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A Lemma on Highly Ramified ε-Factors*

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1. Introduction

(1.1) Let F be a non-archimedean local field and G_r the group $\mathrm{GL}(r, F)$. Fix a non trivial additive character ψ of F. For convenience, we will assume that the largest ideal on which ψ is trivial is the ring of integers $\mathfrak{R}$; we will denote by $\mathfrak{P}$ the maximal ideal in $\mathfrak{R}$ and by q the cardinality of the residual field $\mathfrak{R}/\mathfrak{P}$. Let π be an irreducible admissible representation of G_r on a complex vector space V. To π we can attach functions $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ [G–J]. They have the form

$$\begin{gathered} L(s,\pi)=P(q^{-s})^{-1}, \quad P\in\mathbb{C}[x]; \\ \varepsilon(s,\pi,\psi)=cq^{-fs}. \end{gathered} \tag{1}$$

A simple but useful property of these functions is the following one: suppose π_1 and π_2 are two such representations with the *same* central character ω; then, if χ is a multiplicative character of conductor $\mathfrak{P}^a$ we have

$$\begin{gathered} L(s,\pi_1\otimes\chi)=L(s,\pi_2\otimes\chi)=1, \\ \varepsilon(s,\pi_1\otimes\chi,\psi)=\varepsilon(s,\pi_2\otimes\chi,\psi), \end{gathered} \tag{2}$$

provided a is large enough. As a matter of fact, we have used this property several times. For the sake of completeness, we give the (standard) proof in Sect. 2.

More generally, if π is an irreducible admissible representation of G_r and σ an irreducible admissible representation of G_t, then one can define factors $L(s, \pi\times\sigma)$ and $\varepsilon(s, \pi\times\sigma, \psi)$ and they have the property analogous to (2). Again we used this fact before. The purpose of this paper is to give a proof of it (Sect. 4). In Sect. 3 we give a property of the *conductor* of a representation π, that is of the ideal $\mathfrak{P}^f$, with f as in (1); it is used in an essential way in Sect. 4.

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2. The Case of one Representation

(2.1) Let again π be an irreducible admissible representation of G_r. If f is a matrix coefficient of π and Φ a Bruhat-function on the vector space M_r of r by r matrices we set

$$Z(\Phi, s, f) = \int_{G_r} \Phi(x) f(x) |\det x|^s d^\times x, \tag{1}$$

where $d^\times x$ is a Haar measure on G_r. The factor $L(s, \pi)$ is the "g.c.d." of the integrals $Z(\Phi, s+\frac{1}{2}(r-1), f)$. As for the ε-factor it is defined by the functional equation of the integrals (1); namely if we set

$$\gamma(s, \pi, \psi) = \varepsilon(s, \pi, \psi) L(1-s, \tilde{\pi}) L(s, \pi)^{-1} \tag{2}$$

then

$$\begin{aligned} &Z(\Phi^\wedge, 1-s+\tfrac{1}{2}(r-1), f^\vee) \\ &\quad = \gamma(s, \pi, \psi) Z(\Phi, s+\tfrac{1}{2}(r-1), f), \end{aligned} \tag{3}$$

where $f^\vee(g) = f(g^{-1})$ and $\Phi^\wedge$ is the Fourier-transform of Φ:

$$\Phi^\wedge(x) = \int \Phi(y) \psi(\operatorname{Tr}(yx)) dy, \tag{4}$$

dy being the self-dual Haar measure on M_r.

(2.2) **Proposition.** *Suppose π is as above; let ω be its central character and $\chi_1, \chi_2, \ldots, \chi_r$ characters of $F^\times$ whose product is ω. There is integer $A > 0$ with the following property: if $a \geqq A$ and χ is a character of $F^\times$ with the conductor P^a, then:*

$$L(s, \pi \otimes \chi) = 1, \tag{1}$$

$$\varepsilon(s, \pi \otimes \chi, \psi) = \prod_{i=1}^{r} \varepsilon(s, \chi_i \chi, \psi). \tag{2}$$

(2.3) *Proof of* (2.2.1). Because of the functorial property of the L-factor with respect to induction, it suffices to prove our assertion for an "atom" of the theory, that is, for a supercuspidal π. If $r=1$ any character π of $F^\times$ is a supercuspidal representation and $L(s, \pi) = 1$, as soon as π is ramified at all. If $r > 1$ and π is supercuspidal then $L(s, \pi) = 1$. □

