CHAPTER 15

INTEGRAL REPRESENTATION OF WHITTAKER FUNCTIONS

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To Joseph Shalika

1. Introduction. Recently, the converse theorem has been used to prove spectacular results in the theory of automorphic representations for the group $GL(n)$ (see [KS], [CKPSS] for instance). The converse theorem ([CPS1] & [CPS2]) is based in part on a careful analysis of the properties of the Rankin-Selberg integrals at infinity ([JS]). The simplest example of such an integral takes the form

$$\Psi(s, W, W') = \int_{N_n\backslash G_n} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] \mid \det g \mid^{s-1/2} W'(g)dg.$$

Here W is in the Whittaker model $\mathcal{W}(\pi, \psi)$ of a unitary generic representation π of $GL(n, F)$ and W' in $\mathcal{W}(\pi', \overline{\psi})$ where π' is a unitary generic representation of $GL(n-1, F)$ (see below for unexplained notations). One of the difficulties of the theory is that the representations π and π' need not be tempered. Thus one is led to consider holomorphic fiber bundles of representations (π_u) and $(\pi'_{u'})$, for instance, non-unitary principal series. Correspondingly, the functions $W = W_u$ and $W' = W'_{u'}$ depend also on u and u'. They are associated with sections of the fiber-bundle of representations at hand. Rather than standard sections (with a constant restriction to the maximal compact subgroup), we consider convolutions of standard sections with smooth functions of compact support. It is difficult to prove that the integrals $\Psi(s, W_u, W'_{u'})$ are meromorphic functions of (s, u, u'). The elaborate technics of [JS] were designed to go around this difficulty. In particular, there, the analytic properties of the integral as functions of s, as well as their functional equations, were found to be equivalent to a family of identities (depending on (u, u')) which were then established by analytic continuation with respect to the parameters (u, u'). In the present note, we first find integral representations for W_u and $W'_{u'}$ which converge for **all values** of the parameters (u, u'). In particular, it is easy to obtain estimates for W_u and $W'_{u'}$ which are uniform in (u, u'). Then, using these integral representations, we show that the integrals at hand are meromorphic functions of (s, u, u'). This being established, one can use the methods of [JS] to prove the functional equations. We do not repeat this step here because it is now much easier: one first proves that the integrals are meromorphic functions

Research supported in part by NSF grant DMS 9619766.

of (s, u, u') and then one proves the functional equation. In contrast, in [JS], we add to prove simultaneously the analytic continuation and the functional equation. Moreover, it was difficult to obtain estimates for the functions. Having the functional equation at our disposal, we obtain the more precise result that the integral is a holomorphic function of (s, u, u') times the appropriate Γ factor. We emphasize that the Γ factor is itself a meromorphic function of (s, u, u'). Thus the proofs here are much simpler than in [JS].

Another advantage of the present approach is that we obtain directly the properties of the integrals for smooth vectors. In [JS], we first established the properties of the integrals for K_n-finite vectors and then used the automatic continuity theorem (Casselman and Wallach, see [W] 11.4) to extend the results to smooth vectors. Needless to say, we use extensively (and most of the time implicitly) the existence of a canonical topological model for representations of $GL(n)$ ([W], Chapter 11).

In addition, we have now at our disposal the very complete results on the Whittaker integrals contained in the remarkable book of N. Wallach ([W], Chapter 15).

Since the publication of [JS], there have been several papers containing related results ([D], [S], [St]).

In this note we will not discuss the more subtle question of proving that the appropriate Γ factor can be obtained in terms of the integral. In [JS], it is established that the Γ factor, in other words, the factor $L(s, \pi \times \pi')$, is equal to an integral of the form

$$\int W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, g\right] |\det g|^{s-1/2}\, dg,$$

where W is a function on $GL(n, F) \times GL(n-1, F)$ which belongs to the Whittaker model of $\pi \otimes \pi'$. In other words, the function W corresponds to a smooth vector for the representation $\pi \otimes \pi'$ which needs not be of the form $\sum_i v_i \otimes v_i'$. One could use this to prove directly that the global L-function $L(s, \pi \times \pi')$ is entire and bounded in vertical strips and similarly for the other Rankin-Selberg integrals. Of course, this is no longer needed as direct proofs from the theory of Eisenstein series are now available ([GS], see also [RS]). Nonetheless, the following question is still of interest, namely, to show that

$$L(s, \pi \times \pi') = \sum_i \Psi(s, W_i, W_i'),$$

where the functions W_i and W_i' are respectively K_n and K_{n-1} finite. We will show this is the case in another paper. A more subtle question is to identify precisely the functions W_i and W_i' (see [St1] & [St2] for a special case). The integrals attached to the pair of integers (n, n) have analogous properties. However, for the Rankin-Selberg integrals attached to pairs (n, m) with $m < n - 1$ the L-factor cannot be obtained in terms of K-finite vectors but only in terms of smooth vectors in the tensor product representation ([JS]).

Acknowledgements. Finally, it is a pleasure to dedicate this note to Joseph Shalika as a memento of a long, fructuous, and most enjoyable collaboration. The proof given here is similar to, but different of, the original unpublished proof of the results of [JS]. Moreover, a suggestion of Piatetski-Shapiro is used in a somewhat different form. Thus I must thank both of my former collaborators without being able to pinpoint precisely their contribution.

The paper is arranged as follows. In section 2, we review results on Whittaker linear forms for the principal series. In section 3, we present the main ideas of our construction in the case of the group $GL(2)$. Section 4 contains auxiliary but crucial results. In section 5, we give an integral representation for our sections which leads to the integral representation of Whittaker functions in section 6. In section 7 we give the main properties of the Whittaker functions. In sections 8 to 10, we prove the main result on Rankin-Selberg integrals for the pairs (n, n). Finally, in section 11, we give a few indications on how to treat the case of the Rankin-Selberg integrals for the other cases.

2. Whittaker linear form for principal series. Let F be the field of real or complex numbers. We denote by $| z |_F$ or simply $\alpha_F(z)$ the module of $z \in F$. Thus $\alpha_F(z) = z\overline{z}$ if F is complex. We will denote by ψ a non-trivial additive character of F and by dx the corresponding self-dual Haar measure on F. We will set

$$d^\times x = \frac{dx}{| x |_F} L(1, 1_F).$$

We denote by $\mathbb{U}_1$ the subgroup of $z \in \mathbb{C}$ such that $z\overline{z} = 1$.

We will denote by G_n the group $GL(n, F)$, by A_n the subgroup of diagonal matrices, by B_n the group of upper triangular matrices, by N_n the group of upper triangular matrices with unit diagonal. We write $\overline{B}_n$ and $\overline{N}_n$ for the corresponding groups of lower triangular matrices. All these groups are regarded as algebraic groups over F. We denote by K_n the standard maximal compact subgroup of G_n. We often write G_n for $G_n(F)$ and so on for the other groups. We define a character $\theta_n : N_n \to \mathbb{U}_1$ by

$$\theta_n(v) = \psi\left(\sum_{1 \le i \le n-1} v_{i,i+1}\right).$$

When there is no confusion, we often drop the index n from the notations.

We shall say that a character of module 1 of $F^\times$ is normalized, if its restriction to $\mathbb{R}_+^\times$ is trivial. Hence if $F = \mathbb{R}$ every normalized character is either trivial or of the form $t \mapsto t \,| t |^{-1}$. If $F = \mathbb{C}$ then every normalized character has the form

$$z \mapsto \left(\frac{z}{(z\overline{z})^{\frac{1}{2}}}\right)^n \text{ or } z \mapsto \left(\frac{\overline{z}}{(z\overline{z})^{\frac{1}{2}}}\right)^n$$

with $n \in \mathbb{N}$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be a n-tuple of normalized characters of $F^\times$. Given u= $(u_1, u_2, \ldots, u_n)$ a n-tuple of complex numbers, we consider the representation of $G = GL(n, F)$

$$\Xi_u = I(\mu_1\alpha^{u_1}, \mu_2\alpha^{u_2}, \dots, \mu_n\alpha^{u_n}) \tag{1}$$

induced by the quasi-characters $\mu_i\alpha^{u_i}$ and the group $\overline{B}_n$. A function f in the space of the representation is a smooth function f on $G_n(F)$ with complex values such that

$$f(\overline{v}ag) = \mu(a) \prod_{1\leq\leq i\leq n-1} \mid a_i \mid^{u_i - \frac{n-i}{2}} f(g),$$

for all $v \in \overline{N}_n, a = \mathrm{diag}(a_1, a_2, \dots, a_n) \in A_n$ and $g \in G_n$; we have written $\mu(a) = \prod_i \mu_i(a_i)$.

We will use Arthur's standard notations. Therefore let

$$\mathfrak{a}^* = \mathrm{Hom}_F(A_n, F^\times) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$$

be the real vector space generated by the algebraic characters of A_n. Let $\mathfrak{a}$ be the dual vector space. Let $\rho \in \mathfrak{a}^*$ be the half sum of the roots positives for B_n (i.e., the roots in N_n). We also denote by α_i the simple roots. We have a map $H : G_n \to \mathfrak{a}$ defined by $e^{\langle H(a),u\rangle} = \prod \mid a_i \mid^{u_i}$ for $a \in A_n$, and $H(vak) = H(a)$, for $v \in N_n,\ k \in K_n,\ a \in A_n$. We also define $H' : G_n \to \mathfrak{a}$ by $H'(\overline{v}ak) = H(a)$, for $v \in \overline{N}_n, k \in K_n, a \in A_n$. Thus a function f in the space of the representation is a smooth function such that

$$f(g) = f(\overline{v}ak) = \mu(a)f(k)e^{\langle H'(g),u-\rho\rangle>}.$$

Such a function is determined by its restriction to K_n. Let therefore $V(\mu)$ be the space of smooth functions $f : K_n \to \mathbb{C}$ such that

$$f(ak) = \prod \mu_i(a_i)f(k)$$

for all $a \in A \cap K$. If $f \in V(\mu)$ then for every u the function f_u defined by

$$f_u(\overline{v}ak) = f(k)e^{\langle H(a),u-\rho\rangle}$$

is in the induced representation. Such a section of the fiber bundle of the representations is called a **standard section.**

Suppose that f_u is a standard section. Let ϕ be a smooth function of compact support on $G(F)$. Then the function $f_{\phi,u}$ defined by

$$f_{\phi,u}(g) := \int_{G(F)} f_u(gx)\phi(x)dx \tag{2}$$

is, for every u, an element of the corresponding induced representation. More precisely, its value on $g \in K_n$ is given by

$$\int_{G(F)} f_u(x)\phi(g^{-1}x)dx = \int_{N\times K\times A^+} f(k)\phi(g^{-1}\overline{n}ak)e^{\langle H(a),u-\rho\rangle}d\overline{n}dadk.$$

We often say that such a section is a **convolution section.** For $f \in V(\mu)$, we define the **Whittaker integral**

$$\mathcal{W}_u(f) := \int f_u(n)\overline{\theta}(n)dn. \tag{3}$$

It converges when $\Re(u_{i+1} - u_i) > 0$ for $1 \leq i \leq n-1$ and extends to an entire function of u ([W], Chapter 11). We claim the same is true of the integral

$$\int f_{\phi,u}(n)\overline{\theta}(n)dn. \tag{4}$$

Indeed, we appeal to a simple lemma, which will be constantly used in this paper.

LEMMA 1. *Let V be a locally convex complete topological vector space. Let Ω be an open subset of $\mathbb{C}^n$. Suppose that $A : \Omega \to V$ is a continuous holomorphic map. Suppose that we are given a map $\lambda : \Omega \times V \to \mathbb{C}$, which is continuous and such that, for every $v \in V$, the map $s \mapsto \lambda(s, v)$ is holomorphic, and for every s, the map $v \mapsto A(s, v)$ is linear. Then the map $s \mapsto \lambda(s, A(s))$ is holomorphic.*

Proof. Indeed the map $(s_1, s_2) \mapsto \lambda(s_1, A(s_1))$ from $\Omega \times \Omega$ to $\mathbb{C}$ is continuous and separately holomorphic. Hence it is holomorphic. Thus its restriction to the diagonal is holomorphic. □

We will write $\mathcal{W}_u(f_u)$ and $\mathcal{W}_u(f_{\phi,u})$ for the above integrals. Recall ([W], Chapter 11) that for a given u, the integral (3) defines a **non-zero** linear form on the space $V(\mu)$. Moreover, within a constant factor, it is the only continuous linear form $\mathcal{W}$ such that, for all $v \in N_n$,

$$\mathcal{W}(\Xi_u(v)f) = \theta(v)\mathcal{W}(f).$$

We denote by $\mathcal{W}(\Xi_u, \psi)$ the space spanned by the functions

$$g \mapsto \mathcal{W}_u(\Xi_u(g)f_u).$$

At least when Ξ_u is irreducible, this is the **Whittaker model** of Ξ_u.

