ON EULER PRODUCTS AND THE CLASSIFICATION OF AUTOMORPHIC REPRESENTATIONS I*

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0. Introduction and Notations.

(0.1) In this paper we prove two theorems concerning automorphic forms on GL_r, both having their origin in the existence of poles of L-functions.

Let F be a global field and $\mathbf{A}$ be the ring of adeles of F. To each automorphic cuspidal representation π of $GL_r(\mathbf{A})$ one can attach as in [G-J] an L-function $L(s, \pi)$. For $r = 1$, π is an idele-class character and $L(s, \pi)$ the classical L-function of Hecke. For $r > 1$, $L(s, \pi)$ is entire and satisfies a functional equation

$$L(s, \pi) = \epsilon(s, \pi)L(1 - s, \tilde{\pi}),$$

$\tilde{\pi}$ being the contragradient representation. In an earlier paper [J-S], the authors proved that $L(s, \pi)$ does not vanish on $\mathrm{Re}(s) = 1$. One of our results complements this. We prove in Section 5 that the Euler product for $L(s, \pi)$ is absolutely convergent in the half-plane $\mathrm{Re}(s) > 1$. For $r = 2$ this was first proved by R. Rankin [R.R.]. Our proof is based on his. In the same series of papers, Rankin also proves the non-vanishing of $L(s, \pi)$ on $\mathrm{Re}(s) = 1$, by means entirely distinct from ours. With more work his method should also apply to the general case. As in the recent paper of Moreno [C.J.M.], one should be able to obtain zero free regions as well. We also refer the reader to the classical paper of A. Selberg [A.S.].

Our second result concerns the problem of classification of automorphic forms on $GL_r(\mathbf{A})$. We refer to Langlands' Corvallis lecture [R.P.L.] for the exact statement of this problem as well as a discussion of the interesting concept of an isobaric representation. In this paper we consider a special case. We prove that, if S is any finite set of places and if π and π'

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are automorphic cuspidal representations which are locally equivalent outside S, then π and π' are themselves equivalent. For $r = 1$ this is another formulation of the approximation theorem. For $r = 2$ it is a well-known result of Miyake [T.M.]. Piatetskii-Shapiro [I.P.] has also given a proof in the general case. Both authors suppose however that S does not contain any infinite places. Moreover the assumption needed in our paper is somewhat weaker than the one just stated.

In a subsequent paper we will use the results of this paper to establish the full classification theorem.

We give an outline of the proof of our second theorem. Suppose first that $r = 1$ and that χ and χ' are two idele-class characters of absolute value one which agree locally outside S. Set

$$A(s) = \prod_{v \in S} L(s, \chi_v' \bar{\chi}_v), \qquad B(s) = \prod_{v \in S} L(s, 1_v).$$

The hypothesis implies that

$$B(s)L(s, \chi'\bar{\chi}) = A(s)L(s, 1).$$

But $L(s, 1)$ has a pole at $s = 1$. On the other hand, a local L-factor cannot be zero. Thus the right side has a pole at $s = 1$. Next, and this is an essential point, the factor $L(s, 1_v)$ does not have a pole at $s = 1$. Thus $L(s, \chi'\bar{\chi})$ has a pole at $s = 1$ and that implies $\chi'\bar{\chi} = 1$ or $\chi = \chi'$.

Before passing to the case $r \geq 2$, note that one may obtain the analytic properties of $L(s, \chi)$, by utilizing an integral representation of the form

$$L(s, \chi) = \int_{F_{\mathbf{A}}^{\times}/F^{\times}} \chi(t)|t|^s \sum_{\xi \in F^{\times}} \Phi(t\xi) d^{\times}t, \tag{1}$$

where Φ is a suitable Schwartz-Bruhat function on $F_{\mathbf{A}}$. Similarly if χ_v is a character of absolute value one of $F_v^{\times}$ one has

$$L(s, \chi_v) = \int_{F_v^{\times}} \chi_v(t)|t|^s \Phi_v(t) d^{\times}t, \tag{2}$$

where Φ_v is a suitable Schwartz-Bruhat function on F_v. The assertion that $L(s, \chi_v)$ is holomorphic at $s = 1$ may be proved by showing that the integral on the right is convergent for $\mathrm{Re}(s) \geq 1$.

Suppose now $r \geq 2$. The exact analogue of the integral (1) is the integral

$$\int_{G(F)\backslash G(\mathbf{A})} \phi'\bar{\phi}(g)\,|\det g|^s \sum_{\xi\in F^r-\{0\}} \Phi(\xi g)dg, \tag{3}$$

where $G = GL_r$, ϕ and ϕ' are cusp forms belonging to π and π' respectively, and Φ is a Schwartz-Bruhat function on $F_{\mathbf{A}}{}^r$. The analytic properties of this integral are evident. We express this integral in terms of local integrals which are the analogues of (2); the essential difficulty is to prove that these integrals converge for $\mathrm{Re}(s) \geq 1$ (Proposition (1.5) and Proposition (3.17)). For this one needs certain properties of the *unitary* representations π_v (Proposition (1.3) and (3.7)). With this the proof that $\pi \sim \pi'$ is entirely analogous to the case $r = 1$ (Section 4).

In Section 5, we give an application to the absolute convergence in $\mathrm{Re}(s) > 1$ of certain Euler products $L(s, \pi \times \pi')$ attached to pairs of unitary forms respectively on GL_r and GL_p. For $p = 1$ and π' trivial they reduce to the functions $L(s, \pi)$ considered earlier. These functions enter implicitly in the proof of Theorem (4.7). In fact, for $r = p$ they are essentially given by (3) (see also (2.1.2) below). For $r = 2$ they are the classical Rankin-Selberg convolutions.

We note finally that recently, by methods entirely similar to [J-S], but using the full theory of Eisenstein series for GL_r, F. Shahidi [F.S.] has shown that $L(s, \pi \times \pi')$ is non-zero for $\mathrm{Re}(s) = 1$. This together with our results will be used in proving the classification theorem for forms on GL_r.

(0.2) Upon first reading one should concentrate on Section 1, Sections (2.1), (2.2), the statement of Proposition (2.3), then (3.1), the statement of Proposition (3.5), (3.13), and finally Section 4 and Section 5.

(0.3) We now give a list of our most frequently used notations.

The ground field F is either a local field or an $\mathbf{A}$ field. The group $G_r = GL_r = GL(r)$ is regarded as an F-group. We denote by Z_r the center of G_r; often we identify Z_r with the group GL_1. The group of matrices of the form

$$p = \begin{pmatrix} m & u \\ 0 & 1 \end{pmatrix}, \qquad m \in G_{r-1}, \tag{1}$$

will be denoted P_r. The unipotent radical of P_r, the subgroup of $p \in P_r$ with $m = 1$, is noted U_r while the transpose of U_r is often noted $\overline{U}_r$.

A parabolic subgroup R of G_r will always be an F-parabolic subgroup. The unipotent radical of R will be denoted by U_R. For the most part we will only consider standard parabolic subgroups, i.e. groups of matrices of the form

$$g = \begin{pmatrix} m_1 & & & & * \\ & m_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & m_t \end{pmatrix} \tag{2}$$

where $m_i \in G_{r_i}$ and $(r_1, r_2, \ldots, r_t)$ is a t-tuple of integers such that

$$r_1 + r_2 + \cdots + r_t = r.$$

The t-tuple is the *type* of the parabolic subgroup. The transpose of a standard parabolic subgroup is termed a lower standard parabolic subgroup. The standard parabolic of type $(1, 1, \ldots, 1)$ (the standard Borel subgroup) is denoted B_r and its unipotent radical N_r. We denote by A_r the group of diagonal matrices; the set of simple roots Δ is the set of simple roots of A_r with respect to B_r, i.e. the homomorphisms $\alpha_i \in \mathrm{Hom}(A_r, GL(1))$ defined by

$$\alpha_i(a) = a_i/a_{i+1} \qquad \text{if } a = \mathrm{diag}(a_1, a_2, \ldots, a_r), \qquad 1 \le i \le r-1.$$

The index r is often dropped when this does not create confusion.

When F is local we use standard notation: $|x|$ also denoted by $\alpha_F(x)$ or even $\alpha(x)$ is the module. If F is non-archimedean, $\mathfrak{R}_F = \mathfrak{R}$ will denote the ring of integers of F, $\tilde{\omega}$ will be a generator of the unique maximal ideal of $\mathfrak{R}$ and we will set $q = |\tilde{\omega}|^{-1}$. For any F we will denote by $\mathcal{S}(F^r)$ the space of Schwartz-Bruhat functions on F^r. We often view F^r as a space of row vectors. If ψ is a non-trivial additive character of F, the Fourier transform of a function $\Phi \in \mathcal{S}(F^r)$ is the function $\hat{\Phi} \in \mathcal{S}(F^r)$ defined by

$$\hat{\Phi}(x) = \int_{F^r} \Phi(y)\psi(y^t x)dy. \tag{3}$$

Of course dy is the self-dual Haar measure on F^r.

Similarly when F is an $\mathbf{A}$-field we denote by $|x|$, $\alpha_F(x)$ or $\alpha(x)$ the module of F. If v is a place of F we denote by F_v the corresponding local field. We fix, once and for all, a non-trivial character ψ of $F_{\mathbf{A}}/F$. Then we write $\psi = \Pi_v \psi_v$. Again we denote by $\mathcal{S}(F_{\mathbf{A}}{}^r)$ the space of Schwartz-Bruhat functions on $F_{\mathbf{A}}{}^r$ and by $\hat{\Phi}$ the Fourier transform of a function Φ in $\mathcal{S}(F_{\mathbf{A}}{}^r)$.

We will make extensive use of standard terminology in representation theory. We will deal mainly with unitary representations although admissible representations will occasionally make an appearance. If G is a locally compact group and H a closed subgroup, a representation σ of H on a vector space $\mathcal{V}$ induces a representation π of G; it will be denoted by

$$\mathrm{Ind}(G, H; \sigma). \tag{4}$$

The exact definition depends on the context, but in any case the vectors ϕ in the space of π are functions (or classes of functions) on G with values in $\mathcal{V}$ which satisfy

$$\phi(hg) = \delta_H{}^{1/2}(h)\delta_G{}^{-1/2}(h)\sigma(h)\phi(g);\ h \in H, g \in G. \tag{5}$$

Here δ_G(resp. δ_H) is the module of G(resp. H). More precisely if $d_r(g)$ and $d_\ell(g)$ are respectively right and left Haar measures on G, then $d_r(xg) = \delta_G(x)d_r(g)$ and $d_\ell(gx^{-1}) = \delta_G(x)d_\ell(g)$. Of course π acts by right translations:

$$\pi(x)\phi(g) = \phi(gx);\ g, x \in G.$$

For instance, if $G = G_r(F)$ (F being local), if R is the standard parabolic subgroup of type $(r_1, r_2, \ldots, r_t)$ and for each i, $1 \leq i \leq t$, π_i is a representation of $G_{r_i}(F)$, we can define a representation σ of $R(F)$ by

$$\sigma(g) = \otimes\ \pi_i(m_i) \tag{6}$$

for $g \in R(F)$ of the form (2). The representation of $G_r(F)$ induced by σ will be denoted by

$$\mathrm{Ind}(G_r(F), R_r(F); \pi_1, \pi_2, \ldots, \pi_t). \tag{7}$$

If π is a representation of $G_r(F)$ on a complex vector space $\mathcal{V}$ we will denote by $\pi \otimes \alpha^s$ the representation on the same space defined by

$$(\pi \otimes \alpha^s)(g) = |\det g|^s \pi(g). \tag{8}$$

We will denote by $\overline{\mathcal{V}}$ the complex vector space conjugate to $\mathcal{V}$ and by $\bar{\pi}$ the natural representation of $G_r(F)$ on $\overline{\mathcal{V}}$.

If π is admissible we denote by $\tilde{\pi}$ the representation contragredient to π. In case π is irreducible there is a character ω (perhaps not of absolute value one) of $F^\times$ such that

$$\pi(a) = \omega(a)1, \qquad \text{for } a \in Z_r(F) \simeq F^\times. \tag{9}$$

We call ω the central character of π.

If F is local (resp. global) we define a character θ_r (or simply θ) of $N_r(F)$ (resp. $N_r(F_{\mathbf{A}})$) by

$$\theta(n) = \psi\Big(\sum_{1 \leq i \leq r} n_{i,i+1}\Big). \tag{10}$$

Finally when F is local we often write G_r for $G_r(F)$, P_r for $P_r(F)$, etc.

We let K be the standard maximal compact subgroup of $G_r(F)$: $K = O(r, \mathbf{R})$ if $F = \mathbf{R}$, $K = U(r)$ if $F = \mathbf{C}$, $K = GL(r, \mathfrak{R}_F)$ if F is non-archimedean. When F is an $\mathbf{A}$-field we let K be the compact subgroup of $G_r(F_{\mathbf{A}})$ defined by

$$K = \prod_v K_v$$

where K_v is the above subgroup of $G_r(F_v)$.

1. Generic representations: non-archimedean case. In this section F is a non-archimedean local field. We first review the notion of a "generic representation" (Prop. (1.3)) adding appropriate remarks in the case of unitary representations. We then discuss an analogue for $r \geq 2$ of the integral (2) appearing in the introduction (0.1).

(1.1) Let π be an irreducible admissible representation of G on a complex vector space $\mathcal{V}$. We say that π is generic if there exists a linear form $\lambda \neq 0$ on $\mathcal{V}$ such that

$$\lambda(\pi(n)v) = \theta(n)v, \qquad \text{for } n \in N,\ v \in \mathcal{V}, \tag{1}$$

(c.f. (0.10)).

Up to multiplication by a scalar factor this form is unique ([J.S.], [G-K]). We let $\mathcal{W}(\pi; \psi)$ be the space of all functions W of the form

$$W(g) = \lambda(\pi(g)v) \tag{2}$$

where v is in $\mathcal{V}$. The space $\mathcal{W}(\pi;\ \psi)$ is invariant under right translations and the representation of G on that space is equivalent to π. Clearly for any W in $\mathcal{W}(\pi;\ \psi)$ we have

$$W(ng) = \theta(n)W(g), n \in N, g \in G. \tag{3}$$

The map $W \mapsto W|P$ is a bijection of $\mathcal{W}(\pi;\ \psi)$ on a space $\mathcal{K} = \mathcal{K}(\pi;\ \psi)$ of functions ϕ on P_0. Thus we may identify $\mathcal{V}$ with $\mathcal{K}$. Then

$$\pi(p_0)\phi(p) = \phi(pp_0), \phi \in \mathcal{K}, p_0 \in P. \tag{4}$$

Let $\mathcal{K}_0$ be the space of functions ϕ on P which transform on the left like θ under N, are right invariant under some open compact subgroup of P, and have compact support mod N. Let τ_0 be the representation of P on $\mathcal{K}_0$ by right translations. Then for all generic representations π, $\mathcal{K}_0$ is contained in $\mathcal{K}(\pi;\ \psi)$. In particular $\pi|P$ contains, as a subrepresentation, the representation τ_0 (c.f. also [F.R.]).

(1.2) Let π now be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}^\infty$ be the subspace of smooth vectors, that is, the space of vectors fixed by some open subgroup of G. Let π^∞ be the representation of G on $\mathcal{H}^\infty$.

Finally let τ be the unitary representation of P induced—in Mackey's sense—by θ. Let $\mathcal{L}$ be the Hilbert space of τ and $\mathcal{L}^\infty$ the space of smooth vectors in $\mathcal{L}$. Clearly $\mathcal{L}^\infty \supset \mathcal{K}_0$.

(1.3) PROPOSITION. *Notations being as above, the following conditions are equivalent:*

(i) *the restriction of π to P contains τ;*
(ii) *the representation π^∞ is generic.*

Assume these conditions satisfied and identify $\mathcal{W}(\pi;\ \psi)$ with $\mathcal{H}^\infty$. Then there is a constant $c > 0$ such that

$$\int_{N\backslash P} |W|^2(pg)d_r(p) \leq c\,||W||^2 \tag{1}$$

for $W \in \mathcal{W}(\pi;\ \psi)$ and $g \in G$.

