(s, W_v, W_v')$ converges for $\mathrm{Re}(s) \geq 1$ (Prop. (2.6)). Then $\otimes_v W_v$ (resp. $\otimes_{v'} W_v'$) belongs to $\mathcal{W}(\pi; \psi)$ (resp. $\mathcal{W}(\pi'; \psi)$). Let ϕ and ϕ' be the corresponding elements of $\mathfrak{A}^\infty(\pi)$ and $\mathfrak{A}^\infty(\pi')$ respectively. Now for an appropriate normalization of the measure of $N_r(F)\backslash N_r(\mathbf{A})$, by Theorem (4.5) of [S], the isomorphism of (3.1.2) is actually given by $\phi \mapsto V_{0,\phi}$. Thus in fact

$$V_{0,\phi}(g) = \prod_v W_v(g_v), \qquad V_{0,\phi'}(g) = \prod_v W_v'(g_v). \tag{1}$$

Next consider the integral

$$\int_{G_{r'}(F)\backslash G_{r'}(\mathbf{A})} V_{r',\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] \phi'(g)\,|\det g|^{s-(r-r')/2}\,dg. \tag{2}$$

Calculating formally at first, we get by (3.1.7) the equivalent expression

$$\int_{G_{r'}(F)\backslash G_{r'}(\mathbf{A})} \sum_{N_{r'}(F)\backslash G_{r'}(F)} V_{0,\phi}\left[\begin{pmatrix} \gamma g & 0 \\ 0 & 1 \end{pmatrix}\right] \phi'(g)|\det g|^{s-(r-r')/2}\, dg. \tag{3}$$

Since $\phi'(g)|\det g|^{s-(r-r')/2}$ is invariant on the left under $G_{r'}(F)$ we may combine the integral and the sum to get

$$\int_{N_{r'}(F)\backslash G_{r'}(\mathbf{A})} V_{0,\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] \phi'(g)|\det g|^{s-(r-r')/2}\, dg. \tag{4}$$

Next $V_{0,\phi}(ng) = \theta_r(n) V_{0,\phi}(g)$ for $n \in N_r(\mathbf{A})$, $g \in G_r(\mathbf{A})$. Note that for $n \in N_{r'}(F)$,

$$\theta_r\left[\begin{pmatrix} n & 0 \\ 0 & 1_{r-r'} \end{pmatrix}\right] = \theta_{r'}(n).$$

Thus integrating in stages, we get

$$\int_{N_{r'}(\mathbf{A})\backslash G_{r'}(\mathbf{A})} |\det g|^{s-(r-r')/2}\, V_{0,\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] dg$$

$$\times \int_{N_{r'}(F)\backslash N_{r'}(\mathbf{A})} \phi'(ng)\theta_{r'}(n)dn. \tag{5}$$

But one has

$$\int_{N_{r'}(F)\backslash N_{r'}(\mathbf{A})} \phi'(ng)\theta_{r'}(n)dn = V_{0,\phi'}(\epsilon g) \tag{6}$$

and therefore we conclude that

$$\begin{aligned} &\int_{G_{r'}(F)\backslash G_{r'}(\mathbf{A})} V_{r',\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] \phi'(g)|\det g|^{s-(r-r')/2}\, dg \\ &= \int_{N_{r'}(\mathbf{A})\backslash G_{r'}(\mathbf{A})} V_{0,\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] V_{0,\phi}(\epsilon g)|\det g|^{s-(r-r')/2}\, dg. \end{aligned} \tag{7}$$

Here as before $\epsilon = \operatorname{diag}(1, -1, 1, -1, \ldots)$.

The calculations are justified when $\mathrm{Re}(s)$ is sufficiently large. In effect it suffices to prove that the integral (4) is absolutely convergent for $\mathrm{Re}(s)$ large. But ϕ' is bounded and as we have noted $V_{0,\phi}$ is majorized by a gauge, say ξ. Thus our integral is majorized by

$$\int \xi\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] |\det g|^{\mathrm{Re}(s)-(r-r')/2}\, dg,$$

the integration being over $N_{r'}(F)\backslash G_{r'}(\mathbf{A})$, or what amounts to the same over $N_{r'}(\mathbf{A})\backslash G_{r'}(\mathbf{A})$. Since ξ is invariant on the right under K, we may calculate this as an integral over $A_r(\mathbf{A})$ in which case it is easily seen to be finite.

(3.4) *Proof of Proposition* (3.3). 2nd *part*. We show first that the integral on the left in (3.3.7) converges for all s, uniformly for $\mathrm{Re}(s)$ in a compact set. This will be a consequence of the following lemma.

LEMMA. (i) *Suppose F is a number field. For all $N \geq 1$ and all compact subsets Ω of $G_r(\mathbf{A})$ there exists a constant C_N such that*

$$\left| V_{r',\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\omega\right] \right| \leq C_N \inf(|\det g|^N, |\det g|^{-N})$$

for all $g \in G_{r'}(\mathbf{A})$ and $\omega \in \Omega$.

(ii) *Suppose F is a function field. For all compact subsets Ω of $G_r(\mathbf{A})$ there exists a constant $C > 0$ such that, for $\omega \in \Omega$, the support of the function*

$$g \mapsto V_{r',\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\omega\right]$$

is contained in the set of $g \in G_{r'}(\mathbf{A})$ satisfying

$$C^{-1} \leq |\det g| \leq C.$$

Proof of the lemma (i).