3. The case of $GL(2)$. For an introduction, we review the case of $GL(2)$ which was considered in previous papers (for instance [GJR]). Let $f_u(\bullet)$ be a section, for instance, a standard section. Thus

$$f_u\left[\begin{pmatrix} a_1 & 0 \\ x & a_2 \end{pmatrix} k\right] = \mu_1(a_1) \mid a_1 \mid^{u_1 - 1/2} \mu_2(a_2) \mid a_2 \mid^{u_2 + 1/2} f_u(k).$$

If ϕ is a smooth function of compact support then the convolution of f_u and ϕ is a new section $f_{\phi,u}$ defined by

$$f_{\phi,u}(g) = \int f_u(gx)\phi(x)dx = \int f(x)\phi(g^{-1}x)dx = \int f_u(x^{-1})\check{\phi}(xg)dx$$

where we set $\check{\phi}(g) := \phi(g^{-1})$. To compute this integral we set

$$x = k\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}.$$

Then

$$dx = dkdyd^\times a_1 d^\times a_2$$

and the integral becomes

$$\int f_u(k^{-1})\check{\phi}\left[k\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}g\right]$$
$$\mu_1(a_1)^{-1} \mid a_1 \mid^{-u_1+1/2} \mu_2(a_2) \mid a_2 \mid^{-u_2-1/2} dkdyd^\times a_1 d^\times a_2.$$

Let us set

$$g_u(g) = \int f(k^{-1})\check{\phi}\left[k\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & a_2\end{pmatrix}g\right]$$
$$\mu_2\left(\det\left[\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & a_2\end{pmatrix}g\right]\right)\left|\det\left[\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & a_2\end{pmatrix}g\right]\right|^{-u_2-1/2}$$
$$dkdyd^\times a_2.$$

This function of g is invariant on the left under the subgroup P of matrices of the form

$$\begin{pmatrix}1 & 0\\ * & *\end{pmatrix}.$$

Thus there is a function $\Phi_u[\bullet]$ on F^2 such that

$$\Phi_u[(1,0)g] = g_u(g).$$

In particular, the function

$$(x, y) \mapsto \Phi_u[(x, y)]$$

has support in a fixed compact set of $F^2 - 0$. Then

$$f_{\phi,u}(g) = \int g_u\left[g\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\right]\mu_2\mu_1^{-1}(t) \mid t \mid^{u_2-u_1+1} d^\times t\, \mu_2(\det g) \mid \det g \mid^{u_2+1/2}$$
$$= \int \Phi_u[(t,0)g] \mid t \mid^{u_2-u_1+1} d^\times t\, \mu_2(\det g) \mid \det g \mid^{u_2+1/2}.$$

We now compute

$$\mathcal{W}_u(f_{\phi,u}) = \int f_{\phi,u}\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]\psi(-x)dx$$
$$= \int\int \Phi_u\left[(t, tx)\right]\mu_2\mu_1^{-1}(t) \mid t \mid^{u_2-u_1+1} d^\times t\psi(-x)dx.$$

This double integral converges absolutely for $\Re u_2 > \Re u_1$. If we change x to xt^{-1} we get:

$$\int \mathcal{F}_1(\Phi_u)[(t, t^{-1})] \mid t \mid^{u_2-u_1} d^\times t$$

where we denote by a $\mathcal{F}_1$ (or simply $\hat{\Phi}$) the partial Fourier transform with respect to the second variable:

$$\mathcal{F}_1(\Phi)(x, y) = \int \Phi(x, z)\psi(-zx)dz.$$

It is then clear that this new integral converges for all values of u. For the case of $GL(2)$, it is the integral representation alluded to in the introduction.

To obtain a more general formula, we introduce two representations l_2 and $\hat{l}_2$ of $GL(2, F)$ on $\mathcal{S}(F^2)$ by

$$l_2(g)\Phi(X) = \Phi(Xg)\,,\ \hat{l}_2(g)\mathcal{F}_1(\Phi) = \mathcal{F}_1(l_2(g)\Phi).$$

Then $W_u(g) := \mathcal{W}_u(\rho(g)f_{\phi,u})$ is given by

$$W_u(g) = \int \hat{l}_2(g)\mathcal{F}_1(\Phi_u)[(t, t^{-1})]\mu_2\mu_1^{-1}(t) \mid t \mid^{u_2-u_1} d^\times t$$
$$\mu_2(\det g) \mid \det g \mid^{u_2+\frac{1}{2}}\,.$$

For instance, if $\alpha = \operatorname{diag}(\alpha_1, \alpha_2)$, then

$$W_u(\alpha) = \int \mathcal{F}_1(\Phi_u)\big[\big(t\alpha_1, t^{-1}\alpha_2^{-1}\big)\big]\mu_2\mu_1^{-1}(t) \mid t \mid^{u_2-u_1} d^\times t$$
$$\mu_2(\alpha_1\alpha_2) \mid \alpha_1\alpha_2 \mid^{u_2} \mid \alpha_1 \mid^{1/2} \mid \alpha_2 \mid^{-1/2}\,.$$

4. A reduction step. We will need an elementary but crucial result which in the case of $GL(2)$ describes the space of function represented by the integrals $\mathcal{W}_u(\Xi_u(g)f_{\phi,u})$. We will set $\mathbb{Z}_F = \mathbb{Z}$ if F is real and $\mathbb{Z}_F = \mathbb{Z}/2$ if F is complex.

Proposition 1. *Let Ω be an open, connected, relatively compact set of $\mathbb{C}$, the closure of which is contained in $\mathbb{C} - \mathbb{Z}_F$. Let $\mathbb{X}$ be an auxiliary real vector space of finite dimension. Given $\Phi \in \mathcal{S}(F^2 \oplus \mathbb{X})$ and $u \in \Omega$ there are $\Phi_{1,u}$ and $\Phi_{2,u}$ in $\mathcal{S}(F \oplus \mathbb{X})$ such that, for all $u \in \Omega$,*

$$w_{\Phi,u}(t_1, t_2 : X) := \int \Phi(t_1t, t_2t^{-1} : X) \mid t \mid^u \mu(t)d^\times t$$

is equal to

$$\Phi_{1,u}(t_1t_2 : X) \mid t_1 \mid^{-u} \mu^{-1}(t_1) + \Phi_{2,u}(t_1t_2 : X) \mid t_2 \mid^u \mu(t_2).$$

One can choose the functions $\Phi_{i,u}(t : X)$ in such a way that the maps $u \mapsto \Phi_{i,u}(\bullet : \bullet)$ are holomorphic maps from Ω to $\mathcal{S}(F \oplus \mathbb{X})$. Furthermore, if Φ remains in a bounded set, then the functions $\Phi_{i,u}$ remain in a bounded set. Finally, if $\Phi = \Phi_s$ depends *holomorphically on $s \in \Omega'$, Ω' open in $\mathbb{C}^n$, and remains in a bounded set for all values of s, one can choose the functions Φ_* to depend holomorphically on (u, s) and to remain in a bounded set.*

Proof. We begin the proof with a variant of the Borel lemma, as expounded in [H], Theorem 1.2.6. □

LEMMA 2. *Let f_j be a sequence of holomorphic functions on an open, connected, set Ω of $\mathbb{C}^n$. Assume each f_j is bounded. For any $\epsilon > 0$ there is a smooth function $f(u, t)$ on $\Omega \times]-\epsilon, \epsilon[$ with the following properties. The function is holomorphic in u and, for each $j \geq 0$,*

$$\frac{\partial f^j}{\partial t^j}(u, 0) = f_j(u).$$

Finally the support of f has a compact projection on the second factor.

Proof of the Lemma. We choose a smooth function g of compact support contained in $]-\epsilon, \epsilon[$, with $g(t) = 1$ for t sufficiently close to 0. Next we choose $0 < \epsilon_j < 1$ and set

$$g_j(u, t) = g\left(\frac{t}{\epsilon_j}\right)\frac{t^j}{j!} f_j(u).$$

We have then, for $\alpha < j$,

$$\left|\frac{\partial^\alpha g_j(u, t)}{\partial t^\alpha}\right| \leq C_{\alpha,j} \sup_\Omega \mid f_j \mid \epsilon_j^{j-\alpha} \leq C_j \sup_\Omega \mid f_j \mid \epsilon_j$$

where $C_j = \sup_{\alpha<j} C_\alpha$ depends only j (and our choice of g). If

$$\epsilon_j \leq \frac{1}{C_j \sup_\Omega \mid f_j \mid 2^j}$$

then, for all $\alpha < j$,

$$\left|\frac{\partial t^\alpha g_j(u, t)}{\partial t^\alpha}\right| \leq 2^{-j}.$$

The function

$$f(u, t) = \sum_j g_j(u, t)$$

has the required property. □

In the same way, one proves the following lemma.

LEMMA 3. *Let $f_{j,k}$ be a double sequence of holomorphic functions on an open set Ω of $\mathbb{C}^n$. Assume each $f_{j,k}$ is bounded. For any open disc D of center 0 in $\mathbb{C}$,* *there is a smooth function $f(u, t)$ on $\Omega \times D$ with the following properties. The function is holomorphic in u and, for each $j \geq 0, k \geq 0$,*

$$\frac{\partial f^{j+k}}{\partial t^j \partial \bar{t}^k}(u, 0) = f_{j,k}(u).$$

Finally the support of f has a compact projection on the second factor.

Another variant of the Borel lemma is as follows.

Lemma 4. *Let f_j be a sequence of Schwartz functions on some vector space $\mathbb{U}$. For any $\epsilon > 0$ there is a Schwartz function f on $\mathbb{U} \oplus \mathbb{R}$, supported on $\mathbb{U} \times [-\epsilon, \epsilon]$ such that, for every $j \geq 0$,*

$$\frac{\partial f^j}{\partial t^j}(u, 0) = f_j(u).$$

Indeed, we may view a Schwartz function on $\mathbb{U}$ as a smooth function on the sphere S in $\mathbb{U} \oplus \mathbb{R}$ which, in addition, vanishes as well as all its derivatives at some point $P_0 \in S$. We construct the required function f as before as a sum of a series

$$f(s, t) = \sum_j g\left(\frac{t}{\epsilon_j}\right)\frac{t^j}{j!} f_j(s)$$

which can be differentiated term wise. In particular f and all its derivatives in the s variables vanish at any point of the form (P_0, t).

We go back to the proof of the proposition. To be definite we assume that F is complex. The real case is somewhat simpler. We recall that an integral

$$\int \Phi(t) \mid t \mid^s d^\times t$$

converges absolutely for $\Re s > 0$ and extends meromorphically with a simple pole at $s = 0$ and residue $\Phi(0)$. Let ξ be a normalized character. To find the poles and residues of an integral of the form

$$\int \Phi(t) \mid t \mid^s \xi(t) d^\times t$$

in the half plane $\Re s > -M - 1$, we choose a smooth function of compact support ϕ_0 on F equal to 1 near 0. We write

$$\begin{aligned}\Phi(t) &= \phi_0(t)\Phi(t) + (1 - \phi_1)\Phi(t)\\ &= \phi_0(t) \sum_{n_1+n_2 \leq M} \frac{t^{n_1}\bar{t}^{n_2}}{n_1!n_2!} \frac{\partial^{n_1+n_2}\Phi}{\partial t^{n_1} \partial \bar{t}^{n_2}}(0) + r(t).\end{aligned}$$

Then the integral is the sum of

$$\int r(t) \mid t \mid^s \xi(t) d^\times t$$

which has no pole in the half plane and

$$\sum_{n_1+n_2 \le M} \frac{1}{n_1! n_2!} \frac{\partial^{n_1+n_2}\Phi}{\partial t^{n_1} \partial \overline{t}^{n_2}}(0) \int \phi_0(t) \mid t \mid^s \xi(t) t^{n_1} \overline{t}^{n_2} d^\times t.$$

Each term contributes (at most) one simple pole at any point s such that there are integers $n_1 \ge 0, n_2 \ge 0$ with

$$\mid t \mid^s \xi(t) t^{n_1} \overline{t}^{n_2} \equiv 1.$$

The residue is then

$$\frac{1}{n_1! n_2!} \frac{\partial^{n_1+n_2}\Phi}{\partial t^{n_1} \partial \overline{t}^{n_2}}(0).$$

For instance, if $\xi(z) = (z/\sqrt{z\overline{z}})r$ with $r \ge 0$, there is a pole at any point $-\frac{r}{2} - n$ with $n \ge 0$ integer. The residue is given by the above formula with $n_1 = n$ and $n_2 = n + r$.