(2.4) *Proof of* (2.2.2). Assume first $r > 1$ and π supercuspidal. The definition of the ε-factor can then be reformulated as follows: let f be a matrix coefficient of π and φ an element of $C_c^\infty(F^\times)$ (locally constant functions of compact support). Then the function Φ on M_r defined by

$$\Phi(g) = f(g^{-1}) \varphi(\det g), \quad \text{if} \quad \det g \neq 0; = 0, \text{ otherwise,} \tag{1}$$

is a Bruhat-function; its Fourier transform is similarly given by

$$\Phi^\wedge(g) = f(g) H_\pi \varphi(\det g) \quad \text{if} \quad \det g \neq 0; = 0, \text{ otherwise,} \tag{2}$$

where

$$H_\pi : C_c^\infty(F^\times) \to C_c^\infty(F^\times) \tag{3}$$

is a linear map, depending on π. Then:

$$\gamma(s,\pi,\psi)\int\varphi(a)|a|^{s+\frac{1}{2}(r-1)}d^\times a \\ =\int H_\pi\varphi(a)\,|a|^{1-s+\frac{1}{2}(r-1)}d^\times a\,, \tag{4}$$

for any $\varphi\in C_c^\infty(F^\times)$. The proof is an easy exercise which uses Schur-orthogonality relations and Lemma 5.3 p. 59 in [G-J]. Formal properties of the map H_π are:

$$H_\pi(\chi^{-1}\varphi)=\chi H_{\pi\otimes\chi}(\varphi)\,, \tag{5}$$

for any character χ of $F^\times$, and

$$(H_{\tilde\pi}H_\pi\varphi)(x)=\omega(-1)\varphi[(-1)^r x]\,, \tag{6}$$

where ω is the central character of π and $\tilde\pi$ the representation contragredient to π. We shall use (4) to prove our result.

Before we embark on the proof we remark the following: suppose we have proved equality (2.2.2) up to a positive factor, depending only on r. Then the equation

$$\varepsilon(s,\pi,\psi)\varepsilon(1-s,\tilde\pi,\psi)=\omega(-1)$$

will tell us that the factor is actually one. Thus we may, and will, ignore such positive factors. In particular, we do not bother normalizing Haar measures.

(2.5) We let φ be the characteristic function of $\mathfrak{R}^\times$ in $F^\times$ and compute $H_{\pi\otimes\chi}\varphi$. We denote by $\tilde\omega$ a prime element of $\mathfrak{R}$ and by $\mathfrak{R}_r$ the subring of M_r of matrices with integral entries. For $i>0$ we denote by K_i the congruence subgroup

$$K_i=1+\tilde\omega^i R_r \tag{1}$$

in G_r. There is an $i>0$ and a matrix coefficient f of π such that $f(e)\neq 0$ and f is invariant on both sides under K_i. On the other hand, we let k be an integer – to be taken sufficiently large. We set

$$j=k/2\quad\text{if}\quad k\text{ is even},\qquad j=(k+1)/2\quad\text{if}\quad k\text{ is odd}. \tag{2}$$

We let χ be a character of conductor $\mathfrak{P}^k$. Since in any case $2j\geqq k$, the map

$$a\mapsto h=1+a$$

defines a group-isomorphism:

$$\mathfrak{P}^j/\mathfrak{P}^k\simeq(1+\mathfrak{P}^j)/(1+\mathfrak{P}^k)\,.$$

In particular, there is a c in $\tilde\omega^{-k}\mathfrak{R}^\times$ such that

$$\chi(h)=\psi(ca)\,,\quad\text{for}\quad a\in\mathfrak{P}^j\,. \tag{3}$$

On the other hand, if h is in

$$K_j=1+\tilde\omega^j\mathfrak{R}_r$$

then it has the form

$$h=1+a\,,\qquad a\in\tilde\omega^j\mathfrak{R}_r$$

and

$$\det(h)=1+\mathrm{Tr}(a)\mod\tilde\omega^{2j}\mathfrak{R}\,,$$
$$\mathrm{Tr}(a)=0\mod\tilde\omega^j\mathfrak{R}\,.$$

It follows that

$$\chi(\det h) = \psi[\mathrm{Tr}(ca)]\,. \tag{4}$$

The functions f and φ being as above we set

$$\begin{aligned} \Phi(g) &= f(g^{-1})\chi(\det g)^{-1}\varphi(\det g)\,, \quad \text{if} \quad \det g \neq 0\,, \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise}\,. \end{aligned} \tag{5}$$

Then, for k large enough,

$$\hat{\Phi}(g) = \int_{G_0/K_k} f(x^{-1})\chi(\det x)^{-1}dx \int_{K_k} \psi(\mathrm{Tr}(gxh))dh\,, \tag{6}$$

where G_0 is the group of $g \in G_r$ such that $|\det g| = 1$. The inner integral can also be written as

$$\psi(\mathrm{Tr}(gx)) \int_{\tilde{\omega}^k R_r} \psi(\mathrm{Tr}(gxa))da\,;$$

it vanishes unless x belongs to the set

$$X_g = g^{-1}\tilde{\omega}^{-k}\mathfrak{R}_r\,. \tag{7}$$

Thus

$$\begin{aligned} \hat{\Phi}(g) &= \int_{X_g/K_k} \varphi(\det x) f(x^{-1})\chi(\det x)^{-1}\psi(\mathrm{Tr}(gx))dx \\ &= \int_{X_g/K_j} \varphi(\det x) f(x^{-1})\chi(\det x)^{-1}dx \\ &\quad \cdot \int_{K_j/K_k} \chi^{-1}(\det(h))\psi(\mathrm{Tr}(gxh)dh\,. \end{aligned} \tag{8}$$