The unipotent group $N_{r,r'}$ may be described as the set of all matrices of the form

$$\begin{pmatrix} 1_{r'} & u \\ 0 & v \end{pmatrix} \tag{1}$$

where the first column of $u \in M(r' \times r - r')$ is zero and v belongs to $N_{r-r'}$. Thus this group is the semi-direct product of the groups

$$N' = \left\{\begin{pmatrix} 1_{r'} & u \\ 0 & 1_{r-r'} \end{pmatrix}\right\}, \qquad N'' = \left\{\begin{pmatrix} 1_{r'} & 0 \\ 0 & v \end{pmatrix}\right\}, \tag{2}$$

u and v being as before, N' being normal in $N_{r,r'}$ and N'' commuting with all matrices of the form

$$h = \begin{pmatrix} g & 0 \\ 0 & 1_{r-r'} \end{pmatrix}, \qquad g \in G_{r'}. \tag{3}$$

Denote by $N^*_{r,r'}$ the quotient $N_{r,r'}(F)\backslash N_{r,r'}(\mathbf{A})$ and use a similar notation for N', N''. We have, immediately from the definition,

$$|V_{r',\phi}(h\omega)| \leq \int_{N^*_{r,r'}} |\phi(nh\omega)|\, dn = \int_{N''^*} dn'' \int_{N'^*} |\phi|(n'n''h\omega)dn'. \tag{4}$$

Now $n''h = hn''$. Let Ω' be a compact subset of N'' projecting onto N''^*. Replacing Ω by $\Omega'\Omega$ it will be sufficient to estimate the integral

$$\int_{N'^*} |\phi|(n'h\omega)dn', \tag{5}$$

where $\omega \in \Omega$ and h is of the form (3). Let τ be the permutation matrix defined by

$$\tau = \begin{pmatrix} 1_{r'} & 0 \\ 0 & \mu \end{pmatrix}, \qquad \mu = \begin{bmatrix} 0 & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & 0 \end{bmatrix}. \tag{6}$$

Clearly τ commutes with any matrix of the form (3). Since in addition ϕ is automorphic, we have

$$\int_{N'^*} |\phi|(n'h\omega)dn' = \int_{N'^*} |\phi|(\tau n'\tau^{-1}h\tau\omega)dn'.$$

Now $\tau n'\tau^{-1}$ is contained in the group of matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, \qquad g \in G_{r-1}(\mathbf{A}). \tag{7}$$

Finally, since h also has this form and N'^* is compact, it suffices to show that, for all $N > 0$, there is a C_N such that

$$|\phi(h\omega)| \leq C_N \operatorname{Inf}(|\det h|^{-N}, |\det h|^N) \tag{8}$$

for all h of the form (7) and all $\omega \in \Omega$. By reduction theory we have

$$G_{r-1}(\mathbf{A}) = G_{r-1}(F)A_{r-1}(\mathbf{A})\Omega_0$$

where Ω_0 is a compact subset of $G_{r-1}(\mathbf{A})$. Thus replacing Ω by a sufficiently large compact subset of $G_r(\mathbf{A})$, it will suffice to prove (8) when h is a diagonal matrix of the form

$$a = \operatorname{diag}(a_1, a_2, \ldots, a_{r-1}, 1). \tag{9}$$

Next let Λ be the function defined on all of A_r by

$$\Lambda = \prod_{1 \leq 1 \leq r-1} \alpha_i^{\,i}. \tag{10}$$

For a of the special form (9), note that

$$\Lambda(a) = \det a. \tag{11}$$

For $a \in A_r(\mathbf{A})$, there exists a permutation matrix w such that $a = wbw^{-1}$ where b is in the positive chamber, that is to say $|\alpha_i(b)| \geq 1$ for $1 \leq i \leq r - 1$. On the other hand by the general theory, if we set

$$\kappa(b) = \sup_i |\alpha_i(b)|,$$

then, for all $N \geq 1$, there is a constant C_N such that

$$|\phi(bw^{-1}\omega)| \leq C_N \kappa(b)^{-N}$$

for all $\omega \in \Omega$, b in the positive chamber. Thus

$$|\phi(a\omega)| = |\phi(wbw^{-1}\omega)| = |\phi(bw^{-1}\omega)| \leq C_N \kappa(b)^{-N}.$$

On the other hand, it is easily verified that there is a constant $m > 0$ such that

$$\kappa(b)^{-m} \leq |\Lambda(wbw^{-1})| \leq \kappa(b)^m \tag{12}$$

for all b in the positive chamber and all w. Thus one has for all $a \in A(\mathbf{A})$, $\omega \in \Omega$,

$$|\phi(a\omega)| \leq C_{mN} \inf(|\Lambda(a)|^{-N}, |\Lambda(a)|^N)$$

and combining this with (11), the assertion (8) follows at once. This completes the proof of the first part of the lemma.

Proof of the lemma (ii). We will use the following fact due to G. Harder ([G.H.] (1.2)). Suppose F is a function field. Let ϕ be a cusp form on $G_r(\mathbf{A})$ and Ω a fixed compact subset of $G_r(\mathbf{A})$. *Then there exists a constant $d > 0$, such that for all b in the positive chamber, $\omega \in \Omega$, and w a permutation matrix, the relation $\phi(bw^{-1}\omega) \neq 0$ implies*

$$|\alpha_i(b)| \leq d, \qquad 1 \leq 1 \leq r - 1.$$

The proof of (ii) proceeds as the proof of the first part. One is reduced to proving that, for a given compact subset Ω of $G_r(\mathbf{A})$, there exists a constant $C > 0$ such that $\phi(a\omega) \neq 0$, $\omega \in \Omega$, $a = \operatorname{diag}(a_1, a_2, \ldots, a_{r-1}, 1)$ implies $C^{-1} \leq |\det a| \leq C$. Again there exists a matrix w such that $a = wbw^{-1}$ where b is in the positive chamber. One then has

$$\phi(a\omega) = \phi(bw^{-1}\omega).$$

Thus by the result we quoted $\kappa(b) \leq d$, and finally by (12)

$$d^{-m} \leq |\Lambda(wbw^{-1})| \leq d^m.$$

Since $|\det a| = |\Lambda(a)| = |\Lambda(wbw^{-1})|$ our assertion follows.