Coming back to our integral, we will first prove the Proposition when there is no auxiliary space $\mathbb{X}$ and no dependence on some complex parameter s. We remark that, after a change of variables, we can reduce ourselves to the case where $t_2 = 1$, in other words, study the function $w_u(a) := w_{\Phi,u}(a, 1)$. We then have to show that

$$w_u(a) = \Phi_{1,u}(a) \mid a \mid^{-u} \mu^{-1}(a) + \Phi_{2,u}(a).$$

After a change of variables, we find

$$\begin{aligned} &\int w_u(a) \mid a \mid^s \xi(a) d^\times a \\ &= \int\int \Phi(t_2, t_1) \mid t_2 \mid^s \xi(t_2) d^\times t_2 \mid t_1 \mid^{s-u} \xi\mu^{-1}(t_1) d^\times t_1. \end{aligned}$$

Consider for one moment the following function of two variables:

$$A(s_2, s_1) := \int\int \Phi(t_2, t_1) \mid t_2 \mid^{s_2} \xi(t_2) d^\times t_2 \mid t_1 \mid^{s_1} \xi\mu^{-1}(t_1) d^\times t_1.$$

It is meromorphic function of (s_2, s_1) with hyperplane singularities. There are singularities on some of the hyperplanes

$$s_2 + z_2 = 0,\ z_2 \in \mathbb{Z}_F, z_2 \ge 0$$

and on some of the hyperplanes

$$s_1 + z_1 = 0,\ z_1 \in \mathbb{Z}_F, z_1 \ge 0.$$

Now consider the function $A(s, s-u)$. In $\mathbb{C}^2$, a singular hyperplane

$$s + z_2 = 0$$

and a singular hyperplane

$$s - u + z_1 = 0$$

intersect only at points where $u \in \mathbb{Z}_F$. In particular, if u is not in $\mathbb{Z}_F$, then $A(s, s-u)$, viewed as a function of s, has only simple poles. At a pole where s is such that

$$| t_2 |^s \ \xi(t_2) t_2^{n_1} \overline{t}_2^{n_2} \equiv 1$$

the residue is equal to

$$\frac{1}{n_1! n_2!} \int \frac{\partial^{n_1+n_2}\Phi}{\partial t_2^{n_1} \partial \overline{t}_2^{n_2}}(0, t_1) \mid t_1 \mid^{-u} \ t_1^{-n_1} \overline{t}_1^{-n_2} \mu^{-1}(t_1) d^\times t_1.$$

Note that the integral defines a holomorphic function of u in the complement of $\mathbb{Z}_F$.

Likewise, at a pole where s is such that

$$| t_1 |^{s-u} \ \xi \mu^{-1}(t_1) t_1^{m_1} \overline{t}_1^{m_2} \equiv 1$$

the residue is equal to

$$\frac{1}{m_1! m_2!} \int \frac{\partial^{m_1+m_2}\Phi}{\partial t_1^{m_1} \partial \overline{t}_1^{m_2}}(t_2, 0) \mid t_2 \mid^{u} \ \mu(t_2) t_2^{-m_1} \overline{t}_2^{-m_2} d^\times t_2.$$

Applying the variant of Borel lemma given above, we can find smooth functions of compact support $\Phi_{2,u}(a)$ and $\Phi_{1,u}(a)$, depending holomorphically on u, such that, for all $u \in \Omega$,

$$\text{(5)} \qquad \frac{\partial^{n_1+n_2}\Phi_{2,u}}{\partial a^{n_1} \partial \overline{a}^{n_2}}(0) = \int \frac{\partial^{n_1+n_2}\Phi}{\partial t_2^{n_1} \partial \overline{t}_2^{n_2}}(0, t_1) \mid t_1 \mid^{-u} \ t_1^{-n_1} \overline{t}_1^{-n_2} \mu^{-1}(t_1) d^\times t_1$$

and

$$\text{(6)} \qquad \frac{\partial^{m_1+m_2}\Phi_{1,u}}{\partial a^{m_1} \partial \overline{a}^{m_2}}(0) = \int \frac{\partial^{m_1+m_2}\Phi}{\partial t_1^{m_1} \partial \overline{t}_1^{m_2}}(t_2, 0) \mid t_2 \mid^{u} \ \mu(t_2) t_2^{-m_1} \overline{t}_2^{-m_2} d^\times t_2.$$

The difference between

$$\int w_u(a) \mid a \mid^s \ \xi(a) d^\times a$$

and

$$\text{(7)} \qquad B_u := \int \Phi_{2,u}(a) \mid a \mid^s \ \xi(a) d^\times a + \int \Phi_{1,u}(a) \mid a \mid^{s-u} \ \xi \mu^{-1}(a) d^\times a$$

is then, by construction, an entire function of s. It is rapidly decreasing in a vertical strip and also rapidly decreasing with respect to ξ. It follows there is a smooth function $\Psi_u(a)$ on F, rapidly decreasing at infinity, with zero derivatives at 0 such that the above difference is

$$\int \Psi_u(a) \mid a \mid \xi(a) d^\times a$$

for all ξ. Furthermore, the function depends holomorphically on u. We obtain our claim with $\Phi_{2,u}$ replaced by $\Phi_{2,u} + \Psi_u$.

If there is an auxiliary vector space X and dependence on a complex parameter s we use the variants of the Borel lemma to choose the functions $\Phi_{i,u}(x : X : s)$ in such a way that

$$\begin{aligned} &\frac{\partial^{n_1+n_2}\Phi_{2,u}}{\partial a^{n_1}\partial \overline{a}^{n_2}}(0 : X : s) \\ &\quad = \int \frac{\partial^{n_1+n_2}\Phi}{\partial t_2^{n_1}\partial \overline{t}_2^{n_2}}[(0, t_1) : X : s] \mid t_1 \mid^{-u} t_1^{-n_1}\overline{t}_1^{-n_2}\mu^{-1}(t_1) d^\times t_1 \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^{m_1+m_2}\Phi_{1,u}}{\partial a^{m_1}\partial \overline{a}^{m_2}}(0 : X : s) \\ &\quad = \int \frac{\partial^{m_1+m_2}\Phi}{\partial t_1^{m_1}\partial \overline{t}_1^{m_2}}[(t_2, 0) : X : s] \mid t_2 \mid^{u} \mu(t_2) t_2^{-m_1}\overline{t}_2^{-m_2} d^\times t_2. \qquad \square \end{aligned}$$

In fact, we will need the following supplement to the previous proposition.

Proposition 2. *Let $0 < a$ be real a number not in $\mathbb{Z}_F$ and $\mathfrak{S}$ the strip $\{u \mid -a < \Re u < a\}$. Let $P(u) = P(-u)$ be the polynomial $\prod(u - u_j)$, where the product is over all $u_j \in \mathfrak{S} \cap \mathbb{Z}_F$. Given $\Phi \in \mathcal{S}(F^2 \oplus \mathbb{X})$ set as before*

$$w_{\Phi,u}(t_1, t_2 : X) := \int \Phi(t_1 t, t_2 t^{-1}) \mid t \mid^u \mu(t) d^\times t.$$

For each $u \in \mathfrak{S}$, there are $\Phi_{1,u}$ and $\Phi_{2,u}$ in $\mathcal{S}(F \oplus \mathbb{X})$ such that, for $u \in \mathfrak{S}$,

$$\begin{aligned} &P(u) w_{\Phi,u}(t_1, t_2 : X) \\ &\quad = \Phi_{1,u}(t_1 t_2 : X) \mid t_1 \mid^{-u} \mu^{-1}(t_1) + \Phi_{2,u}(t_1 t_2 : X) \mid t_2 \mid^u \mu(t_2). \end{aligned}$$

One can choose the functions in such a way that the maps $u \mapsto \Phi_{i,u}$ from $\mathfrak{S}$ to $\mathcal{S}(F \oplus \mathbb{X})$ are holomorphic and, furthermore, the functions $\Phi_{i,u}$ remain in a bounded set if Φ does. Finally, if Φ depends holomorphically on $s \in \Omega'$, Ω' open in $\mathbb{C}^m$, and remains in a bounded set for all values of s, one can choose the functions Φ_* to depend holomorphically on (u, s) and to remain in a bounded set.

Proof. The proof is similar to the proof of the previous proposition. The only difference is that we choose the functions $\Phi_{i,u}$ such that they satisfy, instead of (5) and (6) the following relations, for all $u \in \mathfrak{S}$,

$$\frac{\partial^{n_1+n_2}\Phi_{2,u}}{\partial a^{n_1}\partial \overline{a}^{n_2}}(0) = P(u)\int \frac{\partial^{n_1+n_2}\Phi}{\partial t_2^{n_1}\partial \overline{t}_2^{n_2}}(0, t_1) \mid t_1 \mid^{-u} t_1^{-n_1}\overline{t}_1^{-n_2}\mu^{-1}(t_1)d^\times t_1$$

and

$$\frac{\partial^{m_1+m_2}\Phi_{1,u}}{\partial a^{m_1}\partial \overline{a}^{m_2}}(0) = P(u)\int \frac{\partial^{m_1+m_2}\Phi}{\partial t_1^{m_1}\partial \overline{t}_1^{m_2}}(t_2, 0) \mid t_2 \mid^{u} \mu(t_2)t_2^{-m_1}\overline{t}_2^{-m_2}d^\times t_2.$$

Again, the right-hand sides integrals are holomorphic functions of s in $\mathfrak{S}$. To finish the proof as before, we need only check that the difference between

$$A_u(s) := P(u)\int w_u(a) \mid a \mid^s \xi(a)d^\times a$$

and $B_u(s)$ defined in (7) is, for each $u \in \mathfrak{S}$, a holomorphic function of s. For $u \notin \mathbb{Z}_F$, this follows directly as before from the constructions. Consider now a $u_0 \in \mathfrak{S} \cap \mathbb{Z}_F$. Then $A_{u_0}(s) \equiv 0$. Thus we need to check that $B_{u_0}(s)$ is a holomorphic function of s. We write $P(u) = P_1(u)(u - u_0)$. Consider then a potential pole s_0 of the first term in B_u. This means that, for suitable integers $n_1, n_2 \geq 0$,

$$\mid a \mid^{s_0} \xi(a)a^{n_1}\overline{a}^{n_2} \equiv 1. \tag{8}$$

The residue of the first term is then $P_1(u_0)$ times

$$\frac{u-u_0}{n_1!n_2!}\int \frac{\partial^{n_1+n_2}\Phi}{\partial t_2^{n_1}\partial \overline{t}_2^{n_2}}(0, t_1) \mid t_1 \mid^{-u} t_1^{-n_1}\overline{t}_1^{-n_2}\mu^{-1}(t_1)d^\times t_1\bigg|_{u=u_0}.$$

This is zero unless u_0 is a singularity of the integral, that is, for suitable integers $m_1 \geq 0, m_2 \geq 0$,

$$\mid t_1 \mid^{-u_0} t_1^{-n_1}\overline{t}_1^{-n_2}\mu^{-1}(t_1)t_1^{m_1}\overline{t}_2^{m_2} \equiv 1. \tag{9}$$

The residue is then

$$-\frac{P_1(u_0)}{n_1!n_2!m_1!m_2!}\frac{\partial^{n_1+n_2+m_1+m_2}\Phi}{\partial t_2^{n_1}\partial \overline{t}_2^{n_2}\partial t_1^{m_1}\partial \overline{t}_1^{m_2}}(0, 0).$$

On the other hand, we have from (8) and (9)

$$\mid a \mid^{s_0-u_0} \xi\mu^{-1}(a)a^{m_1}\overline{a}^{m_2} \equiv 1.$$

Thus s_0 is a potential pole of the second term with residue

$$P_1(u_0)\,\frac{(u-u_0)}{m_1!m_2!}\int \frac{\partial^{m_1+m_2}\Phi}{\partial t_1^{m_1}\partial \overline{t}_1^{m_2}}(t_2, 0) \mid t_2 \mid^{u} \mu(t_2)t_2^{-m_1}\overline{t}_2^{-m_2}d^\times t_2\bigg|_{u=u_0}.$$

From (8) and (9) we get

$$\mid t_2 \mid^{u_0} \mu(t_2)t_2^{-m_1}\overline{t}_2^{-m_2}t_2^{n_1}\overline{t}_2^{n_2} \equiv 1.$$

We see that u_0 is a pole of this integral and so the residue of the second term at s_0 is

$$\frac{P_1(u_0)}{n_1!n_2!m_1!m_2!}\frac{\partial^{n_1+n_2+m_1+m_2}\Phi}{\partial t_2^{n_1}\partial\overline{t}_2^{n_2}\partial t_1^{m_1}\partial\overline{t}_1^{m_2}}(0,0).$$

We see then that the sum of the two terms has no pole at s_0. The same analysis applies to the second term. Our assertion follows. □

We will need a coarse majorization of our functions.

PROPOSITION 3. *For* $\Phi \in \mathcal{S}(F^2 \oplus \mathbb{X})$ *set*

$$w_{\Phi,u}(t_1, t_2 : X) := \int \Phi(t_1 t, t_2 t^{-1}) \mid t \mid^u \mu(t) d^\times t.$$

Suppose that Φ *is in a bounded set and* $\Re u$ *in a compact set. Then, there is* $M > 0$ *such that, for every* $N > 0$*, there is* $\phi \geq 0$ *in* $\mathcal{S}(\mathbb{X})$ *such that*

$$|w_{\Phi,u}(t_1, t_2 : X)| \leq \mid t_2 \mid^{\Re u} \phi(X) \frac{\mid t_1 t_2 \mid^{-M}}{(1+ \mid t_1 t_2 \mid)^N}$$

If Φ *is in a bounded set and* u *purely imaginary, there is, for every* $N > 0$*, a function* $\phi \geq 0$ *in* $\mathcal{S}(\mathbb{X})$ *such that*

$$|w_{\Phi,u}(t_1, t_2 : X)| \leq \phi(X) \frac{A + B \mid \log \mid a \mid\mid}{(1+ \mid a \mid)^N}.$$

Proof. Say F is complex. We may as well study

$$w_{\Phi,u}(a : X) := \int \Phi(at, t^{-1}) \mid t \mid^u \mu(t) d^\times t.$$

We may bound Φ in absolute value by a product $\Phi_0(t)\Psi(X)$ with fixed non-negative Schwartz functions and replace u by its real part. Then the integral is bounded, for all N_1, N_2, by a constant times $\Psi(X)$ times

$$\int \frac{\mid t \mid^u}{(1+ \mid at \mid)^{N_1} (1+ \mid t^{-1} \mid)^{N_2}} d^\times t \leq \mid a \mid^{-N_1} \int \frac{\mid t \mid^{N_2 - N_1 + u}}{(1+ \mid t \mid)^{N_2}} d^\times t.$$

The first assertion follows then by considering separately the case $\mid a \mid \leq 1$ and the case $\mid a \mid \geq 1$.