Again the map $a \mapsto h = 1 + a$ defines a group isomorphism

$$\tilde{\omega}^j\mathfrak{R}_r/\tilde{\omega}^k\mathfrak{R}_r \simeq K_j/K_k$$

and the inner integral in (8) can be written as

$$\begin{aligned} &\psi(\mathrm{Tr}(gx)) \int \chi^{-1}(\det(1+a))\psi(\mathrm{Tr}(gxa))da \\ &= \psi(\mathrm{Tr}(gx)) \int \psi(\mathrm{Tr}(gx - ca))da\,. \end{aligned}$$

The last integral is over $\omega^j\mathfrak{R}_r$ and vanishes unless $gx - c$ is in $\tilde{\omega}^{-j}\mathfrak{R}_r$ or, what amounts to the same,

$$gx \in cK_{k-j}\,.$$

Thus we find that $\Phi^\wedge$ vanishes outside the set cG_0 and for g in that set is given by

$$\Phi^\wedge(g) = \int f(x^{-1})\chi(\det x)^{-1}\psi(\mathrm{Tr}(gx))dx\,, \tag{9}$$

the integration being over $g^{-1}cK_{k-j}/K_j$.

Now if k is so large that $k - j \geqq i$ this is simply

$$\Phi^\wedge(g) = f(g)\chi(\det g)\,|\det g|^{-r}\tau(\chi, \omega, r) \tag{10}$$

where we have set

$$\tau(\chi, \omega, r) = \omega^{-1}(c)\chi^{-1}(c^r)\eta(\chi, r)\,, \tag{11}$$

$$\eta(\chi, r) = \int_{K_{k-j}/K_j} \chi^{-1}(\det h)\psi(\mathrm{Tr}(ch))dh\,. \tag{12}$$

Thus, we have proved that

$$(H_{\pi\otimes\chi}\varphi)(a)=\varphi(c^{-r}a)\tau(\chi,\omega,r)\,|a|^{-r}, \tag{13}$$

or, by (2.4.1):

$$\varepsilon(s,\pi\otimes\chi,\psi)=q^{-krs}\tau(\chi,\omega,r). \tag{14}$$

The right hand side is now independent of π. It remains to compute it.

(2.6) **Lemma.** *Given r, if k is large enough and χ has conductor $\mathfrak{P}^k$, then:*

$$\eta(\chi,r)=\eta(\chi,1)^r.$$

Proof of Lemma (2.6). We may write an element h of K_{k-j} in the form:

$$h=(1+u)(1+\delta)(1+v),$$

where u and v are strictly upper and lower triangular matrices and δ is diagonal – each belonging to $\tilde{\omega}^{k-j}\mathfrak{R}_r$. We have $\det h=\det(1+\delta)$ and

$$\operatorname{Tr} h=\operatorname{Tr}(1+\delta)+\operatorname{Tr}(uv)+\operatorname{Tr}(u\delta v).$$

Hence, if $3(k-j)\geqq k$, then

$$\begin{aligned}&\chi^{-1}(\det(h))\psi(\operatorname{Tr}(ch))\\&\quad=\chi^{-1}(\det(1+\delta))\psi(\operatorname{Tr}(c(1+\delta)))\psi(\operatorname{Tr}(cuv)).\end{aligned}$$

Thus, if $2(k-j)\geqq k$, we obtain

$$\begin{aligned}\eta(\chi,r)=&\int\chi^{-1}(\det(1+\delta))\psi[\operatorname{Tr}(c(1+\delta))]d\delta\\&\cdot\int\psi[\operatorname{Tr}(cuv)]du\,dv.\end{aligned}$$

The second integral is positive and may be ignored and the first is $\eta(\chi,1)^r$.

This concludes the proof of the lemma.

(2.7) Using the lemma we get

$$\varepsilon(s,\pi\otimes\chi,\psi)=[q^{-ks}\chi^{-1}(c)\eta(\chi,1)]^r\omega^{-1}(c).$$

This formula applies to the case $r=1$ as well (the proof we have just given being then the classical one). Hence, for $1\leqq i\leqq r$, and k large enough:

$$\varepsilon(s,\chi_i\chi,\psi)=q^{-ks}\chi^{-1}(c)\eta(\chi,1)\chi_i^{-1}(c), \tag{1}$$

if χ has conductor $\mathfrak{P}^k$. Then, as claimed:

$$\varepsilon(s,\pi\otimes\chi,\psi)=\prod_i\varepsilon(s,\chi_i\chi,\psi). \tag{2}$$

Thus (2.2) is proved when π is supercuspidal.

The previous proof shows that the right hand side of (2) depends only on r and ω, provided k is large enough (as is well known.) Again the functoriality of the ε-factor shows then that the formula is true in general. □

(2.8) *Remark.* Mutatis mutandis, the proof applies to the L and ε factors attached to an irreducible representation of the multiplicative group of a simple algebra.