Proof. Let A be a non-zero operator in $\mathrm{Hom}_P(\pi|P, \tau)$. Then $A(\mathcal{H}^\infty)$ is contained in $\mathcal{L}^\infty$. Thus for $v \in \mathcal{H}^\infty$, Av is a smooth function on P transforming on the left like θ under N. We set

$$\lambda(v) = A(v)(e), \qquad v \in \mathcal{H}^\infty.$$

We then have

$$\begin{aligned}\lambda(\pi(n)v) &= A(\pi(n)v)(e)\\ &= (\tau(n)A(v))(e) = A(v)(n) = \theta(n)Av(e)\\ &= \theta(n)\lambda(v).\end{aligned}$$

Since $A \neq 0$ and $\mathcal{H}^\infty$ is dense in $\mathcal{H}$ there is a $v \in \mathcal{H}^\infty$ such that $A(v) \neq 0$. Choose $p \in P$ with $A(v)(p) \neq 0$. Then $\lambda(\pi(p)v) = A(v)(p) \neq 0$. Hence $\lambda \neq 0$ and π^∞ is generic. Thus (i) implies (ii).

Now assume π^∞ is generic. Let $(\cdot\ ,\ \cdot)$ denote the Hermitian scalar product on $\mathcal{H}$. We may identify $\mathcal{H}^\infty$ with the space $\mathcal{K}$ defined in (1.1). Then the restriction of the scalar product to $\mathcal{K}_0$ is a positive Hermitian form, invariant under the representation τ_0. It follows from [B-Z], Proposition 3.7, that there is a positive constant c such that

$$(\phi_1, \phi_2) = c \int_{N\backslash P} \phi_1(p)\bar{\phi}_2(p)d_r(p)$$

for $\phi_i \in \mathcal{K}_0$. Thus the injection $\mathcal{K}_0 \to \mathcal{K}$ is continuous for the norm induced on $\mathcal{K}_0$ by $\mathcal{L}$ and the norm induced on $\mathcal{K}$ by $\mathcal{H}$. Thus A extends to a bounded operator $A : \mathcal{L} \to \mathcal{H}$ commuting with P. Thus (ii) implies (i).

Finally assume the equivalent conditions of Proposition (1.3) are satisfied. Again let $A \neq 0$ be in $\mathrm{Hom}_P(\pi|P, \tau)$. If $W \in \mathcal{W}(\pi; \psi)$, let v be the corresponding vector in $\mathcal{H}^\infty$:

$$W(g) = A(\pi(g)v)(e).$$

Fix $g \in G$ and let ϕ be $A(\pi(g)v)$. We have

$$\int_{N\backslash P} |\phi|^2(p)d_r(p) = ||\phi||^2 = ||A\pi(g)v||^2$$

$$\leq ||A||^2||\pi(g)v||^2 = ||A||^2||v||^2.$$

Now $W(pg) = \phi(p)$. Thus

$$\int_{N\backslash P} |W|^2(pg)d_r(p) \leq ||A||^2||v||^2. \qquad \square$$

(1.4) *Remark.* In a future paper, we will show that under the assumptions of (1.3), $\pi|P$ is actually equivalent to τ.

(1.5) Suppose now that π and π' are irreducible *unitary* generic representations of G_r. For $\Phi \in \mathcal{S}(F^r)$, $W \in \mathcal{W}(\pi; \psi)$, and $W' \in \mathcal{W}(\pi'; \psi)$, we set

$$\Psi(s, W', W, \Phi) = \int_{N\backslash G} W'(g)\overline{W}(g)\Phi(\epsilon g)|\det g|^s dg \tag{1}$$

where ϵ is the following row vector:

$$\epsilon = (0, 0, \ldots, 1). \tag{2}$$

PROPOSITION. (i) *The integral* (1) *converges absolutely in the half space* $\mathrm{Re}(s) \geq 1$, *normally for* $\mathrm{Re}(s)$ *in a compact subset of* $[1, \infty[$.

(ii) *Given* $W' \neq 0$ *and* s *with* $\mathrm{Re}(s) \geq 1$, *there exist* W *and* Φ *such that*

$$\Psi(s, W', W, \Phi) \neq 0. \tag{3}$$

Proof. The integral can be written explicitly as

$$\int_K dk \int_{N\backslash P} d_r p\,|\det p|^{s-1} W'(pk)\overline{W}(pk) \int_{F^\times} \Phi(\epsilon a k)|a|^{rs}\omega'\overline{\omega}(a)d^\times a,$$

where ω'(resp. ω) is the central character of π (resp. π'). There is a $\Phi_0 \geq 0$ in $\mathcal{S}(F^r)$ such that

$$|\Phi(xk)| \leq \Phi_0(x)$$

for all $x \in F^r$ and $k \in K$. Then for s real,

$$\int_{F^\times} |\Phi(\epsilon a k)||a|^{rs}|\omega'\overline{\omega}(a)|d^\times a \leq \int_{F^\times} \Phi_0(\epsilon a)|a|^{rs} d^\times a.$$

This is certainly uniformly bounded for s in a compact subset of $[1, \infty[$. Thus it suffices to show that the integral

$$\int_K dk \int_{N\backslash P} d_r(p)|\det p|^{s-1}|W'(pk)||W(pk)|$$

converges uniformly for s in a compact subset of $[1, \infty[$. From (1.3.1) and the Schwartz inequality we already know that

$$\int_K dk \int_{N\backslash P} d_r(p)|W'(pk)||W(pk)| < +\infty. \tag{4}$$

Let ϕ_1(resp. ϕ_2) be the characteristic function of the set of $g \in G_r$ such that $|\det g| \leq 1$ (resp. $|\det g| > 1$). It will be enough to show that the integrals

$$I_i = \int_K dk \int_{N\backslash P} |W'W(pk)|\phi_i(p)|\det p|^{s-1}d_r(p), \qquad i = 1, 2,$$

converge uniformly for s in a compact subset of $[1, \infty[$. In the first case we may take $s = 1$ and our assertion follows from (4). In the second case we may take s large and replace ϕ_2 by the constant function one.

Recall that W and W' are majorized by a "gauge" ([J-P-S] (2.3)): a gauge ξ is a function on G_r invariant on the left under NZ, on the right under K and given on the diagonal matrices by the formula

$$\xi(a) = \phi(\alpha_1(a), \alpha_2(a), \ldots, \alpha_{r-1}(a))|\alpha_1(a)\alpha_2(a) \cdots \alpha_{r-1}(a)|^{-t}$$

where $t \geq 0$ and $\phi \geq 0$ is in $\mathcal{S}(F^{r-1})$. All we have to do then is show that if ξ and ξ' are gauges then

$$\int_{N\backslash P} \xi\xi'(p)|\det p|^{s-1}d_r(p) < +\infty \qquad \text{for } s \text{ large}.$$

This is a straight-forward matter left to the reader.

Proof of (ii). Suppose $W' \neq 0$ and s with $\mathrm{Re}(s) \geq 1$ are given. Then $W'(g_0) \neq 0$ for some $g_0 \in G$. On the other hand

$$\int_{N\backslash G} W'(gg_0)\overline{W}(g)\Phi(\epsilon g)|\det g|^s dg$$

$$= |\det g_0|^{-s} \int_{N\backslash G} W'(g)\overline{W}(gg_0{}^{-1})\Phi(\epsilon gg_0{}^{-1})|\det g|^s dg.$$

Thus at the cost of replacing W' by a right translate we may assume $W'(e) \neq 0$. Since W' is smooth there is a ϕ in the space $\mathcal{K}_0$ such that

$$\int_{N\backslash P} W'(p)\overline{\phi}(p)|\det p|^{s-1}d_r(p) \neq 0.$$

Choosing W in $\mathcal{W}(\pi; \psi)$ such that $W|P = \phi$(c.f. (1.1)) we get the relation

$$\int_{N\backslash P} W'(p)\overline{W}(p)|\det p|^{s-1}d_r(p) \neq 0.$$

For $k \in K$ set

$$F(k) = \int_{N\backslash P} W'(pk)\overline{W}(pk)|\det p|^{s-1}d_r(p).$$

Then $F(e) \neq 0$ and

$$F(pk) = F(k) \qquad \text{for } p \in P \cap K,$$

$$F(ak) = \omega'\overline{\omega}(a)F(k) \qquad \text{for } a \in Z \cap K.$$

It follows that there is a function Φ on F^r such that $\Phi(x) = \overline{F(k)}$ if $x = \epsilon k$ with $k \in K$, $\Phi(x) = 0$ otherwise. Moreover it is clear from the identity $G_r = PZK$ that if $x \in F^r$ is non-zero, then xK_0 is open for any open compact subgroup K_0 of K. It follows readily that $\Phi \in \mathcal{S}(F^r)$. Now, for appropriate normalizations of the measures,

$$\int_{F^\times} \Phi(\epsilon ak)|a|^{rs}\omega'\overline{\omega}(a)d^\times a = \int_{\mathfrak{R}^\times} \Phi(\epsilon ak)\omega'\overline{\omega}(a)d^\times a = \overline{F(k)}.$$

Thus

$$\Psi(s, W', W, \Phi) = \int_K F(k)\overline{F}(k)dk.$$

Since this is clearly non-zero, the proof of Proposition (1.5) is now complete.

(1.6) *Remark.* One can show that the integrals (1.5.1) possess a "g.c.d" in the following sense. Given π and π', an arbitrary pair of generic irreducible representations of G_r, there is a polynomial $P(x)$ such that $P(0) = 1$ with the property that for each triple W, W', Φ as above, there is a polynomial $Q(X, Y)$ such that $\Psi(s, W', \overline{W}, \Phi) = P(q^{-s})^{-1}Q(q^s, q^{-s})$; moreover the vector space $\mathbf{C}[X; Y]$ is spanned by the Q's. We write $L(s, \pi \times \pi') = P(q^{-s})^{-1}$. The L-factor $L(s, \pi \times \pi')$ may be computed explicitly in terms of Langlands' classification. The proposition implies that if π and π' are also unitary, then $L(s, \pi \times \pi') \neq 0$, for $\mathrm{Re}(s) \geq 1$.

2. Euler factors in the unramified case. We compute our basic integral (1.5.1) when W', W and Φ are "unramified."

(2.1) Suppose π is an irreducible, admissible representation of G. We will say that π is *unramified* if it contains the unit representation of K; then the multiplicity of this representation is one. Unramified representations are parametrized by semi-simple conjugacy classes in $GL(r, \mathbf{C})$. In more detail if π is unramified, then, as is well known, π is the unique unramified component of an induced representation of the form

$$\mathrm{Ind}(G_r, B_r; \mu_1, \mu_2, \ldots, \mu_r)$$

where the μ_i's are unramified quasi-characters. The r-tuple $(\mu_1, \mu_2, \ldots, \mu_r)$ is uniquely determined up to permutation. The class A attached to π is by definition

$$A = \mathrm{diag}(\mu_1(\tilde{\omega}), \mu_2(\tilde{\omega}), \ldots, \mu_r(\tilde{\omega})).$$

We set

$$L(s, \pi) = \det(1 - q^{-s}A)^{-1}. \tag{1}$$

If π' is another unramified representation corresponding to A' we set

$$L(s, \pi \times \pi') = \det(1 - q^{-s}A \otimes A')^{-1}. \qquad (2)$$

(The definition is consistent with the one given in (1.6), as we will prove in a subsequent paper.)

It will be useful to remark that if π corresponds to A then the imaginary conjugate $\bar{\pi}$ corresponds to $\overline{A}$. Moreover if ω is the central character of π, then $\omega(\tilde{\omega}) = \det A$.

(2.2) Now suppose π is generic. Suppose also that the largest ideal of F on which ψ is trivial is $\mathfrak{R}$. Then the space $\mathcal{W}(\pi; \psi)$ contains a unique element W invariant under K. Moreover $W(e) \neq 0$ and we may normalize W by requiring that

$$W(e) = 1. \qquad (1)$$

This element will be called the *essential element* in $\mathcal{W}(\pi; \psi)$. There is also an explicit formula for W ([C-S], [T.S.]).

Indeed for each r-tuple J of integers set $\tilde{\omega}^J = \mathrm{diag}(\tilde{\omega}^{j_1}, \tilde{\omega}^{j_2}, \ldots, \tilde{\omega}^{j_r})$, if $J = (j_1, j_2, \ldots, j_r)$. Then every $g \in G_r$ can be written in the form $g = n\tilde{\omega}^J k$ where J is uniquely determined, $n \in N$ and $k \in K$. Thus W is completely determined by its values on the matrices $\tilde{\omega}^J$. Moreover if $T(r)$ is the set of r-tuples $J = (j_1, j_2, \ldots, j_r)$ with $j_1 \geq j_2 \geq \cdots \geq j_r$, then $W(\tilde{\omega}^J) = 0$ unless $J \in T(r)$. This being so, for $J \in T(r)$, let ρ_J be the rational representation of $GL(r, \mathbf{C})$ whose dominant weight is the character Λ_J defined by

$$\Lambda_J(a) = a_1^{j_1} a_2^{j_2} \cdots a_r^{j_r} \qquad \text{if } a = \mathrm{diag}(a_1, a_2, \ldots, a_r).$$

Let δ be the module of the group $A_r(F)N_r(F)$. The formula we have in mind is

$$W(\tilde{\omega}^J) = \delta^{1/2}(\tilde{\omega}^J)\mathrm{Tr}(\rho_J(A)) \qquad \text{if } J \in T(r). \qquad (2)$$

(2.3) Proposition. *Let π and π' be admissible irreducible representations of G_r. Suppose they are generic and unramified. Let W and W' be the essential elements of $\mathcal{W}(\pi; \psi)$ and $\mathcal{W}(\pi'; \psi)$ respectively.*

Suppose Φ *is the characteristic function of* $\mathfrak{R}$. *Then, as meromorphic functions in* $\mathrm{Re}(s) > 1$,

$$\Psi(s, W', W, \Phi) = L(s, \pi \times \bar{\pi}'),$$

the measures being normalized by the conditions $\mathrm{vol}(N \cap K) = 1$, $\mathrm{vol}(K) = 1$.

Proof. For the proof we may suppose $\mathrm{Re}(s)$ large. Because of the Iwasawa decomposition the integral, which converges for $\mathrm{Re}(s)$ large, can be written as an integral on $A_r/A_r \cap K$, or equivalently as a sum over all r-tuples $J = (j_1, j_2, \ldots, j_r)$:

$$\sum_J W'\overline{W}(\tilde{\omega}^J)\delta^{-1}(\tilde{\omega}^J)\Phi(0, 0, \ldots, \tilde{\omega}_r^{j_r})\,|\det \tilde{\omega}^J|^s. \tag{1}$$

As we have noted $W'(\tilde{\omega}^J) = 0$ unless $J \in T(r)$. Moreover $\Phi(0, 0, \ldots, \tilde{\omega}_r^{j_r}) = 1$ if $j_r \geq 0$ and zero otherwise. Call $T_+(r)$ the set of r-tuples $J \in T(r)$ whose last entry j_r is ≥ 0. Then (1) can also be written as

$$\sum_{J \in T_+(r)} W'\overline{W}(\tilde{\omega}^J)\delta^{-1}(\tilde{\omega}^J)\,|\det \tilde{\omega}^J|^s. \tag{2}$$

Set now $\mathrm{tr}(J) = j_1 + \cdots + j_r$ and $X = q^{-s}$. Then $|\det \tilde{\omega}^J|^s = X^{\mathrm{tr}(J)}$ and (2) is also

$$\sum_{J \in T_+(r)} \mathrm{Tr}\,\rho_J(A')\,\overline{\mathrm{Tr}\,\rho_J(A)}\,X^{\mathrm{Tr}(J)}.$$

After replacing A by $\overline{A}$ we can see that what we have to prove is the identity

$$\sum_{J \in T_+(r)} \mathrm{Tr}\,\rho_J(A')\,\mathrm{Tr}\,\rho_J(A)\,X^{\mathrm{Tr}(J)} = \det(1 - A \otimes A'X)^{-1}. \tag{3}$$

Let us denote by S^mA the m^{th} symmetric power of the matrix A. Then the right side is

$$\sum_{m \geq 0} \mathrm{Tr}\,S^m(A \otimes A')X^m.$$

Thus we are reduced to proving the identity

$$\sum_{J \in T_+(r), \mathrm{Tr}(J) = m} \mathrm{Tr}(\rho_J(A') \otimes \rho_J(A)) = \mathrm{Tr}\, S^m(A \otimes A'). \tag{4}$$

(2.4) We will prove a somewhat more general statement which will be used elsewhere to compute similar integrals—but now associated to a product $GL_p \times GL_r$ with r not necessarily equal to p. For that we introduce more notation.

Let T_+ denote the set of all infinite vectors

$$J = (j_1, j_2, \ldots, j_r, 0, 0, \ldots) \tag{4}$$

with only a finite number of non-zero entries and satisfying $j_1 \geq j_2 \geq \cdots \geq j_r \geq 0$. We will identify the element $(j_1, j_2, \ldots, j_r)$ of $T_+(r)$ with the corresponding element (4) of T_+. Then if $p \leq r$, as subsets of T_+, $T_+(p) \subset T_+(r)$. If $J \in T_+(r)$ we write $\rho(r, J)$ for the representation of $GL_r(\mathbf{C})$ with highest weight $\wedge_J$. Let ρ_r denote the standard representation of $GL_r(\mathbf{C})$.