For the second assertion we may assume $u = 0$. We need only consider the estimate for $\mid a \mid \leq 1$. We write

$$\int \frac{1}{(1+ \mid at \mid)^{N_1} \left(1+ \mid t^{-1} \mid\right)^{N_2}} d^\times t$$

as the sum of

$$\int_{|t| \leq 1} \frac{\mid t \mid^{N_2}}{(1+ \mid at \mid)^{N_1} (1+ \mid t \mid)^{N_2}} d^\times t$$

and

$$\int_{|t|\geq 1} \frac{|\ t\ |^{N_2}}{(1+|\ at\ |)^{N_1}\,(1+|\ t\ |)^{N_2}} d^{\times}t.$$

The first integral is bounded independently of a. The second is bounded by

$$\int_{|t|\geq 1} \frac{1}{(1+|\ at\ |)^{N_1}} d^{\times}t = \int_{|t|\geq |a|} \frac{1}{(1+|\ t\ |)^{N_1}} d^{\times}t$$

which is in turn bounded by a polynomial of degree 1 in $\log |\ a\ |$. □

5. Integral representation of sections. Our goal in this section is to obtain an integral representation of sections of the form $f_{\phi,u}$. Assume that $\phi = \phi_1 * \phi_2 * \cdots * \phi_{n-1}$ with $\phi_i \in \mathcal{D}(G_n)$. According to [DM], every element of $\mathcal{D}(G_n)$ is a sum of such products, so there is no loss of generality. Let f_u be a standard section. We consider the section $f_{\phi,u}$ defined by

$$f_{\phi,u}(g) = \int f_u(gx)\phi(x)dx \tag{10}$$

or, more explicitly,

$$f_{\phi,u}(g) = \int f_u[gx_1x_2\cdots x_{n-1}]\phi_1(x_1)\phi_2(x_2)\cdots\phi_{n-1}(x_{n-1})dx_1dx_2\cdots dx_n.$$

We set

$$\mathbb{V}_n := M(1\times 2, F)\times M(2\times 3, F)\times\cdots\times M(n-1\times n, F). \tag{11}$$

We denote by R_i the set of matrices of rank i in $M(i\times i+1, F)$.

PROPOSITION 4. *There is a function*

$$\Phi_u\,[X_1 : X_2 : \cdots : X_{n-1}]$$

on $\mathbb{C}^n\times\mathbb{V}_n$ with the following properties. It is a smooth function, holomorphic in u. The projection of its support on the $i-th$ factor is contained in a fixed compact subset of R_i. Finally, for every u,

$$\begin{aligned} f_{\phi,u}(g) = |\ \det g\ |^{u_n+\frac{n-1}{2}}\ \mu_n(\det g) \\ \times \int \Phi_u\left[(g_1,0)g_2^{-1} : (g_2,0)g_3^{-1} : \cdots : (g_{n-2},0)g_{n-1}^{-1} : (g_{n-1},0)g\right] \\ \prod_{i=1}^{i=n-1} |\ \det g_i\ |^{u_{i+1}-u_i+1}\ \mu_{i+1}\mu_i^{-1}(\det g_i)dg_i\ ; \end{aligned}$$

in this integral, each variable g_i is integrated over $GL(i, F)$.

Note that the integral is convergent for all u under the restricted assumption on the support of Φ. This would not be true for a Schwartz function or even a function of compact support.

In what follows, by abuse of language, we will suppress the characters μ_i from the notations. The reader will easily re-establish them by replacing $|\det g|^{u_n}$ by $\mu_n(\det g)\,|\,g\,|^{u_n}$ and so on. It will be more convenient to prove the result for somewhat more general sections. Namely, we consider functions of the form

$$f_u[g, g_1, g_2, \ldots, g_{n-1}]$$

which are smooth functions on $\mathbb{C}^n \times GL(n)^n$ and holomorphic in u; we assume that, for fixed u and fixed (g_i), the function

$$g \mapsto f_u[g, g_1, g_2, \ldots, g_{n-1}]$$

belongs to the space of Ξ_u. The projection of the support of f on the $i+1$-th factor is contained in a fixed compact set of $GL(n)$. Then

$$g \mapsto \int f_u[gx_1x_2\cdots x_{n-1}, x_1, x_2, \ldots, x_{n-1}]dx_1dx_2\cdots dx_n \tag{12}$$

is the type of section for which we prove our integral representation, by induction on n.

Thus we assume our assertion established for sections of this type and the integer $n-1$. In fact, for the purpose of carrying out our induction, we should consider more generally functions of the form

$$f_{u,s}[g, g_1, g_2, \ldots, g_{n-1}, h]$$

where the s is an auxiliary complex parameter and the last variable h is in some auxiliary manifold S; the function depends holomorphically on s and the projection of its support on S is contained in a fixed compact set. Then the function Φ_u of the Proposition would also depend on s and h with obvious properties. We simply ignore this complication.

Consider a section of the type (12). It is convenient to introduce

$$f_u^1[g, x_1, x_2, \ldots, x_{n-1}; s] = f_u\left[g, x_1^{-1}, x_2, \ldots, x_{n-1}\right]$$

and then the integral defining (12) takes the form

$$\int f_u^1\left[x_1x_2\cdots x_{n-1}, x_1^{-1}g, x_2, \ldots, x_{n-1}\right]dx_1dx_2\cdots dx_{n-1}.$$

Next, we use the Iwasawa decomposition of x_1, in the form

$$\begin{aligned} &x_1 = p_1k_1, \\ &p_1 = \begin{pmatrix} g_{n-1} & 0 \\ X & t \end{pmatrix} = \begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1_{n-1} & 0 \\ X & t \end{pmatrix} \end{aligned} \tag{13}$$

with $k_1 \in K_n$, $g_{n-1} \in GL(n-1)$, $t \in F^\times$. After a change of variables, the integral becomes

$$\int f_u^1 \left[\begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} x_2 \cdots x_{n-1}, k_1^{-1} p_1^{-1} g, k_1^{-1} x_2, \ldots, x_{n-1} \right]$$
$$| \, t \, |^{u_n - \frac{n-1}{2}} \, dk_1 dg_{n-1} d^\times t dX dx_2 \cdots dx_{n-1}.$$

If we set

$$f_u^2(g, x_1, x_2, \ldots, x_{n-1}) = \int f_u^1(g, k_1^{-1} x_1, k_1^{-1} x_2, \ldots, x_{n-1}) dk_1$$

then, after integrating over k_1, we obtain

$$\int f_u^2 \left[\begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} x_2 \cdots x_{n-1}, p_1^{-1} g, x_2, \ldots, x_{n-1} \right]$$
$$| \, t \, |^{u_n - \frac{n-1}{2}} \, dg_{n-1} d^\times t dX dx_2 \cdots dx_{n-1}.$$

To continue we set

$$x_i = \begin{pmatrix} m_i & 0 \\ 0 & 1 \end{pmatrix} p_i, \; m_i \in GL(n-1, F), \; p_i \in GL(n-1, F) \backslash GL(n, F).$$

Then

$$dx_i = dm_i d_r p_i$$

where $d_r p$ is an invariant measure on $GL(n-1, F) \backslash GL(n, F)$. After a change of variables, we get

$$\int f_u^2 \left[\begin{pmatrix} g_{n-1} m_2 m_3 \cdots m_{n-1} & 0 \\ 0 & 1 \end{pmatrix} p_{n-1}, \; p_1^{-1} g, \right.$$
$$\left. \begin{pmatrix} m_2 & 0 \\ 0 & 1 \end{pmatrix} p_2, p_2^{-1} \begin{pmatrix} m_3 & 0 \\ 0 & 1 \end{pmatrix} p_3, \ldots, p_{n-2}^{-1} \begin{pmatrix} m_{n-1} & 0 \\ 0 & 1 \end{pmatrix} p_{n-1} \right]$$
$$| \, t \, |^{u_n - \frac{n-1}{2}} \, dg_{n-1} d^\times t dX \bigotimes_{i=2}^{n-1} dm_i \bigotimes_{i=2}^{n-1} d_r p_i.$$

We introduce a new function f_u^3 on

$$\mathbb{C}^n \times GL(n-1) \times GL(n) \times GL(n-1)^{n-2}$$

which is defined by

$$| \det g_{n-1} |^{-1/2} \prod_{i=2}^{n-1} | \det m_i |^{1/2} \; f_u^3 \, [g_{n-1}, g, m_2, m_3, \ldots, m_{n-1}] :$$
$$= \int f_u^2 \left[\begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} p_{n-1}, \; g, \right.$$
$$\left. \begin{pmatrix} m_2 & 0 \\ 0 & 1 \end{pmatrix} p_2, p_2^{-1} \begin{pmatrix} m_3 & 0 \\ 0 & 1 \end{pmatrix} p_3, \ldots, p_{n-2}^{-1} \begin{pmatrix} m_{n-1} & 0 \\ 0 & 1 \end{pmatrix} p_{n-1} \right] \bigotimes_{i=2}^{i=n-1} d_r p_i.$$

It is clear that f_u^3 is a smooth function, holomorphic in u, and the projection of its support on each linear factor except the first one is contained in a fixed compact set. Moreover, the function

$$g_{n-1} \mapsto f_u^3\,[g_{n-1}, g, m_2, m_3, \ldots, m_{n-1}]$$

belongs to the space of the representation of $GL(n-1)$ determined by $(\mu_1, \mu_2, \ldots, \mu_{n-1})$ and $(u_1, u_2, \ldots u_{n-1})$.

Finally, we set

$$f_u^4(g_{n-1}, g) = \int f_u^3(g_{n-1}m_2m_3\cdots m_{n-1}, g, m_2, m_3, \ldots, m_{n-1})dm_2dm_3\cdots dm_{n-1}.$$

The section (12) can be represented by the integral

$$\int f_u^4\left[g_{n-1}, p_1^{-1}g\right] \mid \det g_{n-1} \mid^{-1/2} \mid t \mid^{u_n - \frac{n-1}{2}} dg_{n-1}d^\times t dX. \tag{14}$$

We apply the induction hypothesis to the function $g_{n-1} \mapsto f_u^4(g_{n-1}, g)$. There is a function Φ_u^1 on

$$\mathbb{C}^n \times \mathbb{V}_{n-1} \times GL(n)$$

such that

$$\begin{aligned} f_u^4(g_{n-1}, g) =& \mid \det g_{n-1} \mid^{u_{n-1}+\frac{n-2}{2}} \\ &\times \int \Phi_u^1\left[(g_1, 0)g_2^{-1} : (g_2, 0)g_3^{-1} : \cdots : (g_{n-3}, 0)g_{n-2}^{-1} : (g_{n-2}, 0)g_{n-1} : g\right] \\ &\prod_{i=1}^{i=n-2} \mid \det g_i \mid^{u_{i+1}-u_i+1} dg_i\,; \end{aligned} \tag{15}$$

the projection of its support on each factor of $\mathbb{V}_{n-1}$ or the factor $GL(n)$ is contained in a fixed compact set; more precisely, for each factor of $\mathbb{V}_{n-1}$, a compact set of R_i.

Finally, we can change variables in the integral (14) where p_1 is given by (13) to obtain

$$\int f_u^4\left[g_{n-1}^{-1}, \begin{pmatrix} 1_{n-1} & 0 \\ X & t \end{pmatrix} \begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} g\right]$$
$$\mid \det g_{n-1} \mid^{1/2} \mid t \mid^{-u_n - \frac{n-1}{2}} dg_{n-1}d^\times t dX.$$

The above integral can be written as

$$\int f_u^4\left[g_{n-1}^{-1}, p\begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} g\right] \left|\det p\begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} g\right|^{-u_n - \frac{n-1}{2}} \tag{16}$$
$$dg_{n-1}d_r p \mid \det g_{n-1} \mid^{u_n + \frac{n}{2}} \mid \det g \mid^{u_n + \frac{n-1}{2}}$$

where the integral in p is over the group P of matrices of the form

$$\begin{pmatrix} 1_{n-1} & 0 \\ * & * \end{pmatrix}$$

and $d_r p$ is a right invariant measure. In terms of Φ_u^1 this can be written as

$$\begin{aligned}
&|\det g\,|^{u_n+\frac{n-1}{2}} \\
&\quad\times \int \Phi_u^1 \left[(g_1,0)g_2^{-1} : (g_2,0)g_3^{-1} : \cdots : (g_{n-3},0)g_{n-2}^{-1} : (g_{n-2},0)g_{n-1}^{-1}\right. \\
&\quad\left. : p\begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} g \right] \left|\det p \begin{pmatrix} g_{n-1} & 0 \\ 0 & 1 \end{pmatrix} g\right|^{-u_n-\frac{n-1}{2}} \\
&\quad \prod_{i=1}^{i=n-1} |\det g_i\,|^{u_{i+1}-u_i+1}\, dg_i\, d_r p.
\end{aligned}$$

There is a smooth function Φ_u on $\mathbb{C}^n \times \mathbb{V}_n$ such that

$$\begin{aligned}
(17) \qquad &\Phi_u(X_1, X_2, \ldots, X_{n-1}, (1_{n-1}, 0)g) \\
&\quad = \int \Phi_u^1(X_1, X_2, \ldots, X_{n-1}, pg)\, |\det pg|^{-u_n-\frac{n-1}{2}}\, d_r p.
\end{aligned}$$

As before, the projection of its support on the $i-th$ linear factor of $\mathbb{C}^n \times \mathbb{V}_n$ is contained in a fixed compact set of R_i. If we combine (14) and the formula just before it we arrive at our conclusion.