3. A Review of the Conductor

(3.1) Let π be an admissible irreducible representation of G_r on a complex vector space V. Assume π is generic. If N_r is the group of upper triangular matrices with unit diagonal we define a character θ or θ_r of N_r by

$$\theta(n)=\psi(n_{12}+n_{23}+\ldots+n_{r-1r}); \tag{1}$$

to say that π is generic means there is a non zero linear form λ on V such that

$$\lambda[\pi(n)v]=\theta(n)v, \quad \text{for} \quad n\in N_r. \tag{2}$$

This linear form is then unique, up to a scalar factor, and we denote by $\mathscr{W}(\pi;\psi)$ the space of functions W on G_r of the form

$$W(g)=\lambda(\pi(g)v), \qquad v\in V. \tag{3}$$

We will need a few auxiliary notations. We let P_r be the group of matrices p in G_r of the form

$$p=\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix}, \quad g\in G_{r-1}$$

and $Z_r\simeq F^\times$ the center of G_r. We also introduce the r by r matrix

$$w_r=\begin{pmatrix} 0 & & \cdot 1 \\ 0 & \cdot^{\cdot^{\cdot}} & \\ \vdots\ 1^{\cdot} & & \\ 1 & & 0 \end{pmatrix} \tag{4}$$

and the row-matrix of length r

$$\eta_r=(0,0,\ldots,0,1). \tag{5}$$

We set

$$K_r=\mathrm{GL}(r,\mathfrak{R})$$

and denote, for $j>0$, by $K_r(j)$ the subgroup of matrices $k\in K_r$ of the form:

$$k=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a\in K_{r-1}, \quad d\in\mathfrak{R}^\times, \quad c\equiv 0 \bmod \mathfrak{P}^j. \tag{6}$$

If $j=0$ we set $K_r(0)=K_r$. Then if ω is a character of $F^\times$ (or $\mathfrak{R}^\times$), trivial on $1+\mathfrak{P}^j$, we define a one-dimensional character ω_j of $K_r(j)$ by setting

$$\begin{aligned} &\omega_j(k)=\omega(d), \quad k \text{ as in (6)}, \quad \text{if} \quad j>0; \\ &\omega_j(k)=1, \quad \text{for all } k, \qquad \text{if} \quad j=0. \end{aligned} \tag{7}$$

If f is the integer in formula (1.1.1) then the ideal $\mathfrak{P}^f$ is called the *conductor* of π ([*J-P-S*] II). If there exists a non-zero vector in V transforming under $K_r(j)$ according to ω_j then ω coincides with the central character of π on $\mathfrak{R}^\times$ and necessarily $j\geqq f$. Moreover, if ω is the central character of π then the conductor of ω is contained in $\mathfrak{P}^f$ and the dimension of the space of vectors in V transforming under $K_r(f)$ according to ω_f is one (loc. cit.; note, however, the notations are slightly different). This property characterizes f.

(3.2) In what follows we will need another characterization of the conductor. It will be based on the following simple lemma. We let H be a smooth complex-valued function on G_r satisfying

$$H(ng)=\bar{\theta}(n)H(g), \quad n\in N_r, \quad g\in G_r, \tag{1}$$

and compactly supported modulo N_r.

Lemma. *With H as above, suppose that the integral*

$$\int_{N_r\backslash G_r} H(g)W(g)dg$$

vanishes, for all admissible irreducible generic representations π of G_r and all $W\in\mathscr{W}(\pi;\psi)$. Then $H=0$.

Proof. With trivial modifications the proof is word for word the same as that of Lemma (3.5) in [J-P-S] II. □

(3.3) Next, for $g\in G_r$ of the form $g=nak$ with $n\in N_r$, $k\in K_r$ and

$$a=diag(a_1,a_2,\dots,a_r),$$

set

$$\varrho(g)=|a_1|. \tag{1}$$

We then have the following corollary:

Corollary. *Fix a character ω of $F^\times$ and an integer $j\geqq 0$. Suppose that H satisfies the conditions of the previous lemma and transforms on the right under $K_r(j)$ according to ω_j^{-1}. Suppose further that H has support in the set of $g\in G_r$ satisfying*

$$\varrho(g)\geqq C, \tag{2}$$

where C is a positive constant. Then the following conditions (A) *and* (B) *are equivalent:*

(A) $$H(g)=\bar{\theta}(n)\omega_j^{-1}(k), \quad \text{if } g=nk, \quad n\in N_r, \quad k\in K_r(j),$$
$$=0 \qquad =0 \quad \text{if } g\notin N_rK_r(j).$$

(B) *For any generic irreducible representation π of G_r with central character ω and conductor $\mathfrak{P}^f$, $f\leqq j$, one has*

$$\int_{N_r\backslash G_r} H(g)W(g)\,|\det g|^s dg=\int_{K_r(j)}\omega_j^{-1}(k)W(k)dk \tag{3}$$

for all $W\in\mathscr{W}(\pi;\psi)$.