PROPOSITION. *Suppose $p \leq r$. For each $m \geq 0$, the representation $S^m(\rho_p \otimes \rho_r)$ of the group $G_p(\mathbf{C}) \times G_r(\mathbf{C})$ is the direct sum with multiplicity one of the representations $\rho(p, J) \otimes \rho(r, J)$ where $J \in T_+(p)$ and $tr(J) = m$.*

When $p = r$, this assertion is equivalent to (2.3.4).

Proof. We first recall some classical results ([H.W.]).

Let $\mathfrak{S}_m$ denote the symmetric group on m letters. By the classical theory of "Young symmetrizers" *there is a one-one correspondence $J \mapsto \sigma_J$* between the *elements J of T_+ of trace m and the irreducible representations of $\mathfrak{S}_m$.*

Let V_r be the standard complex vector space of dimension r. $\mathfrak{S}_m$ operates in a natural way on $\otimes^m V_r$. We denote the corresponding representation by $\sigma_r{}^m$:

$$\sigma_r{}^m(\xi^{-1})(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\xi(1)} \otimes v_{\xi(2)} \otimes \cdots \otimes v_{\xi(m)}.$$

A given representation σ_J of $\mathfrak{S}_m$ occurs with positive multiplicity $\mu_{r,J}$ if and only if $J \in T_+(r)$. Let $V_r(m, J)$ be the corresponding isotypic component.

Then:

$$\otimes^m V_r = \bigoplus_J V_r(m, J),$$

the sum being over all $J \in T_+(r)$ such that $\mathrm{tr}(J) = m$. Next the representations $\rho_r^{\otimes m}$ of $GL(r, \mathbf{C})$ and σ_r^m of $\mathfrak{S}_m$ on $\otimes^m V_r$ commute. Moreover the operators $\sigma_r^m(\xi)$, $\xi \in \mathfrak{S}_m$, generate the commuting algebra of $\rho_r^{\otimes m}$. Thus $V_r(m, J)$ is also a multiple m_J of an irreducible representation of $GL(r, \mathbf{C})$. That representation is precisely the representation $\rho(r, J)$ of highest weight $\wedge_J$. Note that $m_J = \deg \sigma_J$ and therefore $\mu_{r,J} = \deg \rho(r, J)$.

We now proceed to the proof of Proposition (2.4). Let

$$W = \otimes^m(V_p \otimes V_r) \cong (\otimes^m V_p) \otimes (\otimes^m V_r). \tag{1}$$

Let ρ' be the representation $(\rho_p)^{\otimes m} \otimes (\rho_r)^{\otimes m}$ of $GL(p, \mathbf{C}) \times GL(r, \mathbf{C})$ on W. Its decomposition into isotypic components is:

$$W = \oplus\ V_p(m, J) \otimes V_r(m, K), \tag{2}$$

the sum extended over all $J \in T_+(p)$ of trace m and all $K \in T_+(r)$ of trace m.

Now $\mathfrak{S}_m$ acts on W via the representation σ_{p+r}^m. The space W_0 of invariants of this action is just $S^m(V_p \otimes V_r)$ and affords the representation $S^m(\rho_p \otimes \rho_r)$ of $GL_p(\mathbf{C}) \times GL_r(\mathbf{C})$. On the other hand, via (1), the representation of $\mathfrak{S}_m$ on $(\otimes^m V_p) \otimes (\otimes^m V_r)$ is $\sigma_p^m \otimes \sigma_r^m$. Since that representation leaves each term on the right side of (2) invariant, we have

$$W_0 = \oplus\ W_0 \cap V_p(m, J) \otimes V_r(m, K)$$

the sum over J, K as before. Now $W_0 \cap V_p(m, J) \otimes V_r(m, K) \neq 0$ implies that the representation $\sigma_J \otimes \sigma_K$ contains the unit representation of $\mathfrak{S}_m$. Since, as is well known, the irreducible representations of $\mathfrak{S}_m$ are all self-contragradient, this implies that $J = K$ (in T_+). Moreover $\sigma_J \otimes \sigma_J$ then contains the trivial representation of $\mathfrak{S}_m$ exactly once. Thus

$$\mathrm{Dim}(W_0 \cap V_p(m, J) \otimes V_r(m, J)) = \mu_{p,J}\mu_{r,J} = \deg \rho(p, J) \deg \rho(r, J).$$

Since the representation of $GL_p(\mathbf{C}) \times GL_r(\mathbf{C})$ is isotypic of type $\rho(p, J) \otimes \rho(r, J)$ it must be $\rho(p, J) \otimes \rho(r, J)$. Thus we have established that the representation of $GL_p(\mathbf{C}) \times GL_r(\mathbf{C})$ on W_0 is the direct sum, the sum being over all J satisfying $J \in T_+(p)$, $\mathrm{Tr}(J) = m$,

$$\bigoplus_{J \in T_+(p),\ \mathrm{Tr}(J) = m} \rho(p, J) \otimes \rho(r, J),$$

as required.

(2.5) We have the following corollary to Propositions (1.5) and (2.3).

Corollary. Let π be an admissible irreducible representation of G_r. Suppose π is unitary, generic and unramified. Let A be the class of π. Then the eigenvalues of A are in absolute value $< q^{1/2}$.

Proof. We apply Proposition (2.3) with $\pi = \pi'$, $W = W'$ and Φ the characteristic function of $\mathfrak{R}$. Then

$$\det(1 - q^{-s} A \otimes \overline{A}) \Psi(s, W, W, \Phi) = 1,$$

at first for $\mathrm{Re}(s) > 1$. But then, by Proposition (1.5), $\Psi(s, W, W, \Phi)$ has a continuous extension to the half-plane $\mathrm{Re}(s) \geq 1$. Thus $\det(1 - q^{-s} A \otimes \overline{A})$ cannot vanish in this closed half-plane. In particular, if λ is an eigenvalue of A, we must have

$$1 - q^{-\sigma} |\lambda|^2 \neq 0$$

for all $\sigma \geq 1$. Our assertion follows at once.

The proof should be compared with a similar argument of Rankin [R.R.].

(2.6) *Remarks.* (1) The proof of Proposition (3.14) below together with (3.15) can be used to show that the representations

$$\mathrm{Ind}(G_r, B_r; \alpha^s, \alpha^{-s}, 1, \ldots, 1)$$

of G_r are irreducible, unitary, and generic provided $0 < s < 1/2$. Thus the conclusion of Corollary (2.5) is the best possible.

(2) We note that, if π is the trivial representation of G_r, then the maximum eigenvalue of A is $q^{(n-1)/2}$.

(3) We remark finally that the corollary applies to local components of cusp forms.

3. Generic representations: archimedean case. In this section $F = \mathbf{R}$ or $\mathbf{C}$. We extend the results of Section 1 to include the archimedean case. After the analogue of (1.3) is established, this will be an easy matter. However, in order to establish this analogue we will need considerable preliminaries.

(3.1) Let G be a (real) Lie group. Denote by $\mathfrak{L}(G)$, $\mathfrak{U}(G)$, $\mathfrak{Z}(G)$, the Lie algebra of G, the enveloping algebra of $\mathfrak{L}(G) \otimes \mathbf{C}$, and the center of $\mathfrak{U}(G)$ respectively. On $\mathfrak{U}(G)$ we have a natural filtration. We let $\mathfrak{U}^n(G)$ be the n-th term of this filtration.

Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. We will denote by $\mathcal{H}_G{}^\infty$ or, when this does not create confusion, by $\mathcal{H}^\infty$ the space of smooth vectors in $\mathcal{H}$. Thus v is in $\mathcal{H}^\infty$ if and only if the map $g \mapsto \pi(g)v$ from G to $\mathcal{H}$ is C^∞. As is well known this is equivalent to saying that, for all $w \in \mathcal{H}$, the coefficients $(\pi(g)v, w)$ are smooth functions on G. Moreover $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

As is customary, if $X \in \mathfrak{L}(G)$, $v \in \mathcal{H}^\infty$, we set

$$\pi(X)v = (d/dt)\pi(\exp tX)v\,|_{t=0},$$

the limit taken in the topology of $\mathcal{H}$. Then $\pi(X)v$ is again in $\mathcal{H}^\infty$. The semi-norms

$$v \mapsto ||\pi(D)v||, \qquad D \in \mathfrak{U}(G),$$

$||\ ||$ being the given norm on $\mathcal{H}$, define on $\mathcal{H}^\infty$ the topology of a complete locally convex space. The space $\mathcal{H}^\infty$ is invariant under G and the representation π of G on $\mathcal{H}^\infty$ is smooth.

Let $\mathcal{H}'$ be the dual of $\mathcal{H}$ and π' the representation contragradient to π. We may define similarly the dual $\mathcal{H}_G{}^{-\infty}$ of $\mathcal{H}_G{}^\infty$ and the representation $\pi^{-\infty}$ contragradient to π^∞. Since $\mathcal{H}_G{}^\infty$ is dense in $\mathcal{H}$, we may and will regard $\mathcal{H}'$ as a subspace of $\mathcal{H}_G{}^{-\infty}$. Then for $v \in \mathcal{H}'$, we have $\pi'(g)v = \pi^{-\infty}(g)v$. Every vector v in $\mathcal{H}_G{}^{-\infty}$ is a finite sum of the form

$$v = \Sigma\pi^{-\infty}(X_i)v_i, \qquad X_i \in \mathfrak{U}(G), \qquad v_i \in \mathcal{H}'.$$

(3.2) $\mathcal{H} \mapsto \mathcal{H}^\infty$ is a functor: if π_1 and π_2 are unitary representations on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and A belongs to $Hom_G(\pi_1, \pi_2)$ then $A(\mathcal{H}_1{}^\infty) \subset \mathcal{H}_2{}^\infty$ and A defines a continuous operator from $\mathcal{H}_1{}^\infty$ to $\mathcal{H}_2{}^\infty$. In particular:

LEMMA. *Suppose that π is a unitary representation of G on $\mathcal{H}$. Suppose $\mathcal{H}$ is the direct sum of two orthogonal closed invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then $\mathcal{H}_i^\infty = \mathcal{H}^\infty \cap \mathcal{H}_i$ and $\mathcal{H}^\infty$ is the topological direct sum of $\mathcal{H}_1^\infty$ and $\mathcal{H}_2^\infty$.* □

(3.3) We will repeatedly use the following lemma ([P] Theorem (1.3)):

LEMMA. *Suppose $\mathcal{V}$ is a G-invariant subspace of $\mathcal{H}$ dense for the norm topology. Then if $\mathcal{V}$ is contained in $\mathcal{H}_G^\infty$ it is dense in $\mathcal{H}_G^\infty$.*

(3.4) Let R be a closed subgroup of G. Then we may consider the restriction of π to R and the corresponding subspace $\mathcal{H}_R^\infty$ of $\mathcal{H}$. Clearly $\mathcal{H}_G^\infty \subset \mathcal{H}_R^\infty$. Since $\mathcal{H}_G^\infty$ is dense in $\mathcal{H}$, it follows from Lemma (3.3) (applied to R) that $\mathcal{H}_G^\infty$ is dense in $\mathcal{H}_R^\infty$. Thus we may regard $\mathcal{H}_R^{-\infty}$ as a subspace of $\mathcal{H}_G^{-\infty}$.

Now suppose there is a subgroup V of G such that

$$\mathfrak{L}(G) = \mathfrak{L}(R) \oplus \mathfrak{L}(V).$$

Call $\mathcal{H}^{-n}$ the space of vectors $v \in \mathcal{H}_G^{-\infty}$ of the form

$$v = \Sigma \pi(D) w_D; \qquad D \in \mathfrak{U}^n(V), \qquad w_D \in \mathcal{H}_R^{-\infty}.$$

Then:

LEMMA. *With the above notations, each $\mathcal{H}^{-n}$ is an $\mathfrak{U}(R)$ submodule of $\mathcal{H}^{-\infty}$, $\mathcal{H}^{-n} \subset \mathcal{H}^{-(n+1)}$, and*

$$\bigcup_{n \geq 0} \mathcal{H}^{-n} = \mathcal{H}_G^{-\infty}.$$ □

(3.5) In the following paragraphs we will apply the above to the study of the representations of P_r which contain the unitary representation

$$\tau_r = \mathrm{Ind}(P_r, N_r, \theta_r) \tag{1}$$

induced by the character θ_r.

We recall first an elementary lemma from direct integral theory. Let G be a separable locally compact group and V a closed normal subgroup:

LEMMA. *Let π be a unitary representation of G. Suppose $\pi \mid V$ does not contain the trivial representation of V. Then π is a direct integral of irreducible representations*

$$\pi = \int_X \xi_x d\mu(x)$$

where, for almost all $x \in X$, $\xi_x \mid V$ is not the trivial representation.

(3.6) We describe what Mackey's semi-direct product theory tells us about the representations of P_r. Let η_r be the restriction of θ_r to U_r. Since U_r is normal in P_r, the group P_r operates on U_r as well as its dual group $\hat{U}_r$. There are two orbits for P_r acting on $\hat{U}_r$. One is the orbit of the trivial character and the other is the orbit of η_r. The fixer of η_r is the semi-direct product of U_r and the subgroup $P_{r-1} \subset G_{r-1}$ identified with a subgroup of P_r via the map

$$p \mapsto \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

Let ξ be an irreducible unitary representation of P_r. It now follows from Mackey's theory ([G.W.M.], Theorems 14.1 and 14.2) that, either ξ has a U_r fixed vector in which case ξ is really a representation of $P_r/U_r \simeq G_{r-1}$, or else ξ has the form

$$\xi = \operatorname{Ind}(P_r, P_{r-1}U_r; \sigma \otimes \eta_r)$$

where σ is an irreducible unitary representation of P_{r-1}. We may now apply Lemma (3.5) to $G = P_r$, $V = U_r$, noting that inducing commutes with direct integrals, to obtain the following lemma:

LEMMA. *Suppose π is a unitary representation of P_r on a Hilbert space $\mathcal{H}$. Suppose $\mathcal{H}$ does not contain a non-zero vector fixed by $\pi \mid U_r$. Then*

$$\pi \simeq \operatorname{Ind}(P_r, P_{r-1}U_r; \rho \otimes \eta_r)$$

for some unitary representation ρ of P_{r-1}. □

Note that Mackey's theory implies that τ_r is irreducible.

(3.7) *Proposition.* *Let π be a unitary representation of P_r on a Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

(i) *the representation π contains τ_r;*

(ii) *there is a $\lambda \neq 0$ in $\mathcal{H}_P^{-\infty}$ such that*

$$\lambda(\pi(n)v) = \theta(n)\lambda(v) \tag{1}$$

for all v in $\mathcal{H}_P^{\infty}$ and all $n \in N_r$.

Proof. We prove first that (ii) implies (i).

Suppose first that $\pi = \tau_r$. Then every $f \in \mathcal{H}_P^{\infty}$ is actually a C^{∞}-function on P transforming on the left like θ under N. Moreover there are $D_j \in \mathfrak{U}(P)$ such that for every $f \in \mathcal{H}_P^{\infty}$

$$|f(e)| \leq \sum_j ||\tau(D_j)f||.$$

Thus

$$\lambda(f) = f(e)$$

defines a continuous linear form on $\mathcal{H}_P^{\infty}$ which clearly satisfies (1). Of course $\lambda \neq 0$ and so (ii) is true.

More generally suppose π is a unitary representation of P_r on $\mathcal{H}$. Suppose $\mathcal{H}_\tau$ is an invariant subspace affording the representation τ_r. Let A: $\mathcal{H} \to \mathcal{H}_\tau$ be the orthogonal projection. Then by functoriality

$$A(\mathcal{H}^{\infty}) = \mathcal{H}_\tau^{\infty}.$$

Thus

$$\lambda(v) = A(v)(e), \qquad v \in \mathcal{H}^{\infty},$$

defines a non-zero linear form satisfying (1).