We will need estimates on the function Φ_u of the previous proposition.

PROPOSITION 5. *Suppose that ϕ is the convolution product of $(n-1)$ functions, each of which is in a bounded set $\mathcal{B}$ of $\mathcal{D}(G_n)$ and the standard section f_u (or rather its restriction to K_n) is in a bounded set of $\mathcal{V}(\mu)$. Fix a multi strip $\mathfrak{S} = \{u| -A < \Re u_i < A, 1 \le i \le n\}$ in $\mathbb{C}^n$. Then, there is a bounded set $\mathcal{C}$ of $\mathcal{S}(\mathbb{V}^n)$ such that $\Phi_u \in \mathcal{C}$, for all u in $\mathfrak{S}$. Moreover, the map*

$$(f_u, \phi_1, \phi_2, \ldots, \phi_{n-1}) \mapsto \Phi_u$$

is continuous.

Proof. We prove the first assertion. The second assertion has a similar proof. Our construction shows that the functions Φ_u have a support contained in a fixed compact set of $\mathbb{V}_n$. Thus we have only to show that their derivatives are bounded uniformly when u is in $\mathfrak{S}$. This follows from the following lemma. □

LEMMA 5. *Let $\mathfrak{S}$ be a strip in $\mathbb{C}$ and $\mathcal{B}$ a bounded set of $\mathcal{D}(G_n)$. For $\phi \in \mathcal{D}(G_n)$, $u \in \mathbb{C}$, set*

$$g_u(g) = |\det g\,|^u\, \mu(\det g)\phi(g)$$

and

$$g_u^1 = \int_P g_u(pg) d_r p.$$

Let R_{n-1} be the open set of matrices of maximal rank in $M((n-1) \times n, F)$. Define a function $\Phi_u \in \mathcal{S}(M((n-1) \times n, F))$ with compact support contained in R_{n-1} by

$$\Phi_u[(1_{n-1}, 0)g] = g_u^1(g).$$

If u is in a strip $\mathfrak{S}$ and ϕ in a bounded set $\mathcal{B}$, the function Φ_u remains in a bounded set of $\mathcal{S}(M((n-1) \times n, F))$.

Proof of the Lemma. We have to show the derivatives of Φ_u are uniformly bounded. The group $SL(n, F)$ is transitive on R_{n-1} thus the above relation for $g \in SL(n, F)$ already determines Φ_u. If X is in the enveloping algebra of $SL(n, F)$ (viewed as a real Lie group) it is easy to see that for $u \in \mathfrak{S}$ and $\phi \in \mathcal{B}$, the function $\rho(X)g_u$ is uniformly bounded. The same assertion is thus true for the function g_u^1. Since R_{n-1} is isomorphic to $P \cap SL(n, F)\backslash SL(n, F)$ as a manifold, it will suffice to show that for every differential operator ξ on $P \cap SL(n, F)\backslash SL(n, F)$ the function ξg_u^1 is uniformly bounded for $g \in SL(n, F)$ and u in the given strip. This is true for an operator of the form $\xi = \rho(X)$ where X is in the enveloping algebra of $SL(n, F)$. Thus, in turn, it will suffice to show that every differential operator on $P \cap SL(n, F)\backslash SL(n, F)$ can be written as a linear combination of operators of the form $\rho(X)$ with smooth coefficients. For instance, one may use the fact that the map $(g, k) \mapsto gk$ from $SL(n-1) \times K \cap SL(n-1)$ to $SL(n-1, F)$ passes to the quotients and defines an isomorphism of $K_{n-1} \cap SL(n-1, F)\backslash(SL(n-1) \times K \cap SL(n, F))$ with $P \cap SL(n, F)\backslash SL(n, F)$. □

6. Integral representation of Whittaker functions. In this section, our goal is to obtain an absolutely convergent integral formula for $\mathcal{W}_u(f_{\phi,u})$. To that end, we introduce more notations. The groups $GL(n-1, F)$ and $GL(n, F)$ operate on the space of Schwartz functions on $M((n-1) \times n, F)$ as follows:

$$l_n(g_n).\Psi.r_{n-1}(g_{n-1})[X] = \Psi[g_{n-1} X g_n].$$

As the notation indicates, l_n is a left action and r_{n-1} a right action. We may identify the space of $(n-1) \times n$ matrices ($n-1$ rows, n columns) to the direct sum of the space $\mathcal{B}_l(n-1, F)$ of lower triangular matrices of size $(n-1) \times (n-1)$ and the space $\mathcal{B}_u(n-1, F)$ of upper triangular matrices of size $(n-1) \times (n-1)$. For instance, for $n = 3$, the matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix}$$

corresponds to the pair of matrices

$$\left(\begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_{1,2} & x_{1,3} \\ 0 & 0 & x_{2,3} \end{pmatrix} \right).$$

Accordingly, if Φ is a Schwartz function on $M((n-1) \times n, F)$ we define its partial Fourier transform $\mathcal{F}_{n-1}\Phi$ or simply $\hat{\Phi}$ by

$$\hat{\Phi}(b_1, b_2) = \int_{\mathcal{B}_u(n-1,F)} \Phi(b_1 \oplus b')\psi(-\mathrm{tr}(b_2 b'))db'.$$

It is thus a function on $\mathcal{B}_l(n-1, F) \oplus \mathcal{B}_l(n-1, F)$. We can use this partial Fourier transform to define two new representations $\hat{r}_n$, $\hat{l}_{n-1}$ by

$$\mathcal{F}_{n-1}\left(l_n(g_n).\Psi.r_{n-1}(g_{n-1})\right) = \hat{l}_n(g_n).\hat{\Psi}.\hat{r}_{n-1}(g_{n-1}),$$

on the space of Schwartz functions on $\mathcal{B}_l(n-1, F) \oplus \mathcal{B}_l(n-1, F)$.

The following formula will be very useful. Let $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a diagonal matrix of size n. Set

$$\alpha^b = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}),\ \alpha^e = \mathrm{diag}(\alpha_2, \alpha_3, \ldots, \alpha_n). \tag{18}$$

We will denote by α^{-b} and α^{-e} the inverses of the matrices α^b, α^e. Then:

$$l_n(\alpha)\Phi(b_1 \oplus b') = \Phi(b_1\alpha^b \oplus b'\alpha^e) \tag{19}$$

$$\hat{l}_n(\alpha)\hat{\Phi}(b_1, b_2) = \hat{\Phi}(b_1\alpha^b, \alpha^{-e}b_2) \mid \alpha_2 \mid^{-1} \mid \alpha_3 \mid^{-2} \cdots \mid \alpha_n \mid^{-(n-1)}. \tag{20}$$

We can define analogous representations of the appropriate linear groups on the space of Schwartz functions on

$$\mathbb{V}_n := M(1 \times 2, F) \oplus M(2 \times 3, F) \oplus \cdots \oplus M(n-1 \times n, F).$$

They are denoted by $r_1, r_2, \ldots, r_{n-1}, l_2, l_3, \ldots, l_n$. For instance, if $n = 3$, then

$$l_2(g_2)l_3(g_3)\Phi\left[(x_1, y_1) : \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix}\right] r_2(g_2')$$
$$= \Phi\left[(x_1, y_1)g_2 : g_2' \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix} g_3\right].$$

We can also define the partial Fourier transform $\mathcal{F}(\Phi) = \mathcal{F}_1\mathcal{F}_2 \cdots \mathcal{F}_{n-1}(\Phi)$ of a function $\Phi \in \mathcal{S}(\mathbb{V}^n)$. This partial Fourier transform is then a function on the direct sum:

$$\mathbb{U}_n := \tag{21}$$

$$\mathcal{B}_l(1, F) \oplus \mathcal{B}_l(1, F) \oplus \mathcal{B}_l(2, F) \oplus \mathcal{B}_l(2, F) \oplus \cdots \oplus \mathcal{B}_l(n-1, F) \oplus \mathcal{B}_l(n-1, F).$$

We have also representations $\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{n-1}, \hat{l}_2, \ldots, \hat{l}_n$ on the space $\mathcal{S}(\mathbb{U}^n)$. For $\Phi \in \mathcal{S}(\mathbb{V}^n)$ we define $\mathcal{F}^A(\Phi) \in \mathcal{S}(\mathbb{U}_n)$ by

$$\begin{aligned}(22)\quad &\mathcal{F}^A(\Phi)\\ &= \int \hat{l}_2(k_2)\hat{l}_3(k_3)\cdots\hat{l}_{n-1}(k_{n-1})\mathcal{F}(\Phi)\hat{r}_2\big(k_2^{-1}\big)\hat{r}_3\big(k_3^{-1}\big)\cdots\hat{r}_{n-1}\big(k_{n-1}^{-1}\big)\\ &\quad\prod_{i=2}^{i=n-1}\mu_i^{-1}\mu_{i-1}(\det k_i)dk_i\end{aligned}$$

the integral being over the product $K_2 \times K_3 \times \cdots \times K_{n-1}$.

THEOREM 1. *Suppose that $f_{\phi,u}$ is the section represented by the integral of Proposition 4. Set*

$$\Psi_u = \mathcal{F}^A(\Phi_u).$$

Then

$$\begin{aligned}&\mathcal{W}_u(f_{\phi,u})\\ &= \int \Psi_u\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}a_{n-1}^{-b}, a_{n-2}^{-1}a_{n-1}^e : a_{n-1}, a_{n-1}^{-1}\right]\\ &\quad\prod_{i=1}^{i=n-1}\mu_{i+1}\mu_i^{-1}(\det a_i)\mid \det a_i\mid^{u_{i+1}-u_i} da_i.\end{aligned}$$

Here a_i is integrated over A_i.

Before we embark on the proof of the theorem we write down a more general formula, which follows from the theorem.

$$\begin{aligned}(23)\quad &\mathcal{W}_u(\Xi_u(g)f_{\phi,u})\\ &= \int \hat{l}_n(g).\Psi_u\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots\right.\\ &\qquad\left. : a_{n-2}a_{n-1}^{-b}, a_{n-2}^{-1}a_{n-1}^e : a_{n-1}, a_{n-1}^{-1}\right]\\ &\quad\prod_{i=1}^{i=n-1}\mu_{i+1}\mu_i^{-1}(\det a_i)\mid \det a_i\mid^{u_{i+1}-u_i} da_i \times \mid \det g\mid^{u_n+\frac{n-1}{2}} \mu_n(\det g).\end{aligned}$$

In particular, for a diagonal matrix α,

$$\begin{aligned}(24)\quad &\mathcal{W}_u(\Xi_u(\alpha)f_{\phi,u})\\ &= \int \Psi_u\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots\right.\\ &\qquad\left. : a_{n-2}a_{n-1}^{-b}, a_{n-2}^{-1}a_{n-1}^e : a_{n-1}\alpha^b, a_{n-1}^{-1}\alpha^{-e}\right]\end{aligned}$$

$$\prod_{i=1}^{i=n-1} \mu_{i+1}\mu_i^{-1}(\det a_i) \mid \det a_i \mid^{u_{i+1}-u_i} da_i$$
$$\times \mid \det \alpha \mid^{u_n} \mu_n(\det \alpha) e^{\langle H(\alpha),\rho\rangle}.$$

Proof. As before, we suppress the characters μ_i from the notations. Our task can be summarized as follows. □

LEMMA 6. *Suppose Φ is a smooth function on $\mathbb{V}_n$ with compact support contained in $\prod R_i$. Define*

$$f(g) = \int \Phi\left[(g_1,0)g_2^{-1} : (g_2,0)g_3^{-1} : \cdots : (g_{n-2},0)g_{n-1}^{-1} : (g_{n-1},0)g\right]$$
$$\prod_{i=1}^{i=n-1} \mid \det g_i \mid^{u_{i+1}-u_i+1} \mu_{i+1}\mu_i^{-1}(\det g_i) dg_i.$$

Then, for $\Re(u_{i+1}-u_i) > 0$ for all i,

$$\int_{N_n} f(vg)\overline{\theta}_n(v)dv$$
$$= \int \hat{l}_n(g)\Psi\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e :\right.$$
$$\left.\cdots : a_{n-2}a_{n-1}^{-b}, a_{n-2}^{-1}a_{n-1}^e : a_{n-1}, a_{n-1}^{-1}\right]$$
$$\prod_{i=1}^{i=n-1} \mu_{i+1}\mu_i^{-1}(\det a_i) \mid \det a_i \mid^{u_{i+1}-u_i} da_i$$

where $\Psi = \mathcal{F}^A(\Phi)$.