The integral on the left is to be interpreted as a formal Laurent series

$$\sum_m X^m \int_{N_r\backslash G_r} H(g)W(g)\mu_m(g)dg \tag{4}$$

where $X=q^{-s}$, G_r^m is the set of $g\in G_r$ such that $|\det g|=q^{-m}$ and μ_m the characteristic function of G_r^m. Because of the assumption on the support of H the products $H\mu_m$ have compact support modulo N_r, so that the integrals in (4) are well defined (cf. [J-P-S] I and II).

Proof. Clearly (A) implies (B). Assume (B). If π is any irreducible generic representation of G_r with conductor $\mathfrak{P}^j$, $j<f$, then the integral

$$\int_{K_r(f)} \omega_f^{-1}(k)W(gk)dk$$

vanishes for all $W\in\mathscr{W}(\pi;\psi)$ and all $g\in G_r$. Thus both sides of equality (3) are then zero. Similarly, if π is any irreducible generic representation of G_r, both sides of (3) vanish, unless the central character ω_π of π agrees with ω on $\mathfrak{R}^\times$. Thus condition (B) is equivalent to another condition where equality (3) stands for *all* irreducible generic representations π of G_r and all $W\in\mathscr{W}(\pi;\psi)$. In view of the interpretation of the left hand side of (3) this means that

$$\int H\mu_m dg=0 \quad \text{for} \quad m\neq 0,$$

$$\int H\mu_0 dg=\int_{K_r(j)} \omega_j^{-1}(k)W(k)dk.$$

Applying the previous lemma to each function $H\mu_m$ we get assertion A. □

4. The Main Result

(4.1) Let π be an irreducible generic representation of G_r and σ an irreducible generic representation of G_t. In [J-P-S] I we have defined functions $L(s,\pi\times\sigma)$ and $\varepsilon(s,\pi\times\sigma,\psi)$; we have also set

$$\gamma(s,\pi\times\sigma,\psi)=\varepsilon(s,\pi\times\sigma,\psi)L(1-s,\tilde{\pi}\times\tilde{\sigma})/L(s,\pi\times\sigma). \tag{1}$$

We let ω_π (resp. ω_σ) be the central character of π (resp. σ).

Proposition. *Suppose π_i, $i=1,2$, (resp. σ) is an irreducible generic representation of G_r (resp. G_t) and $\omega_{\pi_1}=\omega_{\pi_2}$. There is an integer A with the following property: If χ is a character of $F^\times$ with conductor $\mathfrak{P}^a$, $a\geq A$, then*

$$\gamma(s,(\pi_1\otimes\chi)\times\sigma,\psi)=\gamma(s,(\pi_2\otimes\chi)\times\sigma,\psi). \tag{2}$$

(4.2) We first remark that, by definition, the factors attached to the pair $(\pi_i\otimes\chi,\sigma)$ where σ is a character of $F^\times$ are the same as the factors attached to $\pi_i\otimes\chi\sigma$ (Sect. 1). Thus we already know the proposition in case $t=1$.

We also remark that we may apply this result to an arbitrary irreducible generic representation π_1 of G_r and the irreducible generic component π_2 of a "principal series representation". More precisely, let $\chi_1,\chi_2,\ldots,\chi_r$ be characters of $F^\times$ whose product is ω_{π_1}; let B be the group of upper triangular matrices in G_r and π_2 the generic component of the induced representation

$$\xi=\mathrm{Ind}(G_r,B;\,\chi_1,\,\chi_2,\ldots,\chi_r).$$

By Theorem (3.1) of [J-P-S] I, the right hand side of (4.1.2) is nothing but

$$\prod_i \gamma(s,\sigma\otimes\chi_i\chi,\psi).$$

At this point we may apply the result for $t=1$ and we see that if $\chi_1\chi_2\ldots\chi_r=\omega_\pi$ and $\eta_1\eta_2\ldots\eta_t=\omega_\sigma$ then

$$\gamma(s,(\pi\otimes\chi)\times\sigma,\psi)=\prod_{i,j}\gamma(s,\chi_i\eta_j\chi,\psi), \tag{1}$$

provided a is large enough and χ has conductor $\mathfrak{P}^a$. In particular, we see that

$$\gamma(s, (\pi\otimes\chi)\times\sigma, \psi) = C_0 X^{art}, \qquad X = q^{-s},$$

where χ has conductor $\mathfrak{P}^a$, a is large enough, and C_0 depends only on χ, ω_{π_1}, and ω_σ.