Suppose now π is given and $\lambda \neq 0$ belongs to $\mathcal{H}_P^{-\infty}$ and satisfies (1). Let $\mathcal{H}^U$ be the space of $U(= U_r)$ fixed vectors for π, and let $\mathcal{H}_U$ be the orthogonal complement of $\mathcal{H}^U$ in $\mathcal{H}$. Thus

$$\mathcal{H} = \mathcal{H}^U \oplus \mathcal{H}_U,$$

and by Lemma (3.2)

$$\mathcal{H}^\infty = (\mathcal{H}^U)^\infty \oplus (\mathcal{H}_U)^\infty.$$

Clearly λ vanishes on $(\mathcal{H}^U)^\infty$. Thus we may as well replace $\mathcal{H}$ by $\mathcal{H}_U$. By Lemma (3.6) this amounts to saying that π has the form

$$\pi = \operatorname{Ind}(P_r, P_{r-1}U_r; \sigma \otimes \eta_r) \tag{2}$$

for some unitary representation σ of P_{r-1}. If $r = 2$, then $P_{r-1} = \{1\}$, $U_r = N_r$, $\eta_r = \theta_r$ and π is actually a multiple of τ_r. Suppose then $r > 2$ and the theorem true for $r - 1$. Let $\mathcal{K}$ be the space of σ. The inclusion of G_{r-1} in G_r given by

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

induces an isomorphism $P_{r-1}\backslash G_{r-1} \simeq P_{r-1}U_r\backslash P_r$. It follows that $\pi | G_{r-1}$ is the representation

$$\operatorname{Ind}(G_{r-1}, P_{r-1}; \sigma).$$

Its space $\mathcal{H}$ is the space of all $\mathcal{K}$-valued functions F on G_{r-1} satisfying

$$F(pg) = \delta_{P_{r-1}}^{1/2}(p)\sigma(p)F(g)$$

and such that

$$\int_{P_{r-1}\backslash G_{r-1}} ||F(g)||^2 dg < \infty.$$

Moreover, for $u \in U_r$,

$$(\pi(u)F)(g) = \eta_r(gug^{-1})F(g).$$

Next for $f \in C_c^\infty(G_{r-1}; \mathcal{K}^\infty)$, $g \in G_{r-1}$, set

$$F_f(g) = \int_{P_{r-1}} \delta_{P_{r-1}}^{-1/2}(p)\sigma(p)^{-1}f(pg)d_r(p).$$

Then the space of functions F_f is a dense subspace of $\mathcal{H}^\infty$ (Lemma (3.3)). Moreover $f \mapsto F_f$ is continuous. Thus λ is completely determined by the $\mathcal{K}^\infty$-distribution T defined by

$$T(f) = \int_{G_{r-1}} f(g)dT(g) = \lambda(F_f).$$

Let us denote by χ_u the function on G_{r-1} defined by $\chi_u(g) = \eta_r(gug^{-1})$, $u \in U_r$. We have by (1)

$$\lambda(\chi_u F) = \eta_r(u)\lambda(F), \qquad F \in \mathcal{H}^\infty.$$

Moreover $F_{\chi_u f} = \chi_u F_f$. Hence $T(\chi_u f) = T(f), f \in C_c^\infty(G_{r-1}; \mathcal{K}^\infty)$. It follows that T has support in the fixer P_{r-1} of the character η_r.

A formal manipulation shows that

$$dT(pmn^{-1}) = \sigma(p^{-1})\delta_{P_{r-1}}^{1/2}(p)\theta(n)dT(p), \tag{3}$$

for $m \in G_{r-1}$, $p \in P_{r-1}$, $n \in N_{r-1}$. We now apply the results of (5.2.3) of [G.W.] to our present case. Following their notation we set $M = G_{r-1}$, $G = P_{r-1} \times N_{r-1}$ with G acting on M via

$$g \circ m = pmn^{-1}, \qquad g = (p, n).$$

We let Q be the closed orbit P_{r-1}. The isotropy group H at $1 \in Q$ is the diagonal in $N_{r-1} \times N_{r-1}$. In particular the module of H is 1. Let U be the *differentiable* representation of G on $\mathcal{K}^\infty$ defined by

$$U(p, n) = \delta_{P_{r-1}}^{-1/2}(p)\sigma(p)\bar{\theta}(n), \qquad p \in P_{r-1}, \qquad n \in N_{r-1}.$$

By (3), T is quasi-invariant relative to U. Thus (loc. cit.) we obtain a finite-dimensional vector space V, a continuous bilinear form $z \neq 0$ on $\mathcal{K}^\infty \times V$ and an *algebraic* representation μ of N_{r-1} on V such that

$$z(U(n, n)a, \mu(n)b) = z(a, b),$$

$$a \in \mathcal{K}^\infty, b \in V, n \in N_{r-1},$$

or explicitly

$$z(\sigma(n)a,\ \mu(n)b) = \theta(n)z(a,\ b).$$

There is a vector $b_0 \neq 0$ in V fixed by μ. At the cost of replacing V by a quotient we may assume that the linear form

$$a \mapsto z(a,\ b_0)$$

on $\mathcal{K}^\infty$ is non-zero. Clearly it satisfies (1). Thus, by the induction hypothesis, σ contains τ_{r-1}.

Since π has the form (2), we see that π contains

$$\mathrm{Ind}(P_r,\ P_{r-1}U_r;\ \tau_{r-1} \otimes \eta_r) \simeq \tau_r.$$

This concludes the proof of Proposition (3.7). □

(3.8) We now prove the analogue of Proposition (1.3) for archimedean fields.

Proposition. *Let π be a unitary representation of G_r on a Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

(i) *the restriction of π to P_r contains τ_r;*

(ii) *there is a linear form $\lambda \neq 0$ on $\mathcal{H}_G{}^\infty$ such that*

$$\lambda(\pi(n)v) = \theta(n)v$$

for all $v \in \mathcal{H}_G{}^\infty$ and $n \in N$.

Proof. Suppose (i) is satisfied. By Proposition (3.7) there is a linear form $\lambda \neq 0$ on $\mathcal{H}_P{}^\infty$ satisfying (3.7.1). Since $\mathcal{H}_G{}^\infty$ is dense in $\mathcal{H}_P{}^\infty$ with continuous injection, the restriction of λ to $\mathcal{H}_G{}^\infty$ has the required properties.

Now suppose that (ii) is satisfied. Then λ is an element of $\mathcal{H}_G{}^{-\infty}$ and for any $X \in \mathfrak{U}(N)$

$$\pi^\infty(X)\lambda = \dot{\theta}(X)\lambda \tag{1}$$

where $\dot{\theta}$ denotes the homomorphism $\mathfrak{U}(N) \to \mathbf{C}$ induced by the character θ.

If we could show that this implies that λ is actually in $\mathcal{H}_P{}^{-\infty}$ then (i) would follow from Proposition (3.7). To see what this entails let us use the notation of (3.4) with $G = G_r$, $R = P_r Z_r$, $V = \overline{U}_r = {}'U_r$. Then since Z operates by scalars on $\mathcal{H}_G{}^\infty$ the operators $\pi^\infty(X)$ for $X \in \mathfrak{U}(Z)$ are also scalar. Thus

$$\mathcal{H}_R{}^{-\infty} = \mathcal{H}_{P_r Z_r}{}^{-\infty} = \mathcal{H}_{P_r}{}^{-\infty}$$

as (topological) vector spaces.

Now $\lambda \in \mathcal{H}^{-n}$ for some n by Lemma (3.4). Thus it would suffice to show that λ is in $\mathcal{H}^{-n}$ implies λ is in $\mathcal{H}^{-n+1}$. Call $\mathfrak{J}$ the subalgebra of $\mathfrak{U}(G_r)$ generated by $\mathfrak{U}(P_r)$ and $\mathfrak{Z}(G_r)$. Then each $\mathcal{H}^{-n}$ is an $\mathfrak{J}$-module and our assertion would follow from the following statement:

(2) *Let $\mathcal{V}$ be a $\mathfrak{U}(G)$-module and λ an element of $\mathcal{V}$ such that*

$$X \cdot \lambda = \dot{\theta}(X)\lambda \qquad \text{for } X \in \mathfrak{L}(N).$$

Suppose there is a subspace $\mathcal{W}$ of $\mathcal{V}$ stable under $\mathfrak{J}$ such that λ is in $\mathfrak{L}(\overline{U})\mathcal{W} + \mathcal{W}$. Then λ is in $\mathcal{W}$.

It is easy enough to establish this assertion for $r = 2$. Indeed suppose $F = \mathbf{R}$ and let E^+ and E^- be the following elements of $\mathfrak{L}(G)$:

$$E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By considering the Casimir element of $\mathfrak{Z}(G)$ we see that

$$E^+E^- \equiv E^-E^+ \equiv 0 \pmod{\mathfrak{J}}.$$

Suppose

$$\lambda \equiv E^-\mu \pmod{\mathcal{W}}$$

where μ is in $\mathcal{W}$. Then, applying E^+ to both sides of this congruence, we get

$$\dot{\theta}(E^+)\lambda \equiv E^+E^-\mu \equiv 0 \pmod{\mathcal{W}}.$$

Since $\dot{\theta}(E^+) \neq 0$ we indeed have $\lambda \in \mathcal{W}$. The proof for $r = 2$, $F = \mathbf{C}$ is similar.

Unfortunately we have not been able to prove (2) for all r. What we will do instead is prove (2) for $r = 4$, $F = \mathbf{R}$. At this point we will have proved that (ii) implies (i) in the following cases:

$$r = 2,\ F = \mathbf{R} \text{ or } \mathbf{C} \text{ and } r = 4,\ F = \mathbf{R}.$$

We will then prove that (ii) implies (i) in full generality by using a certain reduction technique (c.f. (3.12)).

(3.9) *Proof of Proposition* (3.8) *when* $r = 4$, $F = \mathbf{R}$.

Let E_{ij} denote the 4×4 matrix with one in the ith row and jth column and zero elsewhere. Let also

$$H_1 = \mathrm{diag}(1, 0, 0, 0), \qquad H_2 = \mathrm{diag}(0, 1, 0, 0),$$

$$H_3 = \mathrm{diag}(0, 0, 1, 0), \qquad H_4 = \mathrm{diag}(0, 0, 0, 1).$$

The E_{ij}, $1 \leq i, j \leq 4$, H_i, $1 \leq i \leq 4$, form a basis for $\mathfrak{L}(G)$. In what follows we will normalize $\dot{\theta}$ by requiring $\dot{\theta}(E_{i,i+1}) = 1$ for $1 \leq i \leq 3$.

Our assumption is that

$$\lambda \equiv E_{41}\mu_1 + E_{42}\mu_2 + E_{43}\mu_3 \pmod{\mathfrak{W}} \tag{1}$$

with $\mu_i \in \mathfrak{W}$. Apply $E_{34} \in \mathfrak{U}(P)$ to this congruence to get

$$E_{34}\lambda = \lambda \equiv E_{34}E_{41}\mu_1 + E_{34}E_{42}\mu_2 + E_{34}E_{43}\mu_3.$$

But mod $\mathfrak{J}$ we have

$$[E_{34}, E_{41}] = E_{31} \equiv 0, \qquad [E_{34}, E_{42}] = E_{32} \equiv 0,$$

$$[E_{24}, E_{42}] \equiv 0, \qquad [E_{14}, E_{41}] \equiv 0,$$

and

$$E_{34}E_{43} + E_{24}E_{42} + E_{14}E_{41} \equiv 0, \tag{2}$$

the last relation from consideration of the Casimir operator. Thus

$$\lambda \equiv E_{41}\nu_1 + E_{42}\nu_2$$

where

$$\nu_1 = E_{34}\mu_1 - E_{14}\mu_3, \qquad \nu_2 = E_{34}\mu_2 - E_{24}\mu_3.$$

Thus changing notations, we may assume, instead of (1),

$$\lambda \equiv E_{41}\mu_1 + E_{42}\mu_2 \pmod{\mathcal{W}} \tag{3}$$

with μ_1, μ_2 in $\mathcal{W}$.

To proceed we need to consider elements of $\mathfrak{Z}(G)$ other than the Casimir operator. Now $\mathfrak{Z}(G)$ is the image under the symmetrization map of the invariant elements of the symmetric algebra $S(\mathfrak{L}(G))$ of $\mathfrak{L}(G)$ in $\mathfrak{U}(G)$. We may use the invariant bilinear form

$$(X, Y) \mapsto \operatorname{tr}(XY)$$

on $\mathfrak{L}(G)$ to identify $S(\mathfrak{L}(G))$ with the polynomial algebra on $\mathfrak{L}(G)$, the coordinate function g_{ij} corresponding to the monomial E_{ji}. Let ω_r denote the *invariant* polynomial function on $\mathfrak{L}(G)$ defined by

$$\omega_r(g) = \operatorname{tr}(\Lambda^r(g)).$$

The image in $S(\mathfrak{L}(G))$ is obtained by formally replacing g by the matrix of E_{ji}'s. We denote by Ω_r the corresponding element of $\mathfrak{Z}(G)$. Thus for example if $r = 3$ we have to compute the sum of the four principal 3×3 minors of

$$\begin{bmatrix} H_1 & E_{21} & E_{31} & E_{41} \\ E_{12} & H_2 & E_{32} & E_{42} \\ E_{13} & E_{23} & H_3 & E_{43} \\ E_{14} & E_{24} & E_{34} & H_4 \end{bmatrix}$$

and then symmetrize. We then find the following expression for Ω_3 mod $\mathfrak{U}(P)$:

$$\Omega_3 \equiv A_1{}'E_{41} + A_2{}'E_{42} + A_3{}'E_{43} \pmod{\mathfrak{U}(P)}$$

where $A_3' = E_{24}E_{32} + 2E_{34} + E_{14}E_{31} - (H_1 + H_2)E_{34}$ and A_1', A_2' are obtained from A_3' by obvious symmetries. Using (2) again, we arrive at the relation

$$A_1E_{41} + A_2E_{42} + A_3E_{43} \equiv 0\,(\mathfrak{J}), \tag{4}$$

where

$$\begin{aligned} A_1 &= E_{34}E_{13} + 2E_{14} + E_{24}E_{12} + H_1E_{14} - H_3E_{14} \\ A_2 &= E_{34}E_{23} + 2E_{24} + E_{14}E_{21} + H_2E_{24} - H_3E_{24} \\ A_3 &= E_{24}E_{32} + 2E_{34} + E_{14}E_{31}. \end{aligned} \tag{5}$$

Using (2), we have the following commutation relations mod $\mathfrak{J}$:

$$\begin{aligned} [A_1, E_{41}] &\equiv 0\,(\mathrm{mod}\,\mathfrak{J}) \\ [A_2, E_{41}] &\equiv 0 \\ [A_2, E_{42}] &\equiv 0 \\ [A_3, E_{43}] &\equiv E_{43}E_{34} \\ [A_3, E_{42}] &\equiv 0 \\ [A_3, E_{41}] &\equiv 0. \end{aligned} \tag{6}$$

From (5), we get immediately that $A_i \in \mathfrak{U}(PZ)$ and in addition, noting that $A_3 = E_{32}E_{24} + E_{31}E_{14}, A_2 = E_{23}E_{34} + E_{21}E_{14} + (H_2 - H_3)E_{24}$,

$$A_2\lambda = \lambda, \qquad A_1\lambda = A_3\lambda = 0. \tag{7}$$

Next multiply (4) by E_{34} on the right to get

$$A_1E_{41}E_{34} + A_2E_{42}E_{34} + A_3E_{43}E_{34} \equiv 0\,(\mathfrak{J}). \tag{8}$$

Using (2) and the fact that $A_i \in \mathfrak{U}(PZ)$, this can be written as

$$(A_1E_{34} - A_3E_{14})E_{41} + (A_2E_{34} - A_3E_{24})E_{42} \equiv 0\,(\mathfrak{J}); \tag{9}$$

or, using (6) for A_1 and A_3, as

$$E_{41}U + VE_{42} \equiv 0\,(\mathrm{mod}\,\mathfrak{J}), \tag{10}$$

where

$$U = A_1 E_{34} - A_3 E_{14}, \qquad V = A_2 E_{34} - A_3 E_{24}. \tag{11}$$

From (7) we get

$$V\lambda = \lambda. \tag{12}$$

Note that U and V belong to $\mathfrak{U}(PZ)$. Thus (3) now implies

$$\lambda \equiv VE_{41}\mu_1 + VE_{42}\mu_3 \pmod{\mathcal{W}},$$

or from (10)

$$\lambda \equiv VE_{41}\mu_1 - E_{41}U\mu_2.$$

Since by (6) V commutes with E_{41} mod $\mathfrak{J}$, this may be written as

$$\lambda \equiv E_{41}(V\mu_1 - U\mu_2).$$

But $V\mu_1 - U\mu_2 \in \mathcal{W}$ and thus we may replace (3) by

$$\lambda \equiv E_{41}\mu \pmod{\mathcal{W}}, \text{ with } \mu \in \mathcal{W}. \tag{13}$$

Before introducing one more element of $\mathfrak{Z}(G)$, using (6), we rewrite (9) in the form

$$(A_1E_{34} - A_3E_{14})E_{41} + E_{42}(A_2E_{34} - A_3E_{24}) \equiv 0 \pmod{\mathfrak{J}},$$

or, as

$$U'E_{41} + E_{42}V' \equiv 0 \pmod{\mathfrak{J}}, \tag{14}$$

where

$$U' = A_1E_{34} - A_3E_{14}, \qquad V' = A_2E_{34} - A_3E_{24}.$$

Once more, by (7),

$$U'\lambda = 0, \qquad V'\lambda = \lambda. \tag{15}$$

Finally we make use of Ω_4. Expanding the determinant and symmetrizing, we get

$$\Omega_4 \equiv B_1E_{41} + B_2E_{42} + B_3E_{43} \,(\mathrm{mod}\ \mathfrak{U}(PZ)) \tag{16}$$

with $B_i \in \mathfrak{U}(PZ)$. Moreover

$$B_1\lambda = c\lambda \text{ with } c \neq 0. \tag{17}$$

It will not be necessary to know the explicit form of the B_i.