(3.5) *Completion of the proof of Prop.* (3.3). We show first that the integral on the left in (3.3.7) converges absolutely for all s, uniformly for s in a vertical strip of finite width. Suppose first that F is a number field. Since ϕ' is bounded, by the lemma the integral is, for each N, dominated by an integral of the form

$$\int_{G_{r'}(F)\backslash G_{r'}(\mathbf{A})} \inf(|\det g|^N, |\det g|^{-N})|\det g|^{s-(r-r')/2}\, dg,$$

with s now real. Let G^0 be group of those $g \in G_{r'}(\mathbf{A})$ satisfying $|\det g| = 1$. Integrating in stages, we get for the previous integral

$$\operatorname{vol}(G_r(F)\backslash G^0) \int_{\mathbf{R}_+{}^\times} \inf(t^N, t^{-N})t^{s-(r-r')/2}\, d^\times t$$

which is convergent for s in a compact if N is sufficiently large. If finally F is a function field ϕ' is again bounded. On the other hand since ϕ is also bounded say by a number d, we have

$$|V_{r',\phi}(g)| \leq \int_{N^*_{r,r'}} |\phi(ug)|\, du \leq d \operatorname{vol}(N^*_{r,r'}).$$

Our integral is then dominated by an integral of the form

$$\int_{c^{-1} \leq |\det g| \leq c} |\det g|^{\operatorname{Re}(s)-(r-r')/2}\, dg.$$

Since the integrand is bounded on the domain of integration and the latter has finite volume, we have the same conclusion.

That being the case the left side of (3.3.7) represents an entire function of s. Thus the right side extends to an entire function of s. Now, using (3.3.1), we have for $\operatorname{Re}(s)$ sufficiently large,

$$\int_{N_{r'}(\mathbf{A})\backslash G_{r'}(\mathbf{A})} V_{0,\phi}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] V_{0,\phi'}(g)|\det g|^{s-(r-r')/2}\, dg$$

$$= \prod_v \Psi(s, W_v, W_v'), \qquad (1)$$

each integral being convergent and the infinite product being absolutely convergent. However, with the particular choice of W_v and W_v' we have made, this product is nothing other than

$$\prod_{v|\infty} \Psi(s, W_v, W_v')L_S(s, \pi \times \pi'). \tag{2}$$

The conclusion is therefore immediate if F is a function field and results from Proposition (2.6) if F is a number field.

(3.6) **Proposition.** *Suppose that $r = r'$ and the representations π and π' are automorphic, cuspidal, and unitary. Let X be the set of s on the line* $\mathrm{Re}(s) = 1$ *such that $\alpha^{s-1} \otimes \pi$ is equivalent to the contragredient representation $\tilde{\pi}'$ of π'. Then:*

(i) *the function $L_S(s, \pi \times \pi')$ extends to a continuous function on the complement of X in the closed half-space* $\mathrm{Re}(s) \geq 1$;

(ii) *if s_0 is in X, then*

$$\lim_{s \to s_0} (s - s_0)L_S(s, \pi \times \pi'),$$

the limit taken in $\mathrm{Re}(s) \geq 1$, $s \neq s_0$, *exists, is finite, and is non-zero.*

Proof. This follows readily from the results of [J-S, Section 4]. In more detail, for $W \in \mathcal{W}(\pi; \psi)$, $W' \in \mathcal{W}(\pi'; \psi)$ and $\Phi \in \mathcal{S}(\mathbf{A}^r)$, let

$$\Psi(s, W, W', \Phi) = \int_{N_r \backslash G_r} W(g)W'(\epsilon g)\Phi(\eta g)|\det g|^s\, dg. \tag{1}$$

The integral converges for $\mathrm{Re}(s)$ sufficiently large and extends to a meromorphic function of s in the half plane $\mathrm{Re}(s) > 0$, holomorphic in $\mathrm{Re}(s) > 1$. Moreover, except for notation, Lemma (4.6) (loc. cit.) implies that if $\Psi(s, W, W', \Phi)$ has a pole at $s = 1$, then that pole is simple and $\tilde{\pi}$ is equivalent to π'. An application of the identity

$$\Psi(s + i\sigma, W, W', \Phi) = \Psi(s, W \otimes \alpha^{i\sigma}, W', \Phi)$$

(σ real) then shows that the poles of $\Psi(s, W, W', \Phi)$ are simple and contained in X.