(4.3) *Reduction to the Case* $t=r-1$. It is easy to see that we can reduce ourselves to the case when $t=r-1$. If $t<r-1$, we can consider an induced representation

$$\eta = \mathrm{Ind}(G_{r-1}, Q; \sigma, \eta_1, \eta_2, \ldots, \eta_{r-1-t})$$

where the η_i are characters of $F^\times$ and Q is a parabolic subgroup of type $(t, 1, \ldots, 1)$ in G_{r-1}. Then, if σ' is the unique irreducible generic component of η, we have (Theorem (13.1) of [J-S-P] I):

$$\begin{aligned}&\gamma(s, (\pi_i\otimes\chi)\times\sigma, \psi)\\ &\quad = \gamma(s, (\pi_i\otimes\chi)\times\sigma', \psi)\Big/\prod_j \gamma(s, \pi_i\chi\eta_j, \psi).\end{aligned} \tag{1}$$

Suppose Prop. (4.1) true for $t=r-1$. Recall it is true for $t=1$. Then the right hand side of (2) has the same value for $i=1$ and 2, provided the conductor of χ is deep enough and we get our assertion in the case (r, t), $t<r-1$. Similarly, in the case $t \geqq r$ we can consider an induced representation

$$\xi = \mathrm{Ind}(G_r, Q;\ \pi,\ \mu_1,\ \mu_2, \ldots, \mu_{t+1-r}),$$

to reduce ourselves to the case $t=r-1$.

(4.4) Next choose a character ξ of conductor $\mathfrak{P}^B$ such that, for $i=1,2$,

$$L(s, \pi_i\otimes\xi) = L(s, \tilde{\pi}_i\otimes\xi^{-1}) = 1,$$

$$\varepsilon(s, \pi_i\otimes\xi, \psi) = C_i q^{-rBs};$$

this is possible by the case $t=1$. At the cost of restricting a to be larger than B in the proposition, we may replace π_i by $\pi_i\otimes\xi$ and, therefore assume, for $i=1,2$: $L(s,\pi_i) = L(s,\tilde{\pi}_i) = 1$, $\varepsilon(s, \pi_i, \psi) = C_i q^{-fs}$. In particular π_1 and π_2 have the same conductor $\mathfrak{P}^f$. Moreover, the space $\mathscr{W}(\pi_i, \psi)$ contains exactly one vector $W_{i,0}$ transforming under ω_f and taking the value 1 on e: this is the "essential vector" of [J-P-S] II. It is clear from their construction (loc. cit.) that the functions $W_{i,0}$ agree on $P_r Z_r K_r(f)$.

We now prove the identity (4.1.2), provided the conductor of χ is $\mathfrak{P}^a$ where a is large enough. We set $\omega = \omega_{\pi_i}$. A simple formal manipulation shows that the functional equation which defines γ ([J-P-S] I, Theorem (2.7)) can be written in the form:

$$\omega_{\sigma\otimes\chi}(-1)^{r-1}\gamma(s, \pi_i\times(\sigma\otimes\chi))I_i = J_i, \tag{1}$$

with

$$I_i = \int W_i\left[\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix} w_r\right] W'(g)\,|\det g|^{s-1/2}dg, \tag{2}$$

$$J_i = \int W_i\left[\begin{pmatrix} 1 & 0\\ 0 & g\end{pmatrix}\right] W'(gw_{r-1})\,|\det g|^{s-1/2}dg, \tag{3}$$

and where W_i is in $\mathscr{W}(\pi_i;\psi)$ and W' in $\mathscr{W}(\sigma\otimes\chi;\bar{\psi})$. We will take W_i to be of the form

$$W_i(g)=\int W_{i,0}\left[g\begin{pmatrix}1 & u & v\\ 0 & 1 & 0\\ 0 & 0 & 1_{r-2}\end{pmatrix}\right]\varphi(u)\varphi_1(v)du dv\,, \tag{4}$$

where φ and φ_1 are Bruhat functions on F and F^{r-2} to be chosen below. We are going to show that φ, φ_1 and W' may be so chosen that $J_1=J_2\neq 0$ and then check that $I_1=I_2$. This will establish our assertion.

To proceed, in the expression for J_i we set

$$g=t\begin{pmatrix}1 & 0\\ 0 & h\end{pmatrix}\begin{pmatrix}1 & 0\\ x & 1_{r-2}\end{pmatrix}$$

with $t\in F^\times$, h in $N_{r-2}\backslash G_{r-2}$ and then we get

$$J_i=\int W_{i,0}\left[\begin{pmatrix}t^{-1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & h\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & x & 1_{r-2}\end{pmatrix}\right]$$
$$W'\left[\begin{pmatrix}1 & 0\\ 0 & h\end{pmatrix}\begin{pmatrix}1 & 0\\ x & 1_{r-2}\end{pmatrix}w_{r-1}\right]$$
$$\varphi^\wedge(t^{-1})\varphi_1^\wedge[-t^{-1}({}^tx)]\omega\chi^{r-1}\omega_\sigma(t)$$
$$|t|^{(r-1)(s-1/2)}|\det h|^{s+1/2}dx\,d^\times t\,dh\,. \tag{5}$$

Of course $\hat{\varphi}$ is the Fourier transform of φ.