Next multiply (16) by E_{34} on the right to get

$$B_1E_{41}E_{34} + B_2E_{42}E_{34} + B_3E_{43}E_{34} \equiv 0\,(\mathfrak{J}), \tag{18}$$

or, using (2) once more,

$$(B_1E_{34} - B_3E_{14})E_{41} + (B_2E_{34} - B_3E_{24})E_{42} \equiv 0\,(\mathfrak{J}).$$

Multiplying on the right by V' (which is in $\mathfrak{J}$) and using (14) we get

$$(B_1E_{34} - B_3E_{14})E_{41}V' - (B_2E_{34} - B_3E_{24})U'E_{41} \equiv 0\,(\mathfrak{J}).$$

It follows from (6) that V' commutes, mod $\mathfrak{J}$, with E_{41}. Thus we get at last

$$AE_{41} \equiv 0\,(\mathrm{mod}\ \mathfrak{J})$$

where A is the following element of $\mathfrak{J}$:

$$A = (B_1E_{34} - B_3E_{14})V' - (B_2E_{34} - B_3E_{24})U'.$$

From (15) and (17) we now have

$$A\lambda = (B_1E_{34} - B_3E_{14})\lambda = B_1\lambda = c\lambda.$$

But then applying A to both sides of (13) we get

$$c\lambda = A\lambda \equiv AE_{41}\mu \equiv 0\,(\mathrm{mod}\ \mathcal{W}).$$

So λ is in $\mathcal{W}$. This concludes the proof of the statement that (ii) implies (i) in the case $r = 4$, $F = \mathbf{R}$.

(3.10) To complete the proof of Proposition (3.8) in general we will first show that every unitary representation of G_r satisfying (ii) is induced, in the unitary sense, by similar representations of G_4 and G_2. We will then show that (i) is inductive. The conjunction of these two results will show that (ii) implies (i) for all r and F.

We first prove the following proposition:

PROPOSITION. *Suppose R is the standard parabolic of type $(n_1, n_2, \ldots, n_u)$ so that $R/U_R \simeq \Pi_i G_{n_i}$. For each i, let σ_i be a unitary representation of G_{n_i}. Let π be the representation induced—in the unitary sense—by the σ_i:*

$$\pi = \operatorname{Ind}(G, R; \sigma_1, \sigma_2, \ldots, \sigma_u).$$

Then if π satisfies condition (ii) *of* (3.8) *then so does each σ_i.*

Proof. The proof of the corresponding assertion for the p-adic case is due to Rodier ([F.R]). The proof in the archimedean case is essentially the same, except for technical difficulties due to the existence of transversal derivatives.

Let $\mathcal{K}_i$ be the Hilbert space on which σ_i operates and $\mathcal{K} = \hat{\otimes}\, \mathcal{K}_i$ the space of the representation $\sigma = \otimes\, \sigma_i$ of $M \simeq R/U_R \simeq \Pi G_{n_i}$. As in (3.7), given $f \in C_c^\infty(G_r; \mathcal{K}^\infty)$, set for g in G_r,

$$F_f(g) = \int_R \delta_R^{-1/2}(r)\sigma(r^{-1})f(rg)dr$$

where dr is a right-invariant measure on R. Given $\lambda \neq 0$ satisfying (3.8)(ii), define a $\mathcal{K}^\infty$-distribution T on G_r by setting

$$T(f) = \lambda(F_f).$$

Then

$$dT(rgn^{-1}) = \sigma(r^{-1})\delta_R^{1/2}(r)\theta(n)dT(g).$$

We let the group $R \times N$ act on the manifold G by

$$(r, n) \circ g = rgn^{-1}.$$

Then G is a finite union of orbits. For any orbit

$$Q = RgN,$$

let $\Omega(Q)$ be the union of Q with the orbits of strictly larger dimension. Then Q is a closed submanifold of the open set $\Omega(Q)$. Starting with (open) orbits of maximal dimension and proceeding inductively, we find a Q such that $S = T \mid \Omega(Q)$ is non-zero and has support in Q.

We now proceed exactly as in (3.7). We write $Q = RwN$ for some permutation matrix w. The isotropy group H at w is the subgroup of $R \times N$ of elements (r, n) satisfying $rwn^{-1} = w$. This group is isomorphic to $N \cap w^{-1}Rw$ via the map $n \mapsto (wnw^{-1}, n)$. In particular it is unimodular. We now apply (5.2.3) of [G.W.], this time with $M = \Omega(Q)$ and the *differentiable* representation U of $R \times N$ on $\mathcal{K}^\infty$ being given by

$$U(r, n) = \delta_R{}^{-1/2}(r)\sigma(r)\bar{\theta}(n), \qquad r \in R, n \in N.$$

As before we get a finite-dimensional vector space V, a continuous bilinear form $z \neq 0$ on $\mathcal{K}^\infty \times V$ and an algebraic representation μ of H on V such that

$$z(U(h)a, \mu(h)b) = z(a, b),$$

for $a \in \mathcal{K}^\infty$, $b \in V$, $h \in H$, or explicitly

$$z(\sigma(wnw^{-1})a, \mu(n)b) = \theta(n)z(a, b), \tag{1}$$

for all $n \in N \cap w^{-1}Rw$. Suppose α is a simple root and let N_α be the corresponding subgroup of N. If now $wN_\alpha w^{-1}$ is contained in U_R, then since σ is trivial on U_R, we get

$$z(a, \mu(n)b) = \theta(n)z(a, b), \; n \in N_\alpha.$$

But $\mu(n)$ is unipotent. Thus $\theta(n) = 1$ for all $n \in N_\alpha$, a contradiction. Thus there are no simple roots α for which $wN_\alpha w^{-1} \subset U_R$ and this implies that Q is open. In fact $Q = Rw_0N$ where w_0 is the order reversing permutation matrix. Now write $R = MU_R$. Then $w_0Nw_0{}^{-1} \cap M = N'$ is a maximal unipotent subgroup of M. Exactly as before we may, at the cost of replacing V by a quotient, assume there is b_0 in V fixed by $N \cap w^{-1}Rw$ and such that the linear form

$$\lambda'(a) = z(a, b_0), \qquad a \in \mathcal{K}^\infty,$$

is non-zero. From (1), we get

$$\lambda'(\sigma(n')a) = \theta(w_0{}^{-1}n'w_0)\lambda'(a) \tag{2}$$

for $n' \in N'$ and $a \in \mathcal{K}^\infty$.

We may write $M_R = \Pi_i M_i$ with $M_i \simeq G_{n_i}$, $N = \Pi \overline{N_i}$ where $\overline{N_i}$ is the group of lower unipotent matrices in G_{n_i}. By Lemma (3.3) the space spanned by the vectors of the form $\otimes\, v_i$ with $v_i \in \mathcal{K}_i{}^\infty$ is dense in $\mathcal{K}^\infty$. Thus given an index j there are vectors $v_i \in \mathcal{K}_i{}^\infty$, for $i \neq j$, such that the linear form

$$\lambda_j'(v) = \lambda'(v \otimes \otimes_{i\neq j} v_i)$$

on $\mathcal{K}_j{}^\infty$ is non-zero. It clearly satisfies

$$\lambda_j'(\sigma_j(n_j)v) = \theta(w_0{}^{-1}n_jw_0)\lambda_j(v), \qquad n_j \in N_j.$$

Finally let w_j be the order-reversing permutation matrix in G_{n_j}. Set, for $v \in \mathcal{K}_j{}^\infty$,

$$\lambda_j(v) = \lambda_j'(\sigma_j(w_j)v).$$

Then the (non-zero) linear form λ_j on $\mathcal{K}_j{}^\infty$ clearly satisfies condition (ii) of (3.8) for the representation σ_i of G_{n_i}. □

(3.11) We will need an elementary result on induced representations. If $(\xi, \mathcal{V})$ is an admissible irreducible representation of the pair $(\mathfrak{U}(G), K)$ we will often allow ourselves to speak of ξ as a representation of G.

We say that an admissible representation $(\xi, \mathcal{V})$ is *pre-unitary* if it is the underlying representation of some unitary representation π of G on a Hilbert space $\mathcal{H}$. Since any admissible $(\mathfrak{U}(G), K)$ module has finite length, it is easy to see that $(\pi, \mathcal{H})$ is unique (up to unitary equivalence).

Now let R be a parabolic subgroup of G and η an admissible representation of $R/U_R = M$. Then one can define the representation induced by η

$$\xi = \mathrm{Ind}(G, R; \eta) \tag{1}$$

even though η is not a representation of the group M. If η underlies the unitary representation σ of M, then ξ underlies the unitary representation π of G induced by σ. Conversely:

LEMMA. *Notations being as above, suppose $\bar{\eta} \simeq \tilde{\eta}$. Suppose that ξ is irreducible and underlies the unitary representation π. Then η is the underlying admissible representation of a unitary representation σ of M. Moreover π is equivalent to the unitary representation of G induced by σ.*

Proof. Denote by $(\cdot\ ,\ \cdot)$ a non-zero invariant Hermitian form on the space $\mathcal{H}_0$ of η. The assumption on η is precisely that there is such a form. A (K-finite) vector in the space of ξ may be regarded as a K-finite function on K with values in $\mathcal{H}_0$. Thus the formula

$$(f_1, f_2) = \int_K (f_1(k), f_2(k))dk$$

defines a non-zero Hermitian form on the space for ξ. It is invariant and therefore must be proportional to a positive form. Changing the original product on $\mathcal{H}_0$, we may assume it is positive. Thus we see that

$$\int_{M \cap K \backslash K} (f(k), f(k))dk \geq 0 \tag{1}$$

for all (right) K-finite functions on K with values in $\mathcal{H}_0$ satisfying

$$f(k_0 k) = \sigma(k_0) f(k)$$

for all $k_0 \in M \cap K$. In (1) we are free to replace f by $f\phi$, where ϕ is any right K-finite scalar function on $K \cap M \backslash K$. Such functions are precisely of the form

$$\phi(k) = \int_{M \cap K} \psi(mk)dm \tag{2}$$

where ψ is right K-finite. If ϕ is a continuous function on $M \cap K \backslash K$, it may also be expressed in the form (2) for some continuous function ψ on K. By Stone-Weierstrass there is a sequence ψ_n of right K-finite functions on K converging uniformly to ψ. If ϕ_n is the corresponding sequence of functions then clearly ϕ_n tends to ϕ uniformly.

Suppose now α is any positive continuous function on $K \cap M\backslash K$. Choose ϕ_n as above converging uniformly to $\alpha^{1/2}$. Replacing f by $f\phi_n$ in (1) and taking limits we get

$$\int_{M \cap K \backslash K} \alpha(k)(f(k), f(k))dk \geq 0.$$

This implies $(f(e), f(e)) \geq 0$. Since $f(e)$ can be any vector in $\mathcal{H}_0$, we see that $(\cdot\ ,\ \cdot)$ is positive (semi-definite) on $\mathcal{H}_0$. Since η is irreducible it is actually positive definite. Thus η is the underlying admissible representation of a unitary representation σ of M and therefore π is infinitesimally equivalent to the unitarily induced representation

$$\mathrm{Ind}(G, R; \sigma).$$

Since π is unitary, it is actually equivalent in the unitary sense to this induced representation. $\square$

(3.12) To continue we recall Langlands' classification of representations (c.f. [N.W.] for example). Let again R be a standard parabolic subgroup of type $(n_1, n_2, \ldots, n_u)$. As usual set $R/U_R = \Pi M_i$, $M_i \simeq G_{n_i}$. For each i let π_i be a (unitary) *tempered* representation and s_i some real number. Suppose $s_1 > s_2 > \cdots > s_u$. Then the induced representation

$$\xi = \mathrm{Ind}(G, R; \pi_1 \otimes \alpha^{s_1}, \ldots, \pi_u \otimes \alpha^{s_u}) \tag{1}$$

contains a unique maximal invariant subspace $\mathcal{V}$. The representation η of G on $\mathcal{H}/\mathcal{V}$ is denoted by

$$\eta = J(G, R; \pi_1 \otimes \alpha^{s_1}, \ldots, \pi_u \otimes \alpha^{s_u}). \tag{2}$$

Every irreducible representation π of G is equivalent to a representation of the form (2) where R, the class of the π_i, and the s_i are uniquely determined. We shall call ξ the *induced representation associated to* η.

We also need to recall certain results from [B.K.] and [D.V.], specialized to $G = GL_n$. We refer to [D.V.], Section 6, for the concept of a "large" representation. As above we shall refer to a representation of R/U_R of the form

$$(\pi_1 \otimes \alpha^{s_1}) \otimes \cdots \otimes (\pi_u \otimes \alpha^{s_u})$$

where the π_i are tempered and the s_i arbitrary real numbers as *quasi-tempered*. Then we have the following theorem which is a simple restatement of Theorem 6.2 of [D.V.]:

THEOREM A. *Suppose η is an irreducible admissible representation of $G = GL_r(F)$, $F = \mathbf{R}$ or $\mathbf{C}$. Then η is large if and only if η has the form*

$$\eta = \mathrm{Ind}(G, R; \sigma)$$

where R is a parabolic in G and σ is a quasi-tempered representation of R/U_R.

We shall also need the following Theorem—a special case of Theorem D of [B.K.]:

THEOREM B. *Suppose $(\eta, \mathcal{V})$ is an irreducible admissible representation of $G = GL_r(F)$. Suppose there is a non-zero linear form λ on $\mathcal{V}$ such that*

$$\lambda(\eta(X)v) = \dot{\theta}(X)\lambda(v)$$

for all $X \in \mathfrak{L}(N)$. Then η is large.

Combining the two theorems in a trivial way we obtain the proposition we need.

PROPOSITION. *Let $(\eta, \mathcal{V})$ be an irreducible admissible representation of $GL_r(F)$. Suppose there is a non-zero linear form λ on $\mathcal{V}$ such that*

$$\lambda(\eta(X)v) = \dot{\theta}(X)\lambda(v)$$

for all $X \in \mathfrak{L}(N)$. Then the induced representation ξ associated to η is irreducible (and thus equivalent to η).

Proof. By the results of Kostant and Vogan we have

$$\eta = \mathrm{Ind}(G, R; \sigma)$$

where R is a parabolic in G and σ a quasi-tempered representation of R/U_R. Say

$$\sigma = (\pi_1 \otimes \alpha^{s_1}) \otimes \cdots \otimes (\pi_u \otimes \alpha^{s_u}).$$

Let R' be a parabolic obtained from R by permuting the blocks. Then $R'/U_{R'} \simeq R/U_R$. Let σ' be the representation of R' obtained by transport of structure. Then the representation

$$\eta' = \mathrm{Ind}(G, R'; \sigma')$$

has the same character as η. Since η is irreducible it must be equivalent to η'. Thus we may assume from the outset that $s_1 \geq s_2 \geq \cdots \geq s_u$.