We prove the lemma by induction on n. The case $n=2$ was treated in section 3. We may assume $n>2$ and the lemma true for $n-1$. Since $\mathcal{F}^A(l_n(g)\Phi) = \hat{l}_n(g)\mathcal{F}^A(\Phi)$, it suffices to prove the formula for $g=1$. We first compute

$$w(g) := \int_{N_{n-1}} f\left[\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} g\right]\overline{\theta}_{n-1}(v)du. \tag{25}$$

After a change of variables we find $w(g)$ is equal to

$$\iint \Phi\left[(g_1,0)g_2^{-1} : (g_2,0)g_3^{-1} : \cdots : (g_{n-2},0)vg_{n-1}^{-1} : (g_{n-1},0)g\right]$$
$$\prod_{i=1}^{i=n-1} \mid \det g_i \mid^{u_{i+1}-u_i+1} dg_i\overline{\theta}_{n-1}(v)dv.$$

We are led to introduce

$$\Omega(g_{n-1}:X):=\int \Phi\left[(g_1,0)g_2^{-1}:(g_2,0)g_3^{-1}:\cdots:(g_{n-2},0)g_{n-1}:X\right] \prod_{i=1}^{i=n-2} |\det g_i|^{u_{i+1}-u_i+1}\,dg_i \tag{26}$$

and

$$\Xi(g_{n-1}:X):=\int \Omega_u(vg_{n-1}:X)\overline{\theta}_{n-1}(v)dv. \tag{27}$$

Then

$$w(g)=\int \Xi(g_{n-1}^{-1},(g_{n-1},0)g)\,|\det g_{n-1}|^{u_n-u_{n-1}+1}\,dg_{n-1}.$$

We can apply the induction hypothesis to the function $g_{n-1}\mapsto \Omega_u(g_{n-1}:X)$. Before we do, we remark that the representation $\hat{l}_{n-1}$ of $GL(n-1)$ on $\mathcal{S}(\mathbb{U}_{n-1})$ and the representation r_{n-1} of $GL(n-1)$ on $\mathcal{S}(M(n-1)\times n, F)$ give corresponding representations on the space of Schwartz functions on $\mathbb{U}_{n-1}\oplus M((n-1)\times n, F)$. By the induction hypothesis, there is a Schwartz function Ψ^1 on that direct sum such that

$$\begin{aligned}\Xi(g_{n-1}:X)\\ &=\int \hat{l}_{n-1}(g_{n-1}).\Psi^1\left[a_1a_2^{-b},a_1^{-1}a_2^e:a_2a_3^{-b},a_2^{-1}a_3^e:\cdots:a_{n-2},a_{n-2}^{-1}:X\right]\\ &\quad\prod_{i=1}^{i=n-2}|\det a_i|^{u_{i+1}-u_i}\,da_i.\end{aligned}$$

It follows that

$$\begin{aligned}w(g)=\int &\hat{l}_{n-1}\left(g_{n-1}^{-1}\right).\Psi^1\left[a_1a_2^{-b},a_1^{-1}a_2^e:a_2a_3^{-b},\right.\\ &\left.a_2^{-1}a_3^e:\cdots:a_{n-2},a_{n-2}^{-1}:(g_{n-1},0)g\right]\\ &\prod_{i=1}^{i=n-2}|\det a_i|^{u_{i+1}-u_i}\,da_i\\ &|\det g_{n-1}|^{u_n-u_{n-1}+1}\,dg_{n-1}.\end{aligned} \tag{28}$$

At this point, we use the Iwasawa decomposition of $GL(n-1)$ to write $g_{n-1}=k_{n-1}b_{n-1}$ with $k_{n-1}\in K_{n-1}$ and $b\in B_{n-1}$. Then $dg_{n-1}=dk_{n-1}d_rb_{n-1}$, where d_rb_{n-1} is a right invariant measure on B_{n-1}. Recalling our notational convention, we replace Ψ^1 by the Schwartz function Ψ^2 defined by

$$\Psi^2=\int_{K_{n-1}}\hat{l}_{n-1}\left(k_{n-1}^{-1}\right).\Psi_u^1.r_{n-1}(k_{n-1})\mu_n\mu_{n-1}^{-1}(\det k_{n-1})dk_{n-1}$$

or, somewhat more explicitly,

$$\Psi^2(\bullet,X)=\int_{K_{n-1}}\hat{l}_{n-1}\left(k_{n-1}^{-1}\right).\Psi^1(\bullet,k_{n-1}X)\mu_n\mu_{n-1}^{-1}(\det k_{n-1})dk_{n-1}.$$

We keep in mind the relation

$$\Xi(vg_{n-1}, X) = \Xi(g_{n-1}, X)\theta_{n-1}(v).$$

It is equivalent to the relation

$$\begin{aligned}&\int \hat{l}_{n-1}(vg_{n-1}).\Psi^2\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}, a_{n-2}^{-1} : X\right]\\ &\quad\prod_{i=1}^{i=n-2} \mid \det a_i \mid^{u_{i+1}-u_i}\, da_i\\ &\quad= \theta_{n-1}(v)\\ &\quad\times \int \hat{l}_{n-1}(g_{n-1}).\Psi^2\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}, a_{n-2}^{-1} : X\right]\\ &\quad\prod_{i=1}^{i=n-2} \mid \det a_i \mid^{u_{i+1}-u_i}\, da_i.\end{aligned}$$

Formula (28) for $w(g)$ becomes

(29)

$$\begin{aligned}&\int \hat{l}_{n-1}\left(b_{n-1}^{-1}\right).\Psi^2\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}, a_{n-2}^{-1} : (b_{n-1}, 0)g\right]\\ &\prod_{i=1}^{i=n-2} \mid \det a_i \mid^{u_{i+1}-u_i}\, da_i\\ &\mid \det b_{n-1} \mid^{u_n-u_{n-1}+1}\, d_r b_{n-1}.\end{aligned}$$

Now we set

$$b_{n-1} = \begin{pmatrix} a_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n-1}\\ 0 & a_{2,2} & x_{2,3} & \cdots & x_{2,n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & a_{n-2,n-2} & x_{n-2,n-1}\\ 0 & \cdots & \cdots & 0 & a_{n-1,n-1}\end{pmatrix},$$

$$a_{n-1} = \mathrm{diag}\left(a_{1,1}, a_{2,2}, \ldots, a_{n-1,n-1}\right).$$

Then

$$d_r b_{n-1} = da_{n-1} \bigotimes dx_{i,j} \mid a_{2,2} \mid^{-1} \mid a_{3,3} \mid^{-2} \cdots \mid a_{n-1,n-1} \mid^{-(n-2)}.$$

We now use formula (29) to compute

$$\int f(v)\overline{\theta}_n(v)dv = \int w\left[\begin{pmatrix} 1_{n-1} & U\\ 0 & 1\end{pmatrix}\right]\psi(-\epsilon_{n-1}U)dU$$

where U is integrated over the space of $(n-1)$-columns and

$$\epsilon_{n-1} = (0, 0, \ldots, 0, 1).$$

We find

$$\int \hat{l}_{n-1}\big(b_{n-1}^{-1}\big).\Psi_u^2\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}, a_{n-2}^{-1} : (b_{n-1}, b_{n-1}U)\right]$$

$$\prod_{i=1}^{i=n-2} \mid \det a_i \mid^{u_{i+1}-u_i} \, da_i$$

$$\mid \det b_{n-1} \mid^{u_n-u_{n-1}+1} \, d_r b_{n-1}\psi(-\epsilon_{n-1}U)dU.$$

We change U to $b_{n-1}^{-1}U$. We get

$$\int \hat{l}_{n-1}\big(b_{n-1}^{-1}\big).\Psi_u^2\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}, a_{n-2}^{-1} : (b_{n-1}, U)g\right]$$

$$\prod_{i=1}^{i=n-2} \mid \det a_i \mid^{u_{i+1}-u_i} \, da_i$$

$$\mid \det b_{n-1} \mid^{u_n-u_{n-1}} \, d_r b_{n-1}\psi\big(- \epsilon_{n-1}b_{n-1}^{-1}U\big)dU$$

or, more explicitly, denoting by U_i, $1 \le i \le n-1$, the entries of U,

$$\int \hat{l}_{n-1}\big(a_{n-1}^{-1}\big).\Psi^2\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}, a_{n-2}^{-1} : (b_{n-1}, U)\right]$$

$$\prod_{i=1}^{i=n-2} \mid \det a_i \mid^{u_{i+1}-u_i} \, da_i$$

$$\psi\left(-a_{1,1}^{-1}x_{1,2} - a_{2,2}^{-1}x_{2,3}\cdots - a_{n-2,n-2}^{-1}x_{n-2,n-1} - a_{n-1,n-1}^{-1}U_{n-1}\right)$$

$$\mid \det b_{n-1} \mid^{u_n-u_{n-1}} \, d_r b_{n-1}dU.$$

By (20), we may bring $\hat{l}_{n-1}(a_{n-1}^{-1})$ "inside" to get

$$\hat{l}_{n-1}\big(a_{n-1}^{-1}\big).\Psi^2\big[\bullet : a_{n-2}, a_{n-2}^{-1} : \bullet\big]$$
$$= \Psi2\big[\bullet : a_{n-2}a_{n-1}^{-b}, a_{n-2}^{-1}a_{n-1}^e : \bullet\big] \mid a_{2,2} \mid\mid a_{3,3} \mid^2 \cdots \mid a_{n-1,n-1} \mid^{n-1} .$$

Thus we find for our integral

$$\int \Psi_u^2\left[a_1a_2^{-b}, a_1^{-1}a_2^e : a_2a_3^{-b}, a_2^{-1}a_3^e : \cdots : a_{n-2}a_{n-1}^{-b}, a_{n-2}^{-1}a_{n-1}^e : b_{n-1}, U)\right]$$

$$\prod_{i=1}^{i=n-1} \mid \det a_i \mid^{u_{i+1}-u_i} \, da_i$$

$$\psi\left(-a_{1,1}^{-1}x_{1,2} - a_{2,2}^{-1}x_{2,3}\cdots - a_{n-2,n-2}^{-1}x_{n-2,n-1} - a_{n-1,n-1}^{-1}U_{n-1}\right)$$

$$\bigotimes d_{x_{i,j}} \bigotimes dU_i.$$

Finally, we set $\Psi = \mathcal{F}_{n-1}(\Psi^2)$. Then the above integral can be written as

$$\int \Psi\left[a_1 a_2^{-b}, a_1^{-1} a_2^{e} : a_2 a_3^{-b}, a_2^{-1} a_3^{e} : \cdots : a_{n-2} a_{n-1}^{-b}, a_{n-2}^{-1} a_{n-1}^{e} : a_{n-1}, a_{n-1}^{-1})\right]$$
$$\prod_{i=1}^{i=n-1} \mid \det a_i \mid^{u_{i+1}-u_i} da_i .$$

To finish the proof of the lemma we remark that

$$\begin{aligned}
\Psi &= \mathcal{F}_{n-1}(\Psi^2)\\
&= \mathcal{F}_{n-1}\left(\int \hat{l}_{n-1}\left(k_{n-1}^{-1}\right)(\Psi^1) r_{n-1}(k_{n-1}) \mu_n \mu_{n-1}^{-1}(\det k_{n-1}) dk_{n-1}\right)\\
&= \int \mathcal{F}_{n-1}\left(\hat{l}_{n-1}\left(k_{n-1}^{-1}\right)(\Psi^1) r_{n-1}(k_{n-1})\right) \mu_n \mu_{n-1}^{-1}(\det k_{n-1}) dk_{n-1}\\
&= \int \hat{l}_{n-1}\left(k_{n-1}^{-1}\right)\left(\mathcal{F}_{n-1}(\Psi^1)\right) \hat{r}_{n-1}(k_{n-1}) \mu_n \mu_{n-1}^{-1}(\det k_{n-1}) dk_{n-1}
\end{aligned}$$

so that inductively we see that $\Psi = \mathcal{F}^A(\Phi)$ as claimed.