Next we choose φ and φ_1 in such a way that

$$\hat{\varphi}(t)=\omega_\sigma\chi^{r-1}(t)\,,\quad \text{if}\quad |t|=1\,;\quad =0\ \text{otherwise}. \tag{6}$$

$$\hat{\varphi}_1(tx)=\omega(t)\hat{\varphi}_1(x)\,,\quad \text{if}\quad |t|=1\,. \tag{7}$$

$$\hat{\varphi}_1(x)\neq 0\quad \text{implies that} \tag{8}$$

$$\begin{pmatrix}1 & 0\\ {}^tx & 1_{r-2}\end{pmatrix}\in K_{r-1}(f)\,.$$

Then (5) becomes

$$J_i=\int_{N_{r-2}\backslash G_{r-2}} W_{i,0}\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & h\end{pmatrix}\right]F(h)\,|\det h|^{s+1/2}dh\,, \tag{9}$$

where we have set, for $h\in G_{r-2}$,

$$F(h)=\int W'\left[\begin{pmatrix}1 & 0\\ 0 & h\end{pmatrix}\begin{pmatrix}1 & 0\\ x & 1_{r-2}\end{pmatrix}w_{r-1}\right]\hat{\varphi}_1[-{}^tx]dx\,. \tag{10}$$

Furthermore the value of the right hand side of (9) does not change if we replace F by the function F_0 defined by

$$F_0(g)=\int_{K_{r-2}(f)} F(gk)\omega_f(k)dk\,. \tag{11}$$

The next step is to choose W' and φ_1 in such a way that

$$F_0(h)=\bar{\theta}_{r-2}(n)\omega_f^{-1}(k), \quad \text{if} \quad g=nk \tag{12}$$

with $n\in N_{r-2}$ and $k\in K_{r-2}(f)$;

$$=0 \quad \text{otherwise}.$$

Of course φ_1 is still constricted to satisfy (7) and (8). Then we will get from (9) the simple relation

$$J_i=cW_{i,0}(e), \quad i=1,2,$$

where c is a positive constant independent of i. Since $W_{i,0}(e)=1$ (loc. cit.) we will get $J_1=J_2\neq 0$, as required.

To proceed we remark that F_0 *satisfies the hypotheses of Corollary* (3.3), for it is a sum of functions of the form

$$h\mapsto W_j\begin{pmatrix}1 & 0\\ 0 & h\end{pmatrix}, \quad \text{with} \quad W_j\in\mathscr{W}(\sigma;\bar{\psi}).$$

By this corollary, in order to obtain condition (1.2), we need only choose φ_1 and W' in such a way that for any irreducible generic representation τ of G_{r-2}, with central character ω and conductor $\mathfrak{P}^j$, $j\leqq f$, the identity

$$\int_{N_{r-2}\backslash G_{r-2}} F_0(g)W''(g)\,|\det g|^s dg = \int_{K_{r-2}(f)} \omega_f^{-1}(k)W''(k)dk, \tag{13}$$

stands for any $W''\in\mathscr{W}(\tau;\psi)$. Without loss of generality, we may even consider only those W'' which, under right-shifts, transforms according to the character ω_f of $K_{r-2}(f)$. Then we may replace back F_0 by F without changing the left-hand side of (13).

To check that this choice of W' and φ_1 is possible, we start with the functional equation which defines the factor γ for the pair $(\sigma\otimes\chi,\tau)$. We take it in the form [cf. (1)]

$$\omega(-1)^{r-2}\gamma(s,(\sigma\otimes\chi)\times\tau,\psi)$$
$$\cdot\int W'\left[\begin{pmatrix}g & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}w_{r-2} & 0\\ 0 & 1\end{pmatrix}w_{r-1}\right]W''(g)\,|\det g|^{s-1/2}dg$$
$$=\int W'\left[\begin{pmatrix}1 & 0\\ 0 & g\end{pmatrix}\right]W''(g)\,|\det g|^{s-1/2}dg.$$

Taking into account the definition of F ((10)), we easily deduce the following identity:

$$\int F(g)W''(g)\,|\det g|^s dg=\omega(-1)^{r-2}\gamma(s+\tfrac{1}{2},(\sigma\otimes\chi)\times\tau,\psi)$$
$$\int W'\left[\begin{pmatrix}g & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}w_{r-2} & 0\\ 0 & 1\end{pmatrix}\right]\varphi_1[\eta_{r-2}g]W''(g)\,|\det g|^s dg; \tag{14}$$

and we have to prove this is equal to the right-hand side of (13), for an appropriate choice of W' and φ_1. Now by the induction hypothesis [cf. (4.2.2)]:

$$\gamma(s,(\sigma\otimes\chi)\times\tau,\psi)=C_0X^{a(r-1)(r-2)}, \quad X=q^{-s}, \tag{15}$$

where C_0 depends solely on χ, ω_σ, and $\omega_\tau=\omega$.