Suppose R is of type $(n_1, n_2, \ldots, n_u)$. Suppose say $t_1 = s_1 = \cdots = s_p > s_{p+1}$. Let $m_1 = n_1 + n_2 + \cdots + n_p$ and R_1 be the parabolic of type $(n_1, n_2, \ldots, n_p)$ in G_{m_1}. Let σ_1 be the tempered representation of G_{m_1} defined by

$$\sigma_1 = \mathrm{Ind}(G_{m_1}, R_1; \pi_1, \pi_2, \ldots, \pi_p).$$

Repeating this procedure for each string of equal exponents s_i we obtain a new partition $(m_1, \ldots, m_\ell)$ of r, for each j, $1 \leq j \leq \ell$, an exponent t_j, and a tempered representation σ_j of GL_{m_j}. By construction $t_1 > t_2 > \cdots > t_\ell$. Let Q be the (standard) parabolic in G_r associated to this partition. Then, by the lemma on inducing in stages, we have

$$\eta = \mathrm{Ind}(G, Q; \sigma_1 \otimes \alpha^{t_1}, \ldots, \sigma_\ell \otimes \alpha^{t_\ell}),$$

and our proof is complete. □

(3.13) Combining Lemma (3.11) and Proposition (3.12) we will obtain the following result:

Lemma. *Suppose π is an irreducible unitary representation of G_r satisfying condition* (ii) *of Proposition* (3.8). *Then there is a partition $(n_1, n_2, \ldots, n_u)$ of r with $n_i = 1, 2$ or 4 if $F = \mathbf{R}$, $n_i = 1, 2$ if $F = \mathbf{C}$ and for each i an irreducible unitary representation π_i of G_{n_i} so that*

$$\pi = \mathrm{Ind}(G, R; \pi_1, \pi_2, \ldots, \pi_u),$$

where R is the parabolic subgroup of type $(n_1, n_2, \ldots, n_u)$.

Proof. Let π be an irreducible unitary representation of G. By the results stated in (3.12) we can choose R, π_i and the s_i so that π is (infinitesimally equivalent) to the representation η of (3.12.2). Clearly

$$\bar{\eta} = J(G, R; \bar{\pi}_1 \otimes \alpha^{s_1}, \bar{\pi}_2 \otimes \alpha^{s_2}, \ldots, \bar{\pi}_u \otimes \alpha^{s_u}).$$

If $(n_1, n_2, \ldots, n_u)$ is the type of R, call Q the parabolic of type $(n_u, \ldots, n_2, n_1)$. Then

$$\tilde{\eta} = J(G, Q; \overline{\pi}_u \otimes \alpha^{-s_u}, \ldots, \overline{\pi}_2 \otimes \alpha^{-s_2}, \overline{\pi}_1 \otimes \alpha^{-s_1}).$$

Since π is unitary we must have $\overline{\eta} \simeq \tilde{\eta}$. This can happen only if $R = Q$ and

$$(\pi_1 \otimes \alpha^{s_1}, \pi_2 \otimes \alpha^{s_2}, \ldots, \pi_u \otimes \alpha^{s_u})$$
$$= (\pi_u \otimes \alpha^{-s_u}, \ldots, \pi_2 \otimes \alpha^{-s_2}, \pi_1 \otimes \alpha^{-s_1}).$$

Assume furthermore that π satisfies (3.8)(ii). Then it satisfies the conditions of Proposition (3.12) and so π is actually (infinitesimally) equivalent to the representation

$$\xi = \mathrm{Ind}(G, R; \pi_1 \otimes \alpha^{s_1}, \pi_2 \otimes \alpha^{s_2}, \ldots, \pi_u \otimes \alpha^{s_u}).$$

In particular ξ is irreducible.

Let $m = [u/2]$ and, for $1 \leq i \leq m$, let R_i be the parabolic subgroup of G_{2n_i} of type (n_i, n_i). Set also

$$\eta_i = \mathrm{Ind}(G_{2n_i}, R_i; \pi_i \otimes \alpha^{s_i}, \pi_i \otimes \alpha^{-s_i}).$$

If now u is even, $u = 2m$ and ξ has the same character as the induced representation

$$\xi_1 = \mathrm{Ind}(G, S; \eta_1, \eta_2, \ldots, \eta_m),$$

where S is the parabolic subgroup of type $(2n_1, 2n_2, \ldots, 2n_m)$. Since ξ is irreducible it is equivalent to ξ_1 and each η_i must be irreducible also. But

$$\overline{\eta}_i = \mathrm{Ind}(G_{n_i}, R_i; \overline{\pi}_i \otimes \alpha^{s_i}, \overline{\pi}_i \otimes \alpha^{-s_i})$$
$$\tilde{\eta}_i = \mathrm{Ind}(G_{n_i}, R_i; \overline{\pi}_i \otimes \alpha^{-s_i}, \overline{\pi}_i \otimes \alpha^{-s_i}).$$

Thus $\overline{\eta}_i$ and $\tilde{\eta}_i$ have the same character. Since they are irreducible $\overline{\eta}_i = \tilde{\eta}_i$. We can now appeal to Lemma (3.11): each η_i is the underlying admissible representation of an irreducible unitary representation η_i' and π is equivalent to the unitary representation

$$\mathrm{Ind}(G, S; \eta_1', \eta_2', \ldots, \eta_m').$$

If u is odd, $u = 2m + 1$ and we obtain a similar conclusion. Each η_i, $1 \leq i \leq m$, (resp. π_{m+1}) underlies an irreducible unitary representation η_i' (resp. π_{m+1}') and, denoting by S the parabolic subgroup of G_r of type $(2n_1, 2n_2, \ldots, 2n_m, n_{m+1})$, π is equivalent to the unitary representation

$$\mathrm{Ind}(G, S; \eta_1', \eta_2', \ldots, \eta_m', \pi_{m+1}').$$

Now π_{m+1}' is a tempered representation, thus certainly satisfies the conclusion of (3.13) (c.f. [H.J.] for example). If we could prove similarly that each η_i' satisfies the conclusion of (3.13) it would follow from the transitivity of Mackey's construction that π itself satisfies it.

Thus we are reduced to proving the following fact. Let π be a tempered representation of G_n, $s > 0$, R the parabolic subgroup of type (n, n) in G_{2n}. Suppose that the induced representation

$$\xi = \mathrm{Ind}(G_{2n}, R; \pi \otimes \alpha^s, \pi \otimes \alpha^{-s})$$

is irreducible and equivalent to a unitary representation η. Then η satisfies the conclusion of (3.13).

Indeed π has the form

$$\pi = \mathrm{Ind}(G_n, S; \pi_1, \pi_2, \ldots, \pi_u),$$

where S is the parabolic subgroup of type $(n_1, n_2, \ldots, n_u)$ in G_n, $n_i = 1, 2$ if $F = \mathbf{R}$ and $n_i = 1$ if $F = \mathbf{C}$, and π_i is an irreducible square-integrable (unitary) representation of G_{n_i}. Let then T be the parabolic subgroup of type $(2n_1, 2n_2, \ldots, 2n_u)$ in G_{2n}. Since ξ is irreducible it is equivalent to the representation

$$\mathrm{Ind}(G_{2n}, T; \sigma_1, \sigma_2, \ldots, \sigma_u),$$

where

$$\sigma_i = \mathrm{Ind}(G_{2n_i}, R_i; \pi_i \otimes \alpha^s, \pi_i \otimes \alpha^{-s}),$$

R_i again being the parabolic subgroup of type (n_i, n_i) in G_{2n_i}. Since ξ is irreducible so are the σ_i. Once more $\tilde{\sigma}_i = \bar{\sigma}_i$ and again by Lemma (3.11) the representation σ_i underlies a unitary representation σ_i' and η is equivalent to the unitary representation

$$\mathrm{Ind}(G_{2u}, T; \sigma_1', \sigma_2', \ldots, \sigma_u').$$

This completes the proof of Lemma (3.13). □

(3.14) *End of the proof of Proposition* (3.8).

Suppose that π satisfies condition (3.8)(ii). Then Lemma (3.13) applies to π. The representations π_i of that lemma must then satisfy (3.8)(ii) by Proposition (3.10). (Perhaps it would be useful to observe that conditions (i) and (ii) of Proposition (3.8) are empty for $r = 1$.) By what we have seen in (3.8) and (3.9) they must also satisfy condition (i) of Proposition (3.8). Thus Proposition (3.8) will at last be a consequence of the following proposition.

PROPOSITION. *Suppose R is a parabolic subgroup of type $(r_1, r_2, \ldots, r_u)$ and for each i, $1 \leq i \leq u$, π_i is a unitary representation of G_i. Let π be the unitary representation of G induced by the π_i's:*

$$\pi = \mathrm{Ind}(G, R; \pi_1, \pi_2, \ldots, \pi_u).$$

If each π_i satisfies condition (3.8)(i) *so does π.*

Proof. By induction on u and the transitivity of inducing representations it suffices to prove the theorem when $u = 2$. It will be more convenient to take for R the lower standard parabolic of type (r_1, r_2). Suppose first $r_2 = 1$. Then R is the transpose of $P_r Z_r$. In any case the complement of $RU(U = U_r)$ has measure zero in G_r. Thus all functions in the space of π are determined by their restriction to U. More precisely, let us identify U with the group F^{r-1} of column vectors of size $r - 1$. Then we may regard π as acting on

$$L^2(F^{r-1}, \mathcal{W}),$$

where $\mathcal{W}$ is the space of π_1, the action of $\pi | P_r$ being given by:

$$\pi \begin{pmatrix} 1 & u_0 \\ 0 & 1 \end{pmatrix} f(u) = f(u + u_0),$$

$$\pi \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} f(u) = \pi_1(m)|\det m|^{-1/2} f(m^{-1}u).$$

Let us use the Fourier transform on F^{r-1} to obtain a new representation $\hat{\pi}$ of P_r—on the same space—equivalent to $\pi|P_r$. More precisely, $\pi(g)\hat{f} = (\hat{\pi}(g)f)\hat{}$ for $g \in P_r$. We have

$$\hat{\pi} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f(v) = f(v)\psi({}^t uv)$$

and

$$\hat{\pi} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} f(v) = \pi_1(m)|\det m|^{1/2} f({}^t mv).$$

Let $\eta = {}^t(0, 0, \ldots, 1) = {}^t\epsilon$. We have

$$F^{r-1} = G_{r-1}(F)\eta \cup \{0\},$$

the stabilizer of η being just $\overline{P_{r-1}}(F)$. Hence every f is determined by the function ϕ from $G_{r-1}(F)$ to $\mathcal{W}$ defined by

$$\phi(m) = \pi_1(m)|\det m|^{1/2} f({}^t m\eta).$$

Such ϕ satisfy

$$\phi(pm) = \pi_1(p)|\det p|^{1/2}\phi(m), \qquad p \in P_{r-1}(F),$$

$$\int ||f(v)||^2 dv = \int_{P_{r-1}(F)\backslash G_{r-1}(F)} ||\phi(m)||^2 dm.$$

Moreover on the ϕ's the representation, say π', corresponding to $\hat{\pi}$ has the form:

$$\pi' \begin{pmatrix} m' & 0 \\ 0 & 1 \end{pmatrix} \phi(m) = \phi(mm'), \qquad \pi' \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \phi(m) = \phi(m)\psi(\epsilon mu).$$

This shows that $\pi | P_r$ is equivalent to the induced representation

$$\xi = \mathrm{Ind}(P_r, P_{r-1}U_r; \sigma \otimes \eta_r),$$

where P_{r-1} is identified with a subgroup of M via the map

$$p \mapsto \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

and $\sigma_1 = \pi_1 | P_{r-1}$. The assumption is that σ contains the representation τ_{r-1} of P_{r-1}. It follows that ξ contains the induced representation

$$\mathrm{Ind}(P_r, P_{r-1}U_r; \tau_{r-1} \otimes \eta_r),$$

a representation equivalent to τ_r. Since $\pi | P_r$ is equivalent to ξ, we are done in this case.

Suppose now $r_2 > 1$. Consider the subgroup $R \cap P_r$ and the representation σ of that group defined by

$$\sigma \begin{pmatrix} m_1 & 0 \\ u & \\ & p \\ 0 & \end{pmatrix} = \pi_1(m_1) \otimes \pi_2(p);$$

here $m_1 \in G_{r_1}$, $p \in P_{r_2}$, and u is a matrix with $r_2 - 1$ rows and r_1 columns. Again functions in the space of π are determined by their restrictions to P_r. Thus $\pi | P_r$ is just the representation

$$\mathrm{Ind}(P_r, R \cap P_r; \sigma).$$

Since the restriction of π_2 to P_{r_2} contains τ_{r_2} we see that $\pi | P_{r_2}$ contains the induced representation

$$\xi' = \mathrm{Ind}(P_r, R \cap P_r; \sigma'),$$

where σ' is now the representation of $R \cap P_r$ defined by

$$\sigma'\begin{pmatrix} m_1 & 0 \\ u & \\ & p \\ 0 & \end{pmatrix} = \pi_1(m_1) \otimes \tau_{r_2}(p).$$

All we have to show is that ξ' contains τ_r. This is an assertion independent of π_2, provided $\pi_2|P_{r_2}$ contains τ_{r_2}. Thus it is enough to prove it for one particular choice of π_2. For instance, we may take π_2 in the unitary principal series; as is very well known $\pi_2|P_{r_2}$ is then equivalent to τ_{r_2} (c.f. for example [H.J.]).

Thus it suffices to prove that $\pi|P_r$ contains τ_r whenever π is a representation of G_r of the form

$$\pi = \mathrm{Ind}(G, R; \pi_1, \pi_2);$$

where $\pi_1|P_{r_1}$ contains τ_{r_1},

$$\pi_2 = \mathrm{Ind}(G_{r_2}, B_{r_2}; \mu_1, \mu_2, \ldots, \mu_{r_2}),$$

B_{r_2} is the parabolic subgroup of type $(1, 1, \ldots, 1)$, and μ_i a character of $F^\times$. We have proved this assertion when $r_2 = 1$. The transitivity of inducing unitary representations allows us to prove it in full generality by induction on r_2. This completes the proof of Proposition (3.8). □

(3.15) *Remarks.* (1) Suppose π is an irreducible unitary representation which satisfies the equivalent conditions of Proposition (3.8). Then in fact:

(iii) *$\pi|P_r$ is equivalent to τ_r.*

Indeed the uniqueness of λ can be used first of all to show that $\pi|P_r$ contains τ_r with multiplicity one. If $r = 2$ we see that $\pi|P_r$ is the sum of τ and a representation trivial on U; but a representation of G_2 cannot contain the trivial representation of U without being one-dimensional. Thus $\pi|P_2 \simeq \tau_2$. If $r = 4$ and $F = \mathbf{R}$, a similar but much more complicated argument can be used to show that $\pi|P_4 \simeq \tau_4$. In general, by Lemma (3.13) we see that π has the form indicated in Proposition (3.14) where the π_i satisfy the stronger condition that $\pi_i|P_{r_i} = \tau_{r_i}$. Then the proof of Proposition (3.14) shows in fact that $\pi|P_r$ is equivalent to τ_r.

(2) In general if π is any irreducible unitary representation of G then, by extending the results of (3.6), one can show that $\pi|P_r$ is a *finite* sum of

irreducible representations. Kirillov has sketched a proof of the fact that $\pi | P_r$ is irreducible but we have not been able to fill in the details.

(3.16) From now on an irreducible unitary representation π of G will be said to be *generic* if it satisfies the equivalent conditions of Proposition (3.8). Let $\mathcal{H}$ be the (Hilbert) space of π and λ any element of $\mathcal{H}^{-\infty}$ satisfying (3.8)(ii). We will denote by $\mathcal{W}(\pi; \psi)$ the space the functions on G of the form

$$W(g) = \lambda(\pi(g)v),$$

where v is in $\mathcal{H}^{\infty}$; we will also denote by $\mathcal{W}_0(\pi; \psi)$ the subspace of those W for which v is in $\mathcal{H}_0$, the space of K-finite vectors in $\mathcal{H}$. Clearly $v \mapsto W$ defines an isomorphism $\mathcal{H}_0 \xrightarrow{\sim} \mathcal{W}_0(\pi; \psi)$ in the admissible category. We may then identify these two spaces. Then exactly as in Proposition (1.5) we have the following Proposition which we will use repeatedly in the next section.

PROPOSITION. *Notations being as above, there is a positive constant c such that*

$$\int_{N\backslash P} |W|^2(pg)d_r(p) \leq c||W||^2$$

for all $w \in \mathcal{W}_0(\pi; \psi)$ *and all* $g \in G$. □

(3.17) Let now π and π' be two irreducible unitary generic representations of G_r. We again introduce, as in (1.5), the integrals

$$\Psi(s, W', W, \Phi)$$

for $W' \in \mathcal{W}_0(\pi'; \psi)$, $W \in \mathcal{W}_0(\pi; \psi)$, and Φ in $\mathcal{S}(F^r)$.

PROPOSITION. (i) *The integral* $\Psi(s, W', W, \Phi)$ *converges absolutely in the half space* $\mathrm{Re}(s) \geq 1$, *normally for* $\mathrm{Re}(s)$ *in a compact subset of* $[1, \infty[$.