Now choose again a finite set S of places (containing the places at infinity) such that π_v and π_v' are unramified outside S. Assume also that, for $v \notin S$, the largest ideal on which ψ_v is trivial is $\mathfrak{R}_v$. Suppose also that $W = \Pi_v W_v$, $W' = \Pi_v W_v'$, and $\Phi = \Pi_v \Phi_v$, where for $v \notin S$, W_v (resp. W_v') is the essential vector of $\mathcal{W}(\pi_v; \psi_v)$ (resp. $\mathcal{W}(\pi_v'; \psi_v)$) and Φ_v the characteristic function of $\mathfrak{R}_v^r$. Then we have, as in (4.7) (loc. cit.), for Re(s) large,

$$\Psi(s, W, W', \Phi) = \prod_v \Psi(s, W_v, W_v', \Phi_v),$$

the local factor being defined by analogy with (3.2.1). Moreover, for $v \notin S$,

$$\Psi(s, W_v, W_v', \Phi_v) = L(s, \pi_v \times \pi_v')$$

(Section 2 loc. cit.). Thus for Re(s) sufficiently large,

$$\Psi(s, W, W', \Phi) = \prod_{v \in S} \Psi(s, W_v, W_v', \Phi_v) L_S(s, \pi \times \pi').$$

By Proposition (1.5) and (3.17) of [J-S] each of the functions $\Psi(s, W_v, W_v', \Phi_v)$ is continuous in the closed half-plane $\mathrm{Re}(s) \geq 1$. Moreover for such an s, we may choose W_v, W_v', and Φ_v such that $\Psi(s, W_v, W_v', \Phi_v) \neq 0$. The first assertion follows.

To prove the second assertion, let $s_0 \in X$. Replacing π by $\alpha^{1-s_0} \otimes \pi$ we may assume $s_0 = 1$. Let ϕ (resp. ϕ') be the element of $\mathcal{A}^\infty(\pi)$ (resp. $\mathcal{A}^\infty(\pi')$) corresponding to W (resp. W'). Then combining (4.5.5) and (4.3.2) of [J-S] we find

$$\lim_{s \to 1} (s - 1)\Psi(s, W, W', \Phi) = c\hat{\Phi}(0) \int_{Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}} \phi(g)\phi'(g)dg,$$

where c is a non-zero constant. Choosing W_v, W_v', and Φ_v once more so that $\Psi(1, W_v, W_v', \Phi_v) \neq 0$ for $v \in S$ we see that

$$\lim_{s \to 1} (s - 1)L_S(s, \pi \times \pi')$$

exists (and is finite). Call κ this limit. Then we have, for any choice of W_v, W_v', and Φ_v ($v \in S$)

$$\kappa \prod_{v \in S} \Psi(1, W_v, W_v', \Phi_v)$$

$$= c\hat{\Phi}(0) \int \phi(g)\phi'(g)dg$$

$$= c \prod_{v \in S} \hat{\Phi}_v(0) \int \phi(g)\phi'(g)dg.$$

Since π' is the representation contragredient to π, we may take $W_v'(g) = \overline{W_v(\epsilon g)}$. Then by (3.3.1), $\phi'(g) = \overline{\phi(\epsilon g)} = \phi(g)$. We may also assume ϕ is non-zero. Then for any choice of the Φ_v we have

$$\kappa \prod_{v \in S} \Psi(1, W_v, W_v', \Phi_v) = c \prod_{v \in S} \hat{\Phi}_v(0) \int_{Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}} |\phi(g)|^2 dg.$$

Since we may choose Φ_v so that, for all $v \in S$, $\hat{\Phi}_v(0) \neq 0$, we see that $\kappa \neq 0$. □

(3.7) A THEOREM OF SHAHIDI. Let S be any finite set of places containing the places at infinity and those places for which π_v and π_v' are ramified. By (3.2), $L_S(s, \pi \times \pi')$ has no zeros in $\mathrm{Re}(s) > 1$.

By abuse of language we shall call the points s_0 in $\mathrm{Re}(s) \geq 1$ for which

$$\lim_{s \to s_0, \mathrm{Re}(s) \geq 1} (s - s_0) L_S(s, \pi \times \pi') \neq 0$$

the poles of $L_S(s, \pi \times \pi')$ in $\mathrm{Re}(s) \geq 1$. These poles of course lie on $\mathrm{Re}(s) = 1$. Thus the function $L_S(s, \pi \times \pi')$ is defined and continuous outside the set of poles in $\mathrm{Re}(s) \geq 1$.

THEOREM. *With the hypotheses of* (3.3) *or* (3.6), *the function* $L_S(s, \pi \times \pi')$ *is not zero in* $\mathrm{Re}(s) \geq 1$.

We have already observed the validity of our assertion for $\mathrm{Re}(s) > 1$. The non-vanishing on $\mathrm{Re}(s) = 1$ is due to F. Shahidi [F.S.] and is based on the general theory of Eisenstein series. Shahidi's proof is similar in spirit to that of [J-S].

Remark (3.8). One can actually prove, either by our methods or those of Shahidi, that $L_S(s, \pi \times \pi')$ is *globally meromorphic*. However, we shall not need this in what follows.

4. Classification Theorems.

(4.1) As before F is an A-field and S is a large finite set of places containing those at infinity. Our results will not change if we replace S by a larger set; in particular we may assume that any representation (or finite set of representations) is unramified outside of S.

Let $\pi = \otimes_v \pi_v$ (resp. $\pi' = \otimes_v \pi_v'$) be an automorphic, cuspidal, unitary representation of GL_r (resp. $\mathrm{GL}_{r'}$). As in [J-S], we set

$$\pi(v^n) = \operatorname{tr}(A_v{}^n) \tag{1}$$

if A_v is, as before, the class associated to π_v. Similarly for π'. Next for σ real, $\sigma \geq 1$, consider the double series

$$l(\sigma, \pi \times \pi') = \sum_{v \notin S} \sum_{n \geq 1} \frac{\bar{\pi}(v^n)\pi'(v^n)}{n q_v{}^{n\sigma}}. \tag{2}$$

In [J-S, Section 4] we proved that this series is absolutely convergent for $\sigma > 1$. Clearly it is then uniformly convergent for $\sigma > 1 + \epsilon$ and hence represents a continuous function in the half-line $\sigma > 1$. Moreover,

$$L_S(\sigma, \pi \times \pi') = \exp l(\sigma, \pi \times \pi'), \tag{3}$$

also for $\sigma > 1$.