To proceed, we choose W' in such a way that the function

$$H'(g)=W'\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w_{r-2} & 0 \\ 0 & 1 \end{pmatrix}\right] \tag{16}$$

has support in the set $\tilde{\omega}^{-a(r-1)}N_{r-2}K_{r-2}(f)$ and is such that

$$H'(\tilde{\omega}^{-a(r-1)}k)=C_1C_0^{-1}q^{-a(r-1)(r-2)/2} \tag{17}$$

with

$$C_1=\omega(-1)^{r-2}\omega(\tilde{\omega}^{a(r-1)})$$

and C_0 as in (15). Finally, we specify φ_1 by requiring that, for $x\in F^{r-3}$, $y\in F$,

$$\varphi_1(x,y)=\omega^{-1}(d), \quad \text{if}$$

$$x\equiv 0 \bmod \mathfrak{P}^{-a(r-1)+f}, \quad y=\tilde{\omega}^{-a(r-1)}d, \quad |d|=1, \tag{18}$$

and that φ_1 be zero otherwise. Then (7) is satisfied in any case. Moreover $\hat{\varphi}_1$ has support in the set of $y\equiv 0 \bmod \mathfrak{P}^{a(r-1)-f}$, so that condition (8) is also satisfied, provided a is large compared to f.

With these choices then we see finally that the right-hand side of (14) reduces to the right-hand side of (13). Hence we have established that $J_1=J_2\neq 0$.

It remains to see that $I_1=I_2$ [cf. (2)]. It suffices to show that the functions

$$g\mapsto W_i\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} w_r\right]$$

agree. Since $W_{1,0}$ and $W_{2,0}$ agree on the subset $P_rZ_rK_r(f)$ and W_i is related to $W_{i,0}$ by (4), it will be enough to show that

$$\varphi(u)\varphi_1(x,y)\neq 0, \quad u\in F, \quad x\in F^{r-3}, \quad y\in F, \tag{19}$$

implies that the matrix

$$w_r\begin{pmatrix} 1 & u & x & y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1_{r-3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

belongs to $P_rZ_rK_r(f)$. But this matrix can also be written as the following product:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & w_{r-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & y^{-1} \\ 0 & 1_{r-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -y^{-1} & 0 & 0 \\ 0 & 1_{r-2} & 0 \\ 0 & 0 & y \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{r-2} & 0 \\ y^{-1} & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & u & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1_{r-3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first three matrices are in $P_r Z_r$. The product of the last two can also be written as

$$\begin{pmatrix} 1 & u & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1_{r-3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1_{r-3} & 0 \\ y^{-1} & y^{-1}u & y^{-1}x & 1 \end{pmatrix}. \tag{21}$$

The first of these two matrices is in P_r. Finally we may assume a is so large that $\mathfrak{P}^a$ is contained in the conductor of ω_σ. Then [formula (6)] $\hat{\varphi}$ is constant on the cosets of $\mathfrak{P}^a$ in F. Thus φ is supported on $\mathfrak{P}^{-a}$ and in the second matrix $u \in \mathfrak{P}^{-a}$. By (18) $y^{-1} \in \tilde{\omega}^{a(r-1)}\mathfrak{R}^\times$. Thus $y^{-1}u \in \mathfrak{P}^{a(r-2)} \subset \mathfrak{P}^a$ since $r>2$. Assuming $a \geqq f$ we have $y^{-1}u \in \mathfrak{P}^f$ and $y^{-1} \in \mathfrak{P}^f$. Finally by (18) again, x is in $\mathfrak{P}^{-a(r-1)+f}$ so $y^{-1}x$ is in $\mathfrak{P}^f$. Thus the second matrix in (21) is in $K_r(f)$ and with that the proof is complete. □

5. Complements

(5.1) **Proposition.** *Let π and σ be two irreducible generic representations of G_r and G_t respectively. If a is large enough and χ has conductor $\mathfrak{P}^a$ then*

$$L(s, (\pi \otimes \chi) \times \sigma) = 1.$$

Proof. Suppose first that π and σ are supercuspidal. One may as well assume they are preunitary and χ is a character of module one. Then, by [J-P-S] II Proposition (8.3), the factors

$$L(s, (\pi \otimes \chi) \times \sigma), \quad L(s, (\tilde{\pi} \otimes \chi^{-1}) \times \tilde{\sigma})$$

have no pole in the region $\mathrm{Re}(s) > 0$. In particular the fraction

$$L(1-s, (\tilde{\pi} \otimes \chi^{-1}) \times \tilde{\sigma}) / L(s, (\pi \otimes \chi) \times \sigma)$$

is in irreducible form. Since it is equal to the γ-factor, up to a monomial factor, we see that it is itself a monomial, if the conductor of χ is deep enough. Then

$$L(s, (\pi \otimes \chi) \times \sigma) = 1.$$

In general π and σ are components of induced representations

$$\xi = \mathrm{Ind}(G_r, Q; \pi_1, \pi_2, \ldots, \pi_m),$$
$$\eta = \mathrm{Ind}(G_t, S; \sigma_1, \sigma_2, \ldots, \sigma_n),$$

where the π_i and the σ_j are supercuspidal. Then:

$$L(s, (\pi \otimes \chi) \times \sigma) = P_\chi(q^{-s}) \prod_{i,j} L(s, (\pi_i \otimes \chi) \times \sigma_j)$$

where P_χ is a polynomial ([J-P-S] I, Theorem (3.1)) and our assertion follows.

(5.2) *Remark.* Even if π and σ are not generic it is possible to define the factors L and ε ([J-P-S] I). Propositions (5.1) and (4.1) are then true for all pairs.

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