7. Properties of the Whittaker functions. In this section we will use the integral representation to obtain a very precise description of the behavior of a Whittaker function on the diagonal subgroup. Our starting point is an investigation of the type of integrals that represent Whittaker functions. We let $\mathbb{A}_n$ be the vector space of diagonal matrices with n-entries, a space we may also identify to F^n. We consider the direct sum

$$\mathbb{W}_n := \mathbb{A}_1 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3 \oplus \mathbb{A}_3 \oplus \cdots \oplus \mathbb{A}_{n-1} \oplus \mathbb{A}_{n-1} \tag{30}$$

and a function $\Psi \in \mathcal{S}(\mathbb{W}_n \oplus \mathbb{X})$, where $\mathbb{X}$ is an auxiliary real vector space. We consider the function on $A_n \times \mathbb{X}$ defined by the following integral:

$$w_{\Psi,u}(m : X) :=\mid \det m \mid^{u_n} \mu_n(\det m) \tag{31}$$

$$\int_{A_1 \times A_2 \times \cdots \times A_{n-1}} \Psi\left[a_1 a_2^{-b}, a_1^{-1} a_2^{e} : a_2 a_3^{-b}, a_2^{-1} a_3^{e} : \cdots :\right.$$
$$\left. a_{n-2} a_{n-1}^{-b}, a_{n-2}^{-1} a_{n-1}^{e} : a_{n-1} m^{b}, a_{n-1}^{-1} m^{-e} : X\right]$$
$$\prod_{i=1}^{i=n-1} \mu_n \mu_{n-1}^{-1}(a_i) \mid \det a_i \mid^{u_n - u_{n-1}} da_i .$$

Let σ_u be the n-dimensional representation of $F^\times$ (or the Weil group of F) defined by

$$\sigma_u(t) = \bigoplus \mu_i(t) \mid t \mid_F^{u_i} .$$

In particular, $\sigma_0 = \bigoplus \mu_i(t)$. For $1 \le j \le n$, the representation $\bigwedge^j \sigma_u$ decomposes into the sum of the characters

$$t \mapsto \mu_{i_1}\mu_{i_2}\cdots\mu_{i_j}(t) \mid t \mid^{u_{i_1}+u_{i_2}+\cdots+u_{i_j}}$$

with

$$1 \le i_1 < i_2 < \cdots < i_j \le n.$$

We consider the algebraic co-characters of A_n defined by the fundamental coweights:

$$\check{\varpi}_j(m) = (\overbrace{m, m, \ldots, m}^{j}, 1, \ldots 1).$$

Thus $\alpha_i(\check{\varpi}_j(m)) = m^{\delta_{i,j}}$. Below we consider over all complex characters ξ of A_n such that, for each $1 \le i \le n$, the character $\xi \circ \check{\varpi}_i$ is a component of $\bigwedge^i \sigma_u$. If we write $\xi(m) = \prod_{1 \le i \le n} \xi_i(m_i)$ where the m_i are the entries of m, this amounts to say that every product $\xi_1\xi_2\cdots\xi_j$ is a component of $\bigwedge^j(\sigma_u)$. The characters ξ depend on u so we will often write them as ξ_u. Thus $\xi_{u,1}(t) = \mu_i(t) \mid t \mid^{u_i}$ for a suitable i and

$$\xi_{u,1}\xi_{u,2}\cdots\xi_{u,n}(t) = \mu_1\mu_2\cdots\mu_n(t) \mid t \mid^{u_1+u_2+\cdots u_n} .$$

PROPOSITION 6. *We fix a multi strip*

$$\mathfrak{S} = \{u \mid -A < \Re u_i < A, 1 \le i \le n\},\ A \notin \mathbb{Z}_F. \tag{32}$$

With the above notations, there is a polynomial P, product of linear factors, with the following properties. For $u \in \mathfrak{S}$,

$$P(u) w_{\Psi,u}(m : X) = \sum \Phi_{\xi_u}(m^b m^{-e} : X)\xi_u(m)$$

where the sum is over all the characters ξ_u of A_n of the above type. Suppose that Ψ remains in a bounded set. Then one can choose the functions Φ_* in a bounded set. Suppose that Ψ depends holomorphically on $s \in \Omega$, where Ω is open in $\mathbb{C}^m$, and remains in a bounded set for all values s. Then one can choose the functions Φ_* to depend holomorphically on (u, s) and to remain in a bounded set.

Proof. We prove the proposition by induction on n. The case $n = 2$ is Proposition 2 of section 4. Thus we assume $n > 2$ and the result true for $(n-1)$. Set $u' = (u_1, u_2, \ldots, u_{n-1})$ and let τ_u be the representation of degree $(n-1)$ of $F^\times$ associated with u' and the characters μ_i, $1 \le i \le n-1$.

As before, we suppress the characters μ_i from the notations. Consider the strip $\mathfrak{S}' = \{u' \mid -A < \Re u_i < A, 1 \le i \le n-1\}$. Let $Q'(u')$ be the polynomial whose existence is guaranteed by the induction hypothesis. Then the product of $Q(u')$ $\mid \det a_{n-1} \mid^{u_{n-1}}$ and the integral

$$\int \Psi\left[a_1 a_2^{-b}, a_1^{-1} a_2^{e} : a_2 a_3^{-b}, a_2^{-1} a_3^{e} : \cdots : a_{n-2} a_{n-1}^{b}, a_{n-2}^{-1} a_{n-1}^{-e} : Y : X\right]$$
$$\prod_{i=1}^{i=n-2} \mid \det a_i \mid^{u_{i+1}-u_i} da_i$$

is a sum of terms of the form

$$\Phi_{u'}(a_{n-1}^{b} a_{n-1}^{-e} : Y : X)\eta(a_{n-1}).$$

The components η_i of η have the property that each product $\eta_1\eta_2\cdots\eta_j$ is a component of $\bigwedge^j \tau_u$. Using this result we see that the product $Q(u')w_{\Psi,u}(m : X)$ is a sum of terms of the following form

$$\mid \det m \mid^{u_n}$$
$$\times \int \Phi_{u'}\left[a_{n-1}^{-b} a_{n-1}^{e} : a_{n-1} m^{b}, a_{n-1}^{-1} m^{e} : X\right] \eta^{-1}(a_{n-1}) \mid \det a_{n-1} \mid^{u_n} da_{n-1}.$$

More explicitly, in terms of the entries b_i of the matrix a_{n-1} and the entries m_i of m, this expression reads

$$\mid m_1 m_2 \cdots m_{n-1} m_n \mid^{u_n} \int \Phi_{u'}\left[b_1^{-1} b_2, b_2^{-1} b_3, \ldots, b_{n-2}^{-1} b_n :\right.$$
$$\left. b_1 m_1, b_1^{-1} m_2^{-1}, b_2 m_2, b_2^{-1} m_3^{-1}, \ldots, b_{n-1} m_{n-1}, b_{n-1}^{-1} m_n^{-1} : X\right]$$
$$\eta_1^{-1}(b_1)\eta_2^{-1}(b_2)\cdots\eta_{n-1}^{-1}(b_n) \mid b_1 b_2 \cdots b_{n-1} \mid^{u_n} \bigotimes d^{\times} b_i.$$

For convenience, we have changed the order of the variables. In general, we remark that if $\Phi(x, y, z, X)$ is a Schwartz function of (x, y, z, X), then the function $\Phi(yz, y, z, X)$ is still a Schwartz function of (y, z, X). Applying this simple remark repeatedly, we see the above expression has the form:

$$\int \Phi_{u'}^{1}\left[b_1 m_1, b_1^{-1} m_2^{-1}, b_2 m_2, b_2^{-1} m_3^{-1}, \ldots, b_{n-1} m_{n-1}, b_{n-1}^{-1} m_n^{-1} : X\right] \tag{33}$$
$$\eta_1^{-1}(b_1)\eta_2^{-1}(b_2)\cdots\eta_{n-1}^{-1}(b_n) \mid b_1 b_2 \cdots b_{n-1} \mid^{u_n}$$
$$\bigotimes d^{\times} b_i \mid m_1 m_2 \cdots m_{n-1} m_n \mid^{u_n}$$

where $\Phi_{u'}^{1}$ is a new Schwartz function. Next we apply Proposition 2 of section 4 repeatedly and we see that there is a polynomial $R(u)$ such that the product of the previous expression and $R(u)$ is a sum of terms of the following form

$$\Phi_u\left(m_1 m_2^{-1}, m_2 m_3^{-1}, \ldots, m_{n-1} m_n^{-1}\right) \xi(m),$$

where the character $\xi(m)$ is obtained as the product of $|\det m|^{u_n}$ and a character obtained from the following table

$$\left\{\begin{matrix}\eta_1(m_1)\mid m_1\mid^{-u_n}\\ \eta_1(m_2)\mid m_2\mid^{-u_n}\end{matrix}\right\}\left\{\begin{matrix}\eta_2(m_2)\mid m_2\mid^{-u_n}\\ \eta_2(m_3)\mid m_3\mid^{-u_n}\end{matrix}\right\}\cdots$$

$$\left\{\begin{matrix}\eta_{n-1}(m_{n-1})\mid m_{n-1}\mid^{-u_n}\\ \eta_{n-1}(m_n)\mid m_n\mid^{-u_n}\end{matrix}\right\}.$$

In each column we choose in any way a character and multiply our choices together. If we evaluate ξ on an element of the form

$$\mathrm{diag}(\overbrace{m_0, m_0, \ldots, m_0}^{j}, 1, 1\ldots, 1)$$

we find $\eta_1\eta_2\cdots\eta_j(m_0)$ if $j<n$ and $\eta_1\eta_2\cdots\eta_{n-1}(m_0)\mid m_0\mid^{u_n}$ if $j=n$. Thus the character ξ has the required properties. $\square$

We can state our main theorem. Indeed, it follows at once from the previous proposition and the integral representation of the Whittaker functions.

THEOREM 2. *Suppose that f_u is a standard section and ϕ the convolution of $(n-1)$ elements ϕ_i of $\mathcal{D}(G_n)$. Fix a multi strip $\mathfrak{S}=\{u|-A<\Re u_i<A, 1\le i\le n\}$ with $A\notin\mathbb{Z}_F$. Then there is a polynomial $P(u)$, product of linear factors, such that in the multi strip*

$$\begin{aligned}&P(u)\mathcal{W}_u(\Xi_u(mk)f_{\phi,u})\\ &\quad=\sum_{\xi_u}\Phi_{\xi_u,u}\left(m_1m_2^{-1}, m_2m_3^{-1},\ldots, m_{n-1}m_n^{-1}), k\right)\xi_u(m)e^{\langle H(m),\rho\rangle}.\end{aligned}$$

For each ξ_u, and each u the function

$$\Phi_{\xi_u,u}(x_1, x_2,\ldots, x_{n-1}, k)$$

is in $\mathcal{S}(F^{n-1}\times K)$ and $u\mapsto\Phi_{\xi_u,u}$ is a holomorphic function with values in $\mathcal{S}(F^{n-1}\times K)$. Its values are in a bounded set if f_u and the functions ϕ_i are each in a bounded set.

If $W_u(g)=\mathcal{W}_u(\rho(mk)f_{\phi,u})$ then we set

$$\widetilde{W}_{-u}:=W_u\left(w_n{}^tg^{-1}\right). \tag{34}$$

Note that $\widetilde{W}_{-u}(ng)=\overline{\theta}(n)\widetilde{W}_{-u}(g)$. Moreover, for $m\in A_n, k\in K_n$,

$$\widetilde{W}_u(mk)=W_u(wm^{-1}wk'),\ k'=w\,{}^tk^{-1}\in K_n.$$

We remark that, for $1\le i\le n$,

$$\bigwedge^{n-j}\sigma_u\otimes\det\sigma_u^{-1}=\bigwedge^{j}\widetilde{\sigma}_u$$

and

$$\check{\varpi}_i(wt^{-1}w) = \check{\varpi}_{n-i}(t)\check{\varpi}_n(t)^{-1}.$$

It follows that the function $\widetilde{W}_u(g)$ has the same properties as the function W_u, except that the representation σ_u is replaced by the contragredient representation, or, what amounts to the same, the characters μ_i are replaced by their inverses and u by $-u$. In particular, we have the following proposition, the proof of which is immediate.

PROPOSITION 7. *Notations being as in the theorem, the product*

$$P(u)\widetilde{W}_{-u}(mk)$$

can be written as

$$\sum_{\xi_u} \widetilde{\Phi}_{\xi_u,u}\left(m_1 m_2^{-1}, m_2 m_3^{-1}, \ldots, m_{n-1}m_n^{-1}), k\right) \xi_u(m)^{-1} e^{\langle H(m),\rho\rangle}$$

where the sum is the same as before and the functions $\widetilde{\Phi}_{\xi_u,u}$ have the same properties as above.

We shall need estimates which are uniform in u for the Whittaker functions. We first consider integrals of the form (31).