Finally let $\delta_{\pi,\pi'} = 1$ if $\pi \simeq \pi'$ and $= 0$ otherwise.

THEOREM. *As σ tends to 1 through values ≥ 1, we have*

$$\lim_{\sigma \to 1} \left(\log \frac{1}{\sigma - 1}\right)^{-1} l(\sigma, \pi \times \pi') = \delta_{\pi,\pi'}.$$

Proof. Suppose first that $\tilde{\pi} \neq \pi'$. Then by Prop. (3.3), $L_S(\sigma, \pi \times \pi')$ has a continuous extension to the half line $\sigma \geq 1$. Moreover by Theorem (3.7), $L_S(1, \pi \times \pi') \neq 0$. Thus there is a determination of the logarithm (principal or otherwise) for which $\log L_{S'}(\sigma, \pi \times \pi')$ is continuous in an interval $1 \leq \sigma < a$. Next from (3) we get

$$\log L_S(\sigma, \pi \times \pi') = l(\sigma, \pi \times \pi') + 2\pi k(\sigma) \tag{4}$$

where k takes integral values and now $1 < \sigma < a$. But then $k(\sigma)$ being the difference of two continuous functions must be constant in $1 < \sigma < a$.

Thus

$$\lim_{\sigma\to 1, \sigma>1} l(\sigma, \pi, \times \pi') \text{ exists,} \tag{5}$$

which is more than is required for the proof of the theorem.

The proof for $\tilde{\pi} \simeq \pi'$ is similar. In fact by Prop. (3.6)

$$\lim_{\sigma\to 1} (\sigma - 1)L_S(\sigma, \pi \times \overline{\pi}) = \kappa \neq 0. \tag{6}$$

In this case $L_S(\sigma, \pi \times \overline{\pi}) > 0$ for $\sigma > 1$. Thus from (3) we get (with real logarithms)

$$\log(\sigma - 1) + \log L_S(\sigma, \pi \times \overline{\pi}) = \log(\sigma - 1) + l(\sigma, \pi \times \overline{\pi}) \tag{7}$$

which tends to $\log \kappa$ as $\sigma \to 1$. Our assertion follows at once. □

(4.2) We shall prove as a corollary to Theorem (4.1) that *any finite set of distinct automorphic cuspidal representations is linearly independent.*

Suppose ρ is an automorphic cuspidal representation of $\mathrm{GL}_r(\mathbf{A})$. Then ρ may be written uniquely in the form

$$\rho = \pi \otimes \beta, \tag{1}$$

where π is unitary and β is a positive quasi-character ($\beta = \alpha^t$, t real). We have locally $\rho_v = \pi_v \otimes \beta_v$; thus if A_v is the class of π_v, then $B_v = A_v q_v^{-t}$ is the class of ρ. As before we set

$$\rho(v^n) = \mathrm{tr}(B_v{}^n). \tag{2}$$

Next let $\mathfrak{A}_r$ denote the set of all equivalence classes of automorphic cuspidal representations of $G_r(\mathbf{A})$. Let $\mathfrak{A} = \cup_{r\geq 1} \mathfrak{A}_r$ (disjoint union). The precise form of our result is as follows:

THEOREM. *Let $\rho_1, \rho_2, \ldots, \rho_p \in \mathfrak{A}$. Let S be a finite set of places containing the infinite places. Suppose that the representations $\rho_1, \rho_2, \ldots, \rho_p$ are unramified outside of S. Let $c_1, c_2, \ldots, c_p$ be complex numbers. Suppose that we have a relation*

$$\sum_{1\leq j\leq p} c_j \rho_j(v^n) = 0 \tag{3}$$

which holds for all $v \notin S$ and all integers $n \geq 1$. Then

$$c_1 = c_2 = \cdots = c_p = 0.$$

Proof. We may assume that all of the c_j's are non-zero. Write $\rho_j = \pi_j \otimes \alpha^{t_j}$ where π_j is unitary and t_j is real. We may assume that $t_1 \leq t_2 \leq \cdots \leq t_p$. From (2) we have

$$\rho_j(v^n) = \pi_j(v^n) q_v{}^{-nt_j}. \tag{4}$$

Thus multiplying (3) through by $q_v{}^{nt_1}$ one may also assume $t_1 = 0$. Then from (3), (4) and (4.1.2) we get immediately

$$\sum_{1 \leq j \leq p} c_j l(\sigma + t_j, \pi_1 \times \pi_j) = 0. \tag{5}$$

If a given t_j is positive, then the function $l(\sigma + t_j, \pi_1 \times \pi_j)$ is continuous in σ near $\sigma = 1$ (c.f. (4.1)). Suppose then that $t_1 = t_2 = \cdots = t_l = 0$ and that $t_{l+1} > 0$. We get then from (5) and Theorem (4.1)

$$\sum_{1 \leq j \leq l} c_j \lim_{\sigma \to 1} \left(\log \frac{1}{\sigma - 1} \right)^{-1} l(\sigma, \pi_1 \times \pi_j) = 0. \tag{6}$$

But by the same theorem this sum is also

$$\sum_{1 \leq j \leq l} c_j \delta_{\pi_1, \pi_j}. \tag{7}$$

Here $\pi_j = \rho_j$ is inequivalent to $\pi_1 = \rho_1$ unless $j = 1$. Thus combining (6) and (7) we get $c_1 \doteq 0$. This contradiction completes the proof of the theorem. □

(4.3) We shall obtain a classification theorem for automorphic forms on GL_r which is a precise analogue for this group of the known results for local groups.