(ii) *Given* $W' \neq 0$ *in* $\mathcal{W}_0(\pi'; \psi)$, *and s with* $\mathrm{Re}(s) \geq 1$, *there exist* W *in* $\mathcal{W}_0(\pi; \psi)$ *and* Φ *in* $\mathcal{S}(F^r)$ *such that*

$$\Psi(s, W', W, \Phi) \neq 0.$$

Proof. The proof of (i) is similar to the proof of (i) in Proposition (1.5). As for (ii) we may assume at the cost of replacing W' by a

K-translate, that $W' \,|P$ is not identically zero. Then by Proposition (3.16) $W' \,|P$ is a non-zero element of the space $\mathcal{K}$ of τ_r. Moreover it follows from the same proposition that the function

$$\phi'(p) = W'(p)|\det p|^{s-1}$$

is also in $\mathcal{K}$, whenever $\mathrm{Re}(s) \geq 1$. In more detail suppose $t \geq 0$ and write the integral

$$\int_{N\backslash P} |W(p)|^2 |\det p|^t d_r(p)$$

as a sum of two integrals, the first over $|\det p| \leq 1$ and the second over $|\det p| \geq 1$. In the first integral we may decrease t to 0 in which case the convergence follows from Proposition (3.16). In the second we may replace W by a gauge and take t large, in which case the convergence readily follows.

On the other hand, let $\mathcal{K}$ be the space of π. $\mathcal{K}_0$ is dense in $\mathcal{K}$. Set for $W \in \mathcal{W}_0(\pi;\, \psi) \simeq \mathcal{K}_0$, $A(W) = W|P$. Then by Proposition (3.16), A is non-zero element of $\mathrm{Hom}_P(\pi|P,\, \tau_r)$. Thus $A(\mathcal{K}_0)$ is dense in $\mathcal{K}$. Otherwise said, the functions $p \mapsto W(p)$ with W in $\mathcal{W}_0(\pi;\, \psi)$ are dense in $\mathcal{K}$. Thus there is a W such that

$$\int_{N\backslash P} W'(p)\overline{W}(p)|\det p|^{s-1} d_r(p) \neq 0.$$

Thus the function F on K defined by

$$F(k) = \int_{N\backslash P} W'(pk)\overline{W}(pk)|\det p|^{s-1} d_r(p)$$

is non-zero and K-finite. As in (1.5),

$$\Psi(s, W', W, \Phi) = \int_K dk \int_{N\backslash P} d_r(p)|\det p|^{s-1} W'(pk)\overline{W}(pk)$$

$$\times \int_{F^\times} \Phi(\epsilon a k)|a|^{rs}\omega'\overline{\omega}(a)d^\times a$$

$$= \int_K \int_{F^\times} F(k)\Phi(\epsilon ak)|a|^{rs}\omega'\bar{\omega}(a)d^\times a dk$$

$$= \int_K \int_{\mathbf{R}_+^\times} F(k)\Phi(\epsilon tk)|t|^{rs}\omega'\bar{\omega}(t)d^\times t dk.$$

Now choose $\phi \in C_c^\infty(\mathbf{R}_+^\times)$ such that $\int_{F^\times} \phi(t)|t|^{rs}\omega'\bar{\omega}(t)d^\times t \neq 0$. Then take Φ to be the following function:

$$\Phi(x) = \phi(t)\overline{F(k)} \text{ if } x = t\epsilon K,\ t > 0,\ k \in K;\ \Phi(x) = 0 \text{ otherwise.}$$

Then $\Phi \in \mathcal{S}(F^r)$ and we get finally

$$\Psi(s, W', W, \Phi) = \int_{F^\times} \phi(t)\omega'\bar{\omega}(t)|t|^{rs}d^\times t \int_K \overline{F(k)}F(k)dk \neq 0. \quad \square$$

4. Global Theory. In this section F is an $\mathbf{A}$-field. Although the main results are stated in full generality, we have found it convenient to give the proofs only in the case of number fields. The case of function fields is easier and left to the reader (see also Remark (4.8)). Accordingly until subsection (4.4) the ground field F is assumed to be a number field.

(4.1) We first recall the properties of certain special Eisenstein series. Let η be a character of $F^\times\backslash F_{\mathbf{A}}^\times$ of absolute value one. For $\Phi \in \mathcal{S}(F_{\mathbf{A}}^r)$ set

$$f(g, s, \eta) = |\det g|^s \int_{F_{\mathbf{A}}^\times} \Phi(a\epsilon g)|a|^{rs}\eta(a)d^\times a, \tag{1}$$

where, as before, ϵ is the row matrix with r entries given by

$$\epsilon = (0, 0, \ldots, 0, 1).$$

This integral converges for $\mathrm{Re}(s) > 1/r$. Set also

$$E(g, \Phi, s, \eta) = \Sigma f(\gamma g, s, \eta) \tag{2}$$

the sum over γ in $Z_r(F)P_r(F)\backslash G_r(F)$. This is the Eisenstein series we have in mind. A useful expression can be found if we write

$$f(g, s, \eta) = |\det g|^s \int_{F_{\mathbf{A}}^\times/F^\times} \sum_{\alpha\in F^\times} \Phi(a\alpha\epsilon g)\,|a|^{rs}\eta(a)d^\times a$$

and replace f by this expression in (2):

$$E(g, \Phi, s, \eta) = \sum_\gamma |\det \gamma g|^s \int_{F_{\mathbf{A}}^\times/F^\times} \sum_{\alpha\in F^\times} \Phi(a\alpha\epsilon\gamma g)|a|^{rs}\eta(a)d^\times a.$$

Now every vector $\xi \neq 0$ in F^r can be written uniquely as $\xi = \alpha\epsilon\gamma$. Thus after exchanging the sum over γ and the integral we get

$$E(g, \Phi, s, \eta) = |\det g|^s \int_{F_{\mathbf{A}}^\times/F^\times} \sum_\xi \Phi(a\xi g)\,|a|^{rs}\eta(a)d^\times a, \qquad (3)$$

the sum extended over all ξ in $F^r - \{0\}$.

It now follows from Lemmas (11.5) and (11.6) of [G-J] that the right side of (3) converges absolutely for $\mathrm{Re}(s) > 1$; the same is therefore true for the right side of (2) and they are equal.

It will be convenient to consider a certain set of conditions for functions on $G(F)\backslash G(\mathbf{A})$ depending on a complex parameter. Let I be the open interval $]0, \infty[$, Ω the set of $s \in \mathbf{C}$ such that $\mathrm{Re}(s) \in I$. For $t > 0$ let A_t be the set of $a \in A_\infty$ such that

$$\det a = 1 \text{ and } |\alpha(a)| \geq t \qquad \text{for all } \alpha \in \Delta.$$

If ϕ is a function on $G(F)\backslash G(\mathbf{A}) \times \Omega$ the conditions we have in mind are the following: (4)

(i) ϕ is continuous;

(ii) for each g, the function $s \mapsto \phi(g, s)$ is holomorphic on Ω;

(iii) (if F is a number field), given a compact subset C of $G(\mathbf{A})$, a compact subset J of Ω, and $t > 0$, there exists a constant $c > 0$ and a positive integer m such that

$$|\phi(ax, s)| \leq c \sup_{\alpha\in\Delta} |\alpha(a)|^m$$

for all $a \in A_t$, $x \in C$, and $s \in J$.

(4.2) Lemma. *The functions $E(g, \Phi, s, \eta)$ extend to meromorphic functions on Ω.*

(i) *If η is non-trivial on the ideles of F of absolute value one, they are holomorphic and the function $(g, s) \to E(g, \Phi, s, \eta)$ satisfies condition* (4.1.4).

(ii) *There is a constant $c \neq 0$ such that if $\eta = \alpha^{i\sigma}$ then*

$$E(g, \Phi, s, \eta) = c \cdot \hat{\Phi}(0) |\det g|^{-i\sigma/r} \cdot (s + i\sigma/r - 1)^{-1} + R(g, s),$$

where R is a function on $G(F)\backslash G(\mathbf{A}) \times \Omega$ satisfying (4.1.4).

Proof. We break (4.1.3) into the sum of two integrals:

$$|\det g|^s \int_{|a| \leq 1} \sum_{\xi \neq 0} \Phi(a\xi g) |a|^{rs} \eta(a) d^\times a, \tag{1}$$

and

$$|\det g|^s \int_{|a| \geq 1} \sum_{\xi \neq 0} \Phi(a\xi g) |a|^{rs} \eta(a) d^\times a. \tag{2}$$

Now by the Poisson summation formula

$$\sum_{\xi \neq 0} \Phi(a\xi g) = \sum_{\xi \neq 0} \hat{\Phi}(a^{-1} {}^t\xi {}^t g^{-1}) |a|^{-r} |\det g|^{-1}$$

$$+ \hat{\Phi}(0) |a|^{-r} |\det g|^{-1} - \Phi(0),$$

where as usual

$$\hat{\Phi}(x) = \int_{\mathbf{A}} \Phi(y) \psi(y {}^t x) dy.$$

Thus, after changing a to a^{-1} where needed, we can write the first integral (1) as a sum of three terms:

$$|\det g|^{s-1} \int_{|a| \geq 1} \sum_{\xi \neq 0} \hat{\Phi}(a {}^t\xi {}^t g^{-1}) |a|^{r(1-s)} \bar{\eta}(a) d^\times a \tag{3}$$

$$|\det g|^{s-1} \hat{\Phi}(0) \int_{|a| \leq 1} |a|^{r(s-1)} \eta(a) d^\times a \tag{4}$$

$$- |\det g|^s \Phi(0) \int_{|a| \le 1} |a|^{rs} \eta(a) d^\times a. \tag{5}$$

This computation is valid for $\mathrm{Re}(s) > 1$.

Now by Lemma (11.5) of [G-J] expressions (2) and (3) are defined for all s and, as functions of (g, s) satisfy the conditions (4.1.4). If now η is non-trivial on the ideles of absolute value one (4) and (5) vanish. This proves the first assertion of the lemma. On the other hand if $\eta = \alpha^{i\sigma}$ where σ is real (4) and (5) are respectively

$$c\,|\det g|^{s-1} \hat{\Phi}(0)(s + i\sigma/r - 1)^{-1}, \tag{6}$$

$$-c\,|\det g|^s \Phi(0)(s + i\sigma/r)^{-1}, \tag{7}$$

where c is a certain positive constant. Clearly (7), as a function of (g, s), satisfies (4.1.4). On the other hand we may write

$$a^s = 1 + \phi(a, s)s \qquad (a > 0)$$

where ϕ is a continuous function on $I \times \mathbf{C}$, holomorphic in s. Then (6) is the sum of two terms:

$$c\,|\det g|^{-i\sigma/r} \hat{\Phi}(0)(s + i\sigma/r - 1)^{-1} \tag{8}$$

and

$$c\,|\det g|^{-i\sigma/r} \hat{\Phi}(0)\phi(|\det g|, s + i\sigma/r - 1). \tag{9}$$

Again (9) satisfies (4.1.4), and we have proved (ii), R being the sum of (2), (3), (7) and (9).

(4.3) In this paragraph we introduce what is basically an integral representation for the L-functions $L(s, \pi \times \pi')$ attached to a pair of cusp forms on $G_r(\mathbf{A})$. While these functions appear somewhat implicitly here they are behind the scene and will make their full appearance in subsequent publications. As we have already said their treatment in the case $r = 2$ is due to Rankin and Selberg..

Suppose then that π and π' are irreducible unitary representations of $G(\mathbf{A})$. Let ω and ω' be their central characters. We assume that π and π'

are "cuspidal" (c.f. [B-J]). Let $\mathfrak{A}$ (resp. $\mathfrak{A}'$) be the corresponding space of cusp forms. Each ϕ in $\mathfrak{A}$ (resp. ϕ' in $\mathfrak{A}'$) is K-finite on the right, transforms like the character ω (resp. ω') under $Z(\mathbf{A})$, and is invariant on the left under $G(F)$. Moreover each ϕ is "rapidly decreasing." Namely in the notation of (4.1.4) for any compact set C in $G(\mathbf{A})$, any $t > 0$, and any positive integer m there is a positive constant c such that

$$|\phi(ax)| \leq c(\sup_{\alpha\in\Delta}|\alpha(a)|)^{-m}$$

for all $a \in A_t$ and x in C. Finally note that ω and ω' are trivial on $F^\times$.

Now set $\eta = \omega'\bar{\omega}$ and consider the integral

$$I(s, \Phi, \phi, \phi') = \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} E(g, \Phi, s, \eta)\phi'(g)\bar{\phi}(g)dg. \tag{1}$$

For $a \in Z(\mathbf{A}) \simeq F_{\mathbf{A}}^\times$ we have

$$E(ag, \Phi, s, \eta) = \bar{\eta}(a)E(g, \Phi, s, \eta).$$

Indeed this is clear for $\operatorname{Re}(s)$ large, and by analytic continuation it is true for all s. Thus the integrand is indeed invariant on the left under $Z(\mathbf{A})G(F)$.

Suppose $\eta = \omega'\bar{\omega}$ has a non-trivial restriction to the ideles of F of absolute value one. Then by (4.2)(i) the integrand is a continuous function on $G(\mathbf{A}) \times \Omega$, holomorphic in s, and uniformly bounded for s in a compact subset of Ω. Since the volume of $G(F)Z(\mathbf{A})\backslash G(\mathbf{A})$ is finite the integral converges for all s in Ω and defines a holomorphic function of s on Ω.

Suppose now that $\eta = \alpha^{i\sigma}$ with σ real. Then

$$I(s, \Phi, \phi, \phi') = c\cdot\hat{\Phi}(0)(s + i\sigma/r - 1)^{-1}$$

$$\cdot\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} |\det g|^{-i\sigma/r}\phi'\bar{\phi}(g)dg + \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} R(g, s)\phi'\bar{\phi}(g)dg. \tag{2}$$

Note that since the function $g \mapsto |\det g|^{-i\sigma/r}$ transforms like $\bar{\eta}$ under $Z(\mathbf{A})$ this is true also for $g \mapsto R(g, s)$. The second integral is again holomorphic on Ω.

(4.4) We will use the above remarks to establish the following lemma.

LEMMA. (i) *Suppose* $\hat{\Phi}(0) \neq 0$, $\pi \simeq \pi'$, *and* $\phi = \phi' \neq 0$. *Then* $I(s, \Phi, \phi, \phi')$ *has a* (simple) *pole at* $s = 1$.

(ii) *Suppose there exist* Φ, ϕ, *and* ϕ' *such that* $I(s, \Phi, \phi, \phi')$ *has a pole at* $s = 1$. *Then* $\pi \simeq \pi'$.

Proof. Under the assumptions of (i) we have $\eta = 1$, $\sigma = 0$, and

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} |\phi(g)|^2 dg \neq 0,$$

and so by (4.3.2) $I(s, \Phi, \phi, \phi')$ has a (simple) pole at $s = 1$.

Suppose $I(s, \Phi, \phi, \phi')$ has a pole at $s = 1$. Then η must be trivial on the ideles of absolute value one, or $\eta = i\sigma$ for some real σ. Then by (4.3) we must have $\sigma = 0$ and so $\omega = \omega'$. We have then

$$\int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \phi'\bar{\phi}(g) dg \neq 0$$

and this implies $\pi \simeq \pi'$. □

(4.5) We transform the integrals of (4.2) into the global analogues of the integrals of (1.3). Recall that if

$$\pi = \bigotimes_v \pi_v$$

then each π_v is generic. Fix once for all a non-trivial character ψ of $F_{\mathbf{A}}$ trivial on F and write

$$\psi = \prod_v \psi_v.$$

Then the space $\mathcal{W}(\pi_v; \psi_v)$ is defined as well as the space $\mathcal{W}_0(\pi_v; \psi_v)$ if v is archimedean. Denote by $\mathcal{W}_0(\pi; \psi)$ the space of functions on $G(\mathbf{A})$ spanned by the products

$$W(g) = \prod_v W_v(g_v), \tag{1}$$

where W_v is in $\mathcal{W}(\pi_v;\, \psi_v)$ if v is non-archimedean, in $\mathcal{W}_0(\pi_v;\, \psi_v)$ if v is archimedean, and, for almost all places v, W_v is the essential element of $\mathcal{W}(\pi_v;\, \psi_v)$ ([J-P-S] Sections 11, 12).

For each W in $\mathcal{W}(\pi;\, \psi)$ the function ϕ defined by

$$\phi(g) = \Sigma\, W(\xi g), \qquad \xi \in N_r(F)\backslash P_r(F), \tag{2}$$

is in $\mathcal{A}$. The function W can be recovered from ϕ:

$$W(g) = \int_{N(\mathbf{A})} \phi(ng)\bar{\theta}(n)dn. \tag{3}$$

Moreover each W is majorized by a gauge (loc. cit.). Recall that a gauge ξ on $G(\mathbf{A})$ is a function ξ on $G(\mathbf{A})$ invariant under $N(\mathbf{A})Z(\mathbf{A})$ on the left, under K on the right, and given on $A(\mathbf{A})$ by

$$\xi(a) = \Phi(\alpha_1(a), \alpha_2(a), \ldots, \alpha_{r-1}(a))\,|\alpha_1(a) \cdots \alpha_{r-1}(a)|^{-t}$$

where $\Phi > 0$ is in $\mathcal{S}(\mathbf{A}^{r-1})$ and $t \geq 0$.