Accordingly let P be a standard parabolic subgroup of G_r of type $(r_1, r_2, \ldots, r_u)$. The quotient of P by its unipotent radical $U = U_P$ is isomorphic to the group

$$M = G_{r_1} \times G_{r_2} \times \cdots \times G_{r_u}. \tag{1}$$

For each j, $1 \leq j \leq u$, let σ_j be an automorphic cuspidal representation of $\mathrm{GL}(m_j, \mathbf{A})$. For each place v the representation $\sigma_v = \otimes_j \sigma_{jv}$ of the group M_v can be regarded as a representation of P_v trivial on U_v; it induces an admissible representation of G_v which we will denote by

$$\xi_v = \mathrm{Ind}(G_v, P_v; \sigma_v). \tag{2}$$

One obtains then a family of irreducible admissible representations of $G_r(\mathbf{A})$ by taking for each v an irreducible component π_v of the representation ξ_v and forming the "tensor product" $\pi = \otimes_v \pi_v$. On the other hand, with $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$, one can define globally an induced representation

$$\xi = \mathrm{Ind}(G_r(\mathbf{A}), P(\mathbf{A}); \sigma). \tag{3}$$

Of course $\xi = \otimes_v \xi_v$. The irreducible representations π which we have described are exactly the components of ξ. In fact the restriction of ξ to the (local) Hecke algebra at v is a discrete multiple of ξ_v. In this way one obtains all of the automorphic representations of $G_r(\mathbf{A})$ [RPL I].

Let Q be another standard parabolic say of type $(s_1, s_2, \ldots, s_w)$ and τ_j an automorphic cuspidal representation of $\mathrm{GL}(s_j, \mathbf{A})$. As before let $\tau_v = \otimes_j \tau_{jv}$,

$$\eta_v = \mathrm{Ind}(G_v, Q_v; \tau_v), \tag{4}$$

and

$$\eta = \mathrm{Ind}(G_r(\mathbf{A}), Q(\mathbf{A}); \tau), \tag{5}$$

where $\tau = \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_w$. We may ask whether ξ and η have a common constituent. Suppose P and Q are associate (in which case $u = w$) and that there is a permutation ϕ of $\{1, 2, \ldots, u\}$ such that $s_j = r_{\phi(j)}$. Suppose moreover that $\tau_j \simeq \sigma_{\phi(j)}$. We will say in this situation that *the pairs* (σ, P) *and* (τ, Q) *are associate.* When this is so the representations ξ_v and η_v have the same character and therefore the same components; in particular if both ξ_v and η_v are unramified then their unique unramified components are the same. In other words the irreducible components of ξ and η are the same.

The following theorem will imply that the converse is also true.

(4.4) THEOREM. *Let P, Q, σ_j and τ_k be as above. Let S be a finite set of places containing all the places at infinity. Suppose that for $v \notin S$ the representations σ_{jv} and τ_{kv} are unramified and that the representations ξ_v and η_v of G_v they induce have the same unramified component. Then the pairs (σ, P) and (τ, Q) are associate.*

Proof. The proof follows directly from Theorem (4.2). In fact, for $v \notin S$, let A_{jv} (resp. B_{kv}) be the semi-simple conjugacy class in $\mathrm{GL}(r_j, \mathbf{C})$ (resp. $\mathrm{GL}(s_k, \mathbf{C})$) associated to σ_{jv} (resp. τ_{kv}). The class associated to the unramified component of ξ_v (resp. η_v) is nothing other than $\oplus_j A_{jv}$ (resp. $\oplus_k B_{kv}$). The hypothesis is equivalent to the assertion

$$\bigoplus_j A_{jv} = \bigoplus_k B_{kv} \tag{1}$$

(equality as conjugacy classes in $\mathrm{GL}_r(\mathbf{C})$). From (1) we deduce immediately the equality

$$\sum_{1 \le j \le u} \sigma_j(v^n) = \sum_{1 \le k \le w} \tau_k(v^n) \tag{2}$$

which holds for all $v \notin S$ and all integers $n \geq 1$. We regard the σ_j and τ_k as elements of $\mathfrak{A}$. Let then $\mathfrak{B} = \{\rho_1, \rho_2, \ldots, \rho_a\}$ (resp. $\mathfrak{B}' = \{\rho_1', \rho_2', \ldots, \rho_b'\}$) be the set of distinct elements of $\mathfrak{A}$ among the σ_j (resp. τ_k). Then (2) may be rewritten as

$$\sum_{1 \le \alpha \le a} m_\alpha \rho_\alpha(v^n) = \sum_{1 \le \alpha \le b} m_\alpha' \rho_\alpha'(v^n) \tag{3}$$

where m_α (resp. m_α') is the number of times ρ_α (resp. ρ_α') occurs in the left (resp. right) side of (2).