Consider now the integral

$$\Psi(s, W', W, \Phi) = \int_{N(\mathbf{A})\backslash G(\mathbf{A})} W'(g)\overline{W}(g)\Phi(\epsilon g)\,|\det g\,|^s dg, \tag{4}$$

where $W \in \mathcal{W}(\pi;\, \psi)$, $W' \in \mathcal{W}(\pi';\, \psi)$ and $\Phi \in \mathcal{S}(\mathbf{A}^r)$. From the estimates we have just recalled it is easy to see that the integral converges for $\mathrm{Re}(s)$ large. Moreover we have for $\mathrm{Re}(s)$ large

$$I(s, \Phi, \phi', \phi) = \Psi(s, W', W, \Phi), \tag{5}$$

where $\phi' \in \mathcal{A}'$ is defined in terms of W' as ϕ is in terms of W. Indeed if we replace in the left side of (4.3.1), $E(g, \Phi, s, \eta)$ by its expression (4.1.2) as a series, we get

$$\int_{Z(\mathbf{A})P_r(F)\backslash G_r(\mathbf{A})} f(g, s, \eta)\phi'(g)\bar{\phi}(g)\,|\det g\,|^s dg.$$

Then from the definition (4.1.1) of f we get

$$I(s, \Phi, \phi', \phi) = \int_{P_r(F)\backslash G_r(\mathbf{A})} \Phi(\epsilon g)\phi'(g)\bar{\phi}(g)\,|\det g\,|^s dg.$$

Now replace ϕ (or rather $\bar{\phi}$) by its expression as the series (2). We get

$$\int_{N(F)\backslash G(\mathbf{A})} \Phi(\epsilon g)\phi'(g)\overline{W}(g)\,|\det g\,|^s dg.$$

Since the functions $g \mapsto \Phi(\epsilon g)$ and $g \mapsto |\det g\,|^s$ are invariant under $N(\mathbf{A})$ while W transforms like θ this is also

$$\int_{N(\mathbf{A})\backslash G(\mathbf{A})} \Phi(\epsilon g)\overline{W}(g)\,|\det g\,|^s dg \int_{N(F)\backslash N(\mathbf{A})} \phi'(ng)\bar{\theta}(n)dn$$

and the inner integral is W'. Thus (5) is established—provided we justify our formal manipulations.

Since ϕ' is bounded and $N(F)\backslash N(\mathbf{A})$ has finite volume it suffices to show that the integral

$$\int_{N(\mathbf{A})\backslash G(\mathbf{A})} \Phi(\epsilon g)\, W(g)\,|\det g\,|^s dg$$

converges absolutely for $\mathrm{Re}(s)$ large. Since W is majorized by a gauge the proof is essentially the same as the convergence of the integral in (4) (for $\mathrm{Re}(s)$ large). □

(4.6) From (4.5.5) and (4.4), we obtain the following lemma:

Lemma. *The function* $\Psi(s, W', W, \Phi)$ *extends to a meromorphic function of* s *on* Ω, *holomorphic for* $\mathrm{Re}(s) > 1$.

(i) *If* $\hat{\Phi}(0) \neq 0$, $\pi = \pi'$ *and* $W = W' \neq 0$ *that function has a* (simple) *pole at* $s = 1$.

(ii) *Conversely, if there exist* W, W' *and* Φ *such that* $\Psi(s, W', W, \Phi)$ *has a pole at* $s = 1$ *then* $\pi \simeq \pi'$. □

(4.7) We now write $\Psi(s, W', W, \Phi)$ as a finite sum of products of the local integrals of Section 1 and Section 3. Suppose W and W' are of the form (4.5.1) and that

$$\Phi(x) = \prod_v \Phi_v(x_v), \tag{1}$$

where Φ_v is in $\mathcal{S}(F_v{}^r)$ for all v and is, for almost all v, the characteristic function of $\mathfrak{R}_v{}^r$ in $F_v{}^r$. Now for $\operatorname{Re}(s) \geq 1$ all the local integrals $\Psi(s, W_v{}', W_v, \Phi_v)$ converge [(1.5) and (3.17)] and so, for $\operatorname{Re}(s)$ large,

$$\Psi(s, W', W, \Phi) = \prod_v \Psi(s, W_v{}', W_v, \Phi_v), \tag{2}$$

the right side being an absolutely convergent product. Our first theorem is as follows:

(4.8) THEOREM. *Suppose π and π' are irreducible unitary cuspidal representations of $G(\mathbf{A})$. Suppose S is a finite set of places containing the infinite places and all the finite places where either π_v or $\pi_v{}'$ is ramified. Suppose that for $v \notin S$* (notation of (2.1.2):

$$L(s, \pi_v{}' \times \bar{\pi}_v) = L(s, \pi_v{}' \times \bar{\pi}_v{}').$$

Then $\pi \simeq \pi'$.

Proof. (for a number field). At the cost of enlarging S we may assume that for $v \notin S$ the largest ideal on which ψ_v is trivial is $\mathfrak{R}_v$. We take W and W' of the form (4.5.1), Φ and Φ' in $\mathcal{S}(\mathbf{A}^r)$ of the form (4.7.1) where

$$\Phi_v{}'(0) \neq 0, \ \Psi(1, W_v{}', W_v, \Phi_v) \neq 0 \qquad \text{for } v \in S$$

(c.f. (1.5) and (3.17));

$$\Phi_v{}' = \Phi_v = \text{the characteristic function of } \mathfrak{R}_v{}^r \text{ for } v \notin S;$$

$W_v{}'$ and W_v are the essential elements of $\mathcal{W}(\pi_v{}'; \psi_v)$ and $\mathcal{W}(\pi_v; \psi_v)$ resp. for $v \notin S$. Then for $v \notin S$, by Proposition (2.3),

$$\Psi(s, W_v{}', W_v{}', \Phi_v{}') = L(s, \pi_v{}' \times \bar{\pi}_v{}'),$$

$$\Psi(s, W_v{}', W_v, \Phi_v) = L(s, \pi_v{}' \times \bar{\pi}_v),$$

and so by hypothesis

$$\Psi(s, W_v{}', W_v{}', \Phi_v{}') = \Psi(s, W_v{}', W_v, \Phi_v) \qquad \text{for } v \notin S. \tag{1}$$

Set

$$A(s) = \prod_{v \in S} \Psi(s, W_v', W_v, \Phi_v),$$

$$B(s) = \prod_{v \in S} \Psi(s, W_v', W_v', \Phi_v').$$

Then $A(s)$ and $B(s)$ are holomorphic in $\mathrm{Re}(s) > 0$, continuous in $\mathrm{Re}(s) \geq 1$ (cf. (1.5) and (3.17)) and $A(1) \neq 0$.

From (1) we get the identity

$$B(s)\Psi(s, W', W, \Phi) = A(s)\Psi(s, W', W', \Phi')$$

at first for $\mathrm{Re}(s)$ large but then, by analytic continuation, for all s with $\mathrm{Re}(s) > 1$.

Now by (4.6)(i), $\Psi(s, W', W', \Phi')$ has a pole at $s = 1$, since by choice $W' \neq 0$ and $\hat{\Phi}'(0) = \Pi_v \hat{\Phi}_v'(0) = \Pi_{v \in S} \hat{\Phi}_v'(0) \neq 0$. Since $A(1) \neq 0$ we have

$$\lim_{s \to 1} |A(s)\Psi(s, W', W', \Phi')| = +\infty$$

the limit taken for $s > 1$. Since $B(1)$ is finite we must have similarly

$$\lim_{s \to 1} |\Psi(s, W', W, \Phi)| = +\infty.$$

Thus $\Psi(s, W', W, \Phi)$ has a pole at $s = 1$. By (4.6)(ii), $\pi \simeq \pi'$. □

(4.9) Remark. For the function field case, the proof is substantially the same, the changes being mostly notational. Since a cusp form has compact support mod $Z(\mathbf{A})G(F)$ no estimate is needed for $E(g, \Phi, s, \eta)$.

(4.10) Corollary. *Suppose π and π' are irreducible unitary representations of $G_r(\mathbf{A})$. Suppose S is a finite set of places such that $\pi_v \simeq \pi_v'$ for $v \notin S$. Then $\pi \simeq \pi'$.* □

(4.11) *Remark.* *Let S* be as in (4.8). Set

$$L_S(s, \pi' \times \bar{\pi}) = \prod_{v \notin S} L(s, \pi_v' \times \bar{\pi}_v).$$

As we noted the product is absolutely convergent for $\mathrm{Re}(s)$ large. One can actually prove that $L_S(s, \pi' \times \bar{\pi})$ is meromorphic—by methods quite distinct from those of the present paper. If one takes this for granted the proof of Theorem (4.8) shows that $L_S(s, \pi' \times \bar{\pi})$ has at most a simple pole at $s = 1$ with non-zero residue if and only if $\pi' \simeq \bar{\pi}$. In the next section we show that the Euler product for $L_S(s, \pi' \times \bar{\pi})$ is absolutely convergent for $\mathrm{Re}(s) > 1$ and any pair (π, π').

5. Convergence of Euler Products. The results of this section complement the earlier non-vanishing theorem for L-functions obtained by the present authors ([J-S]). See also [F.S.]

(5.1) Suppose that π(resp. π') is an irreducible unitary cuspidal representation of $G_r(\mathbf{A})$ (resp. $G_p(\mathbf{A})$). Let S be a finite set of places containing the infinite places and all the finite places where either π_v or π_v' is ramified. As in Section 4, we fix once and for all a non-trivial character ψ on $F_{\mathbf{A}}$ trivial on F. We *assume* throughout this section that S is large enough so that for $v \notin S$ the largest ideal on which ψ_v is trivial is $\mathfrak{R}_v$. As in Section 2 let A_v(resp. A_v') be the (semi-simple) conjugacy class in $G_r(\mathbf{C})$ (resp. $G_p(\mathbf{C})$) associated to π_v (resp. π_v'). We set as before

$$L(s, \pi_v \times \pi_v') = \det(1 - q_v^{-s} A_v \otimes A_v')^{-1}. \tag{1}$$

It follows from (2.5) that, if

$$A_v = \mathrm{diag}(\mu_{1v}(\tilde{\omega}_v), \mu_{2v}(\tilde{\omega}_v), \ldots, \mu_{rv}(\tilde{\omega}_v)), \tag{2}$$

then

$$|\mu_{jv}(\tilde{\omega}_v)| \leq q_v^{1/2} \tag{3}$$

for all $v \notin S$, $1 \leq j \leq r$. Thus it is clear that the Euler product

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi_v') \tag{4}$$

converges absolutely to an analytic function in some right half-plane. Before turning to our second theorem we prove an intermediate lemma.

(5.2) Lemma. *Let π be an irreducible unitary cuspidal representation of $G_r(\mathbf{A})$. Then the function $L_S(s, \pi \times \bar{\pi})$ has an analytic continuation to the half plane* $\mathrm{Re}(s) > 1$.

Proof. We proceed as in (4.8). Fix s_0 with $\mathrm{Re}(s_0) \geq 1$. We take W and W' of the form (4.5.1), Φ in $\mathcal{S}(\mathbf{A}^r)$ of the form (4.7.1) where

$$\Psi(s_0, W_v', W_v, \Phi_v) \neq 0 \qquad \text{for } v \in S$$

(c.f. (1.5) and (3.17));

$$\Phi_v = \text{the characteristic function of } \mathfrak{R}_v{}^r \text{ for } v \notin S;$$

$W_v' = W_v$ is the essential element of $\mathcal{W}(\pi_v; \psi_v)$ for $v \notin S$. Set $A(s) = \Pi_{v \in S} \Psi(s, W_v', W_v, \Phi_v)$. Then as in (4.8) (c.f. also (2.3)) we have

$$\Psi(s, W', W, \Phi) = A(s) L_S(s, \pi \times \bar{\pi})$$

for $\mathrm{Re}(s)$ large. By (4.6) the left side is holomorphic for $\mathrm{Re}(s) > 1$. Since $A(s_0) \neq 0$ and $A(s)$ is also holomorphic for $\mathrm{Re}(s) > 1$, it follows that $L_S(s, \pi \times \bar{\pi})$ is holomorphic in a neighborhood of s_0. This being true now for any s_0 with $\mathrm{Re}(s_0) > 1$ our assertion follows.

(5.3) THEOREM. *Let π (resp. π') be an automorphic cuspidal representation of $G_r(\mathbf{A})$ (resp. $G_p(\mathbf{A})$). Suppose in addition that π and π' are unitary. Then the infinite product*

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi_v')$$

is absolutely convergent in the half-plane $\mathrm{Re}(s) > 1$. *In particular the function $L_S(s, \pi \times \pi')$ does not vanish for* $\mathrm{Re}(s) > 1$.

Proof. We suppose A_v given by (5.1.2) and

$$A_v' = \mathrm{diag}(\nu_{1v}(\tilde{\omega}), \nu_{2v}(\tilde{\omega}), \ldots, \nu_{pv}(\tilde{\omega})).$$

Then

$$L(s, \pi_v \times \pi_v') = \Pi(1 - \mu_i(\tilde{\omega})\nu_j(\tilde{\omega}) q_v{}^{-s})^{-1},$$

the product over $1 \leq i \leq r$, $1 \leq j \leq p$. The individual terms in this finite product are of the form $(1 - z)^{-1}$ with $|z| < 1$. Thus if we set $\pi(v^n) = \mathrm{tr}(A_v{}^n)$, $\pi'(v^n) = \mathrm{tr}(A_v{}'^n)$, we have

$$L(s, \pi_v \times \pi_v') = \exp \sum_{n \geq 1} \pi(v^n)\pi'(v^n)/nq_v^{ns} \tag{1}$$

the series being absolutely convergent for $\mathrm{Re}(s) > 2$. Taking the product over $v \notin S$, and using (5.1.3), we get for $\mathrm{Re}(s)$ large

$$L_S(s, \pi_v \times \pi_v') = \exp \sum_{v \notin S} \sum_{n \geq 1} \pi(v^n)\pi'(v^n)/nq_v^{ns}, \tag{2}$$

the sum over $v \notin S$ being taken in any order.

To proceed further we assume first that $\pi' = \bar{\pi}$. Then

$$f(s) = \sum_{v \notin S} \sum_{n \geq 1} |\pi(v^n)|^2/nq_v^{ns} \tag{3}$$

is a Dirichlet series with non-negative coefficients, converging for $\mathrm{Re}(s)$ large. Let a be the real point of the line of convergence of this series. By a well-known theorem on Dirichlet series [E.C.T.] a is a singularity of $f(s)$. Suppose $a > 1$. Then from (1) we get by analytic continuation

$$L_S(s, \pi \times \bar{\pi}) = \exp f(s) \tag{4}$$

for $\mathrm{Re}(s) > a$. For s real, $s > a$, $f(s) \geq 0$ is increasing as $s \to a$. On the other hand by Lemma (5.2) the left side of (4) approaches a finite limit as $s \to a$. It follows that $L_S(a, \pi \times \bar{\pi}) \geq 1$ and in particular is non-zero. Thus in the intersection of a small disc about a with the half-plane $\mathrm{Re}(s) > a$, we get

$$\log L_S(s, \pi \times \bar{\pi}) = f(s).$$

This contradicts the fact that a is a singularity of f. Thus $a \leq 1$ and the series (3) converges for $\mathrm{Re}(s) > 1$.

Next an application of Cauchy-Schwartz shows that the double series

$$\sum_{v \notin S} \sum_{n \geq 1} \pi(v^n)\pi'(v^n)/nq_v^{ns} \tag{5}$$

also is absolutely convergent for $\mathrm{Re}(s) > 1$. It follows immediately from (2) that the product for $L_S(s, \pi \times \pi')$ is commutatively and therefore absolutely convergent for Re(s) > 1 ([N.B. Chap. 8]). This completes the proof of Theorem (5.3).

(5.4) *Remark.* The theorem applies in particular to the case $p = 1$, $\pi' = 1$. Then:

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} \det(1 - A_v q_v{}^{-s})^{-1}.$$

Since a local L-factor is never zero we conclude from the results of [J-S] that

$$L(s, \pi) = \prod_v L(s, \pi_v)$$

does not vanish in the half-plane Re $s \geq 1$.

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