But by Theorem (4.2) $\mathfrak{B}$, regarded as a set of functions on the set of v^n, $v \notin S$, $n \geq 1$, is linearly independent. Similarly $\mathfrak{B}'$ is linearly independent. Thus we get at once $\mathfrak{B} = \mathfrak{B}'$, $a = b$ and $m_\alpha = m_\alpha'$ for $1 \leq \alpha \leq a$. In particular

$$u = \sum m_\alpha = \sum m_\alpha' = w, \tag{4}$$

and we also see that the maps

$$j \mapsto \sigma_j, \; j \mapsto \tau_j$$

of $\{1, 2, \ldots, u\}$ into $\mathfrak{A}$ differ only by a permutation. $\square$

(4.5) As we have seen in (4.4) any automorphic representation π of $G_r(\mathbf{A})$ is a subquotient of an essentially unique induced representation

$$\mathrm{Ind}(G_r(\mathbf{A}), P(\mathbf{A}); \sigma) \tag{1}$$

where $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_u$ is an automorphic cuspidal representation of $M(\mathbf{A})$. Thus it is natural to associate to π the (formal) sum

$$\sum_{1 \leq j \leq u} \sigma_j, \tag{2}$$

or, if m_ρ is the multiplicity of a given element ρ of $\mathfrak{A}$ in (2), the formal sum

$$\chi_\pi = \sum_{\rho \in \mathfrak{A}} m_\rho \rho. \tag{3}$$

If we define $\deg \pi = r$, we clearly have

$$\deg \pi = \sum m_\rho \deg \rho. \tag{4}$$

Conversely the theory of Eisenstein series allows us, at least when the m_ρ are positive and integral, to attach to a formal sum (3) an automorphic representation of some group $G_r(\mathbf{A})$. More generally we shall refer to a sum (3) where the coefficients m_ρ are allowed to be positive or negative integers or zero as a *virtual form*. Of course we require $m_\rho = 0$ for all but finitely many ρ. If χ is a virtual form we call the ρ which actually appear ($m_\rho \neq 0$) in χ the *components* of χ, and define

$$\deg \chi = \sum m_\rho \deg \rho. \tag{5}$$

The *contragredient* $\tilde{\chi}$ of χ is the virtual form

$$\tilde{\chi} = \sum m_\rho \tilde{\rho}. \tag{6}$$

If $\chi' = \sum m_\rho' \rho$ is another virtual form we define the inner product (χ, χ') to be

$$(\chi, \chi') = \sum_{\rho \in \mathfrak{A}} m_\rho m_\rho'. \tag{7}$$

If S is a finite set of places (containing those at infinity) outside of which the components of χ and χ' are unramified, we may also define the L-function

$$L_S(s, \chi \times \chi') = \prod_{\rho,\tau \in \mathfrak{A}} L_S(s, \rho \times \tau)^{m_\rho m_\tau}. \tag{8}$$

If then we set, for $v \notin S$, $n \geq 1$,

$$\chi(v^n) = \Sigma\, m_\rho \rho(v^n), \qquad \chi'(v^n) = \Sigma\, m_\rho' \rho(v^n), \tag{9}$$

we get at once

$$L_S(s, \chi \times \chi') = \exp \sum_{v \notin S} \sum_{n \geq 1} \frac{\chi(v^n)\chi'(v^n)}{n q_n{}^{ns}} \tag{10}$$

at least if $\mathrm{Re}(s)$ is large.

Suppose now that the components of χ and χ' are *unitary*; then (8) is defined for $\mathrm{Re}(s) \geq 1$. Then by Prop. (3.3), Prop. (3.6), and Theorem (3.7), $L_S(s, \tilde{\chi} \times \chi')$ has a pole at $s = 1$ exactly of order (χ, χ'). More precisely the limit

$$\lim_{s \to 1, s \geq 1} (s-1)^{(\chi,\chi')} L_S(s, \tilde{\chi} \times \chi') \tag{11}$$

exists and is non-zero. In what follows we shall apply this to give a criterion for a Galois representation to be "automorphic." We can only hope that an eventual application will be found.

(4.6) We recall the "strong form" of Artin's conjecture due to Langlands and Weil.

Suppose K is a finite Galois extension of the global field F with Galois group $\mathfrak{G}$. If v, a place of F, is unramified in K, let Fr_v denote the corresponding (Frobenius) conjugacy class in $\mathfrak{G}$. If σ is a representation, irreducible or not, of $\mathfrak{G}$ of degree d, then $\sigma(\mathrm{Fr}_v)$ is a well-defined conjugacy class in $\mathrm{GL}_d(\mathbf{C})$. We shall say that σ is automorphic (or satisfies the strong form of Artin's conjecture) if there is an automorphic representation $\hat{\sigma}$ of $\mathrm{GL}_d(\mathbf{A})$, $\hat{\sigma} = \otimes_v \hat{\sigma}_v$, such that, at each place v of F which is unramified in K, $\hat{\sigma}_v$ is unramified and

$$\sigma(\mathrm{Fr}_v) = A_v, \tag{1}$$

where A_v is the (Langlands) class associated to $\hat{\sigma}_v$. Let χ_σ denote the character (trace) of σ. An equivalent form of (1) is

$$\chi_\sigma(\mathrm{Fr}_v{}^n) = \hat{\sigma}(v^n), \tag{2}$$

with v as above and $n \geq 1$. Note then that by Theorem (4.2), the virtual form

$$\chi_{\hat{\sigma}} = \Sigma\, m_\rho \rho \tag{3}$$

associated to $\hat{\sigma}$ is then uniquely determined. It is easy to see that any component ρ of $\chi_{\hat{\sigma}}$ is unitary. In fact let $\omega = \otimes_v \omega_v$ be the central quasi-character of ρ, and for v outside of a sufficiently large finite set S of places of F, let B_v be the class of ρ_v. Then the eigenvalues of B_v are among those of A_v and thus by (1) have absolute value one. Then

$$|\omega_v(\tilde{\omega}_v)| = 1, \qquad v \notin S, \tag{4}$$

and by Dirichlet's theorem (or (4.2)) ω itself is unitary. Since ρ is a cusp form it must be unitary. We remark that even though a representation σ of $\mathfrak{G}$ may be automorphic it is not clear that the same is true for the irreducible components of σ. We can however establish the following theorem.

(4.7) THEOREM. *Let F be a global field, and K be a finite Galois extension of F. Suppose π is an irreducible representation of* Gal(K/F) *and that as a virtual representation*

$$\pi = \sigma - \tau, \tag{1}$$

where σ and τ are both automorphic. Then π is automorphic and cuspidal.

Proof. Let deg $\sigma = a$, deg $\tau = b$. By hypothesis there are automorphic representations $\hat{\sigma}$ and $\hat{\tau}$ of $\mathrm{GL}_a(\mathbf{A})$ and $\mathrm{GL}_b(\mathbf{A})$ respectively, and a finite set of places S of F such that

$$\chi_\sigma(\mathrm{Fr}_v{}^n) = \chi_{\hat{\sigma}}(v^n), \qquad \chi_\tau(\mathrm{Fr}_v{}^n) = \chi_{\hat{\tau}}(v^n), \tag{2}$$

for all $v \notin S$ and all $n \geq 1$. Let χ be the virtual form

$$\chi = \chi_{\hat{\sigma}} - \chi_{\hat{\tau}}. \tag{3}$$

Then

$$\chi_\pi(\mathrm{Fr}_v{}^n) = \chi_\sigma(\mathrm{Fr}_v{}^n) - \chi_\tau(\mathrm{Fr}_v{}^n) = \chi_{\hat\sigma}(v^n) - \chi_{\hat\tau}(v^n) = \chi(v^n), \tag{4}$$

and consequently

$$\sum_{v \notin S} \sum_{n \geq 1} |\chi_\pi(\mathrm{Fr}_v{}^n)|^2/nq_v{}^{ns} = \sum_{v \notin S} \sum_{n \geq 1} |\chi(v^n)|^2/nq_v{}^{ns}. \tag{5}$$

Exponentiating both sides and using (4.5.8), we get at once

$$L_S(s, \pi \times \tilde\pi) = L_S(s, \chi \times \tilde\chi), \tag{6}$$

the left side being the product of the local Artin L-functions $L_v(s, \pi \times \tilde\pi)$ over the places v of F not in S. Similarly

$$L_S(s, \sigma \times \tilde\sigma) = L_S(s, \chi_{\hat\sigma} \times \tilde\chi_{\hat\sigma}), \qquad L_S(s, \tau \times \tilde\tau) = L_S(s, \chi_{\hat\tau} \times \tilde\chi_{\hat\tau}). \tag{7}$$

There is a well-known analogue of (4.5.11) for Artin L-functions: if ξ is a (unitary) representation of $\mathfrak{G}$ then $L_S(s, \xi)$ has a pole at $s = 1$ of order exactly equal to the multiplicity of the trivial representation in ξ. In particular, $L_S(s, \xi \otimes \tilde\xi)$ has a pole of order (χ_ξ, χ_ξ) at $s = 1$, the inner product normalized to be 1 if ξ is irreducible. Thus from (6), (7) and (4.5.11) we have at once

$$(\chi_{\hat\sigma}, \chi_{\hat\sigma}) = (\chi_\sigma, \chi_\sigma), \qquad (\chi_{\hat\tau}, \chi_{\hat\tau}) = (\chi_\tau, \chi_\tau), \qquad (\chi, \chi) = 1. \tag{8}$$

In particular $\chi = \Sigma_{\rho \in \mathfrak{A}}\, m_\rho \rho$, with $\Sigma\, m_\rho{}^2 = 1$. The m_ρ being integers all but one of them, say m_{ρ_0}, must be zero and $\chi = \pm\rho_0$ is, up to sign, a cuspidal representation. If $\chi = \rho_0$ we are done, noting that, by (3), $\deg \rho_0 = \deg \hat\sigma - \deg \hat\tau = \deg \sigma - \deg \tau = \deg \pi$. Suppose then that $\chi = -\rho_0$ with ρ_0 an automorphic cuspidal representation. Then from (3) we get

$$\chi_{\hat\tau} = \chi_{\hat\sigma} + \rho_0. \tag{9}$$

Since $\hat\sigma$ and ρ_0 are actually automorphic representations and not merely virtual, we get at once from (9)

$$(\chi_{\hat\tau}, \chi_{\hat\tau}) \geq (\chi_{\hat\sigma}, \chi_{\hat\sigma}) + (\rho_0, \rho_0) > (\chi_{\hat\sigma}, \chi_{\hat\sigma}). \tag{10}$$

But in an analogous way we get from (1),

$$(\chi_\sigma, \chi_\sigma) > (\chi_\tau, \chi_\tau). \tag{11}$$

The obvious contradiction which results from combining (8), (10) and (11) shows in fact that χ is automorphic cuspidal and we are done. $\square$

(4.8) *Remark.* With the above notation, if π is a representation of $\mathrm{Gal}(K/F)$, we may always use Brauer's Theorem to write $\pi = \sigma - \tau$ where the representations σ and τ are sums of *monomial representations.* Thus the construction of forms on $\mathrm{GL}(n)$ attached to arbitrary extensions of the ground field combined with the theory of Eisenstein series [R.P.L. I] would lead to a proof of (the strong form of) Artin's conjecture. Unfortunately except in the case of normal cyclic extensions there does not seem to be much hope in proving the existence of these forms in the near future.

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