PROPOSITION 8. *Suppose that Ψ is in a bounded set and u in a multi strip. There are integers N_i, $1 \le i \le n-1$, with the following property. For any integer $M > 0$ there is a majorization*

$$|w_{\Psi,u}(m : X)| \le | m_n |^{\Re(\sum_{1\le i\le n} u_i)} \prod_{i=1}^{i=n-1} \frac{\left|m_i m_{i+1}^{-1}\right|^{-N_i}}{\left(1+\left|m_i m_{i+1}^{-1}\right|\right)^M} \Phi_0(X)$$

where $\Phi_0 \ge 0$ is in $\mathcal{S}(\mathbb{X})$. Suppose that u is purely imaginary. Then there is a polynomial Q in $\log(| m_i/m_{i+1} |)$ *such that, for every integer M, there is a majorization*

$$|w_{\Psi,u}(m : X)| \le \frac{Q(\log | m_i/m_{i+1} |)}{\prod_{i=1}^{i=n-1}\left(1+\left|m_i m_{i+1}^{-1}\right|\right)^M} \Phi_0(X).$$

Proof. We prove the first assertion by induction on n. For the case $n = 2$, we have, dropping the characters μ_i from the notation,

$$w_{\Psi,u}(m : X) = | m_1 m_2 |^{u_2} \int \Psi\left(a_1 m_1, a_1^{-1} m_2^{-1}\right) | a_1 |^{u_2 - u_1} d^\times a_1$$
$$= | m_2 |^{u_1+u_2} | m_1 m_2^{-1} |^{u_2} \int \Psi(a_1 m_1 m_2^{-1}, a_1^{-1}) | a_1 |^{u_2-u_1} d^\times a_1$$

and then our assertion follows from Proposition 3 of Section 4. Thus we may assume $n > 2$ and our assertion true for $n-1$. We may also assume that Ψ is bounded by a function ≥ 0 which is a product and so we may ignore the dependence on X

and assume $\Psi \geq 0$ and the u_i are real. Then, for $a_{n-1} \in A_{n-1}$ with entries b_i, the product of $|\det a_{n-1}|^{u_{n-1}}$ and the integral

$$\int \Psi\left[a_1 a_2^{-b}, a_1^{-1} a_2^{e} : a_2 a_3^{-b}, a_2^{-1} a_3^{e} : \cdots : a_{n-2} a_{n-1}^{b}, a_{n-2}^{-1} a_{n-1}^{-e} : Y\right]$$
$$\prod_{i=1}^{i=n-2} |\det a_i|^{u_{i+1}-u_i}\, da_i$$

is bounded by

$$|b_{n-1}|^{\sum_{1\leq i\leq n-1} u_i} \times \prod_{i=1}^{i=n-2} \frac{\left|b_i b_{i+1}^{-1}\right|^{-N_i}}{\left(1+\left|b_i b_{i+1}^{-1}\right|\right)^M} \Phi_0(Y).$$

Thus $|w_{\Psi,u}(m)|$ is bounded by

$$|\det m|^{u_n} \times \int |\det a_{n-1}|^{u_n} |b_{n-1}|^{-\sum_{1\leq i\leq n-1} u_i} \prod_{i=1}^{i=n-2} \frac{\left|b_i b_{i+1}^{-1}\right|^{N_i}}{\left(1+\left|b_i^{-1} b_{i+1}\right|\right)^M}$$
$$\Phi_0\left(a_{n-1} m^b, a_{n-1}^{-1} m^{-e}\right) da_{n-1}.$$

In turn, this is majorized by

$$|\det m|^{u_n} \times \int |\det a_{n-1}|^{u_n} |b_{n-1}|^{-\sum_{1\leq i\leq n-1} u_i} \prod_{i=1}^{i=n-2} \left|b_i b_{i+1}^{-1}\right|^{N_i+M}$$
$$\Phi_0\left(a_{n-1} m^b, a_{n-1}^{-1} m^{-e}\right) da_{n-1}.$$

This has the form

$$|\det m|^{u_n} \times \int \prod_{i=1}^{i=n-1} |b_i|^{s_i}\ \Phi_0\left(a_{n-1} m^b, a_{n-1}^{-1} m^{-e}\right) da_{n-1}$$

where each s_i belongs to some fixed interval and

$$\sum_{i=1}^{i=n-1} s_i = n u_n - \sum_{i=1}^{i=n} u_i.$$

Explicitly, the integral reads (after changing the order of the variables)

$$|\det m|^{u_n} \times \int \Phi_0\left[b_1 m_1, b_1^{-1} m_2^{-1}, b_2 m_2, b_2^{-1} m_3^{-1}, \ldots, b_{n-1} m_{n-1}, b_{n-1}^{-1} m_n^{-1}\right]$$
$$\prod_{i=1}^{i=n-1} |b_i|^{s_i}\, d^{\times} b_i.$$

Changing variables, we get:

$$| \det m |^{u_n} | m_2 |^{-s_1} | m_3 |^{-s_2} \cdots | m_n |^{-s_{n-1}}$$
$$\times \int \Phi_0 \left[b_1 m_1 m_2^{-1}, b_1^{-1}, b_2 m_2 m_3^{-1}, b_2^{-1}, \ldots, b_{n-1} m_{n-1} m_n^{-1}, b_{n-1}^{-1}\right]$$
$$\prod_{i=1}^{i=n-1} | b_i |^{s_i} \, d^\times b_i,$$

or, finally,

$$\prod_{1 \leq i \leq n-1} | m_i m_{i+1}^{-1} |^{t_i} | m_n |^{\sum_{1 \leq i \leq n} u_i}$$
$$\int \Phi_0 \left[b_1 m_1 m_2^{-1}, b_1^{-1}, b_2 m_2 m_3^{-1}, b_2^{-1}, \ldots, b_{n-1} m_{n-1} m_n^{-1}, b_{n-1}^{-1}\right]$$
$$\prod_{i=1}^{i=n-1} | b_i |^{s_i} \, d^\times b_i,$$

where each one of the exponents t_i is in some fixed interval. Our assertion follows then from repeated use of Proposition 3 in Section 4. This concludes the proof of the first assertion. The proof of the second assertion is similar. □

Using once more the integral representation of the Whittaker functions we obtain at once the following estimates.

PROPOSITION 9. *Suppose that u is in a multi strip, f_u in a bounded set and the functions ϕ_i in a bounded set. Then there are $N_i > 0$, $1 \leq i \leq n-1$, such that, for all $M > 0$,*

$$|\mathcal{W}_u(\Xi_u(mk) f_{\phi,u})| \leq c_M \, | m_n |^{\Re u_n} \, e^{\langle H(m), \rho \rangle} \prod_{i=1}^{i=n-1} \frac{| m_i / m_{i+1} |^{-N_i}}{\left(1 + \left|m_i m_{i+1}^{-1}\right|\right)^M}.$$

If u is purely imaginary, then there is a polynomial Q in $n-1$ variables such that

$$|\mathcal{W}_u \, \Xi_u(mk) f_{\phi,u})| \leq | \det m |^{\Re u_n} \, e^{\langle H(m), \rho \rangle} \frac{Q(\log(| m_i / m_{i+1} |)}{\prod_{i=1}^{i=n-1} \left(1 + \left|m_i m_{i+1}^{-1}\right|\right)^M}.$$

8. Rankin-Selberg integrals for $GL(n) \times GL(n)$. In this section we consider two n-tuples of (normalized) characters

$$(\mu_1, \mu_2, \ldots, \mu_n), \; (\mu_1', \mu_2', \ldots, \mu_n')$$

and two elements u and u' of $\mathbb{C}^n$. We denote by $\mathcal{W}_{u'}'$ the linear form on the space $\mathcal{V}(\mu')$ which is defined by analytic continuation of the integral

$$\mathcal{W}_{u'}'(f_{u'}) = \int_{N_n} int f_{u'}(v) \theta_n(v) dv.$$

We consider two corresponding convolution sections $f_{\phi,u}$ and $f_{\phi',u'}$ and we set

$$W_u(g) = \mathcal{W}_u(\Xi_u(g) f_{\phi,u}),\ W'_{u'}(g) = \mathcal{W}'_{u'}(\Xi_{u'}(g) f_{\phi',u'}). \tag{35}$$

We study the corresponding Rankin-Selberg integral. We set

$$\epsilon_n = (0, 0, 0, \dots, 0, 1). \tag{36}$$

We let and Φ be an element of $\mathcal{S}(F^n)$. We define

$$\Psi(s, W_u, W'_{u'}, \Phi) := \int_{N_n \backslash G_n} W_u(g) W'_{u'}(g') \Phi[(\epsilon_n) g] \mid \det g \mid^s dg. \tag{37}$$

Proposition 10. *If (u, u') are in a multi strip then there is s_0 such that the integral converges absolutely in the right half plane $\Re s > s_0$, uniformly on any vertical strip contained in the half plane. If u and u' are purely imaginary, then the integral converges in the half plane $\Re s > 0$, uniformly on any vertical strip contained in the half plane.*

Proof. We write $g = ak$ with $dg = e^{\langle H(a), -2\rho\rangle} da dk$. We write a in terms of the co-weights:

$$a = \prod_{1 \le i \le n} \check{\varpi}_i(a_i) = \mathrm{diag}(a_1 a_2 \cdots a_n, a_2 \cdots a_n, \dots, a_n)$$

and use the majorization of $W_u(ak)$ and $W'_{u'}(ak)$ from the previous section. We find that

$$W_u(ak) W'_{u'}(ak) e^{\langle H(a), -2\rho\rangle}$$

is majorized by

$$c_M \prod_{1 \le i \le n-1} \frac{\mid a_i \mid^{-M_i}}{(1+ \mid a_i \mid)^N} \mid a_n \mid^{\Re\left(\sum u_i + \sum u'_i\right)}$$

where the integer N is arbitrarily large. On the other hand, for all N, $\Phi[(\epsilon_n) ak]$ is majorized by

$$c'_N \frac{1}{(1+ \mid a_n \mid)^N}.$$

The integral is thus majorized by

$$\int \frac{\mid a_1 \mid^{\Re s - M_1}}{(1+ \mid a_1 \mid)^N} \frac{\mid a_2 \mid^{2\Re s - M_2}}{(1+ \mid a_2 \mid)^N} \cdots \\ \frac{\mid a_{n-1} \mid^{(n-1)\Re s - M_{n-1}}}{(1+ \mid a_{n-1} \mid)^N} \frac{\mid a_n \mid^{\sum \Re u_i + \sum \Re u'_i + n \Re s}}{(1+ \mid a_n \mid)^{N_n}} \otimes d^\times a_i$$

and the first assertion follows. The proof of the second assertion is similar. We simply majorize $W_u(ak) W'_{u'}(ak) e^{\langle H(a), -2\rho\rangle}$ by a polynomial in the variables $\log \mid a_i \mid$, $1 \le i \le n-1$. □

Our next result will be improved upon in the next theorem.

PROPOSITION 11. *Let σ_u and $\sigma'_{u'}$ be the n-dimensional representations of $F^\times$ (or the Weil group) defined by*

$$\sigma_u = \oplus \mu_i \alpha_F^{u_i},\ \sigma'_{u'} = \oplus \mu'_i \alpha_F^{u'_i}.$$

Then $\Psi(s, W_u, W'_{u'}, \Phi)$ extends to a meromorphic function of (s, u, u') which is the product of

$$\prod_{j=1}^{j=n} L\left(js, \bigwedge^j \sigma_u \otimes \bigwedge^j \sigma'_{u'}\right)$$

and an entire function of (s, u, u').

Proof. Choose a multi strip $\mathfrak{S}$ of the form (32) in $\mathbb{C}^{2n}$. For $(u, u') \in \mathfrak{S}$ the integral converges in some halfspace $\mathfrak{A} = \{s | \Re s > s_0\}$ and thus defines a holomorphic function of (s, u, u') in $\mathfrak{A} \times \mathfrak{S}$. As before let us write $g = ak$ and a in terms of the co-weights. There are polynomials P, P', products of linear factors, such that

$$P(u)P(u')W_u(ak)W'_{u'}(ak)e^{\langle H(a), -2\rho\rangle}$$

is a sum of terms of the following form

$$\Phi_u(a_1, a_2, \ldots, a_{n-1}, k)\Phi'_{u'}(a_1, a_2, \ldots, a_{n-1}, k)$$
$$\xi_{1,u}\xi'_{1,u'}(a_1)\xi_{2,u}\xi'_{2,u'}(a_2)\cdots\xi_{n-1,u}\xi'_{n-1,u'}(a_{n-1})\xi_{n,u}\xi'_{n,u'}(a_n)$$

where each $\xi_{i,u}$ (resp. $\xi'_{i,u'}$) is a component of $\bigwedge^i \sigma_u$ (resp. $\bigwedge^i \sigma'_{u'}$). On the other hand

$$|\det ak|^s = |a_1|^s |a_2|^{2s} \cdots |a_n|^{ns}$$

and

$$\Phi(\epsilon_n ak) = \phi(a_n, k)$$

where ϕ is a Schwartz function in one variable depending on k. The contribution of the term at hand to the total integral has thus the form

$$\int \left(\int \Phi_u(a_1, a_2, \ldots, a_{n-1}, k)\Phi'_u(a_1, a_2, \ldots, a_{n-1}, k)\phi(a_n, k)dk\right)$$
$$\prod_{i=1}^{n} \xi_{i,u}\xi'_{i,u'}(a_i) \mid a_i \mid^{is} d^\times a_i.$$

After integrating over K_n we obtain a multivariate Tate integral (where in addition the Schwartz function depends holomorphically on (u, u')). Thus

$$P(u)P(u')\Psi(s, W_u, W'_{u'}, \Phi)$$

is a sum of terms of the form

$$h(s,u,u')\prod_{j=1}^{n}L(js,\xi_j\xi'_j)$$

where h is holomorphic in $\mathbb{C}\times\mathfrak{S}$. This already shows that Ψ extends to a meromorphic function on $\mathbb{C}\times\mathfrak{S}$. The only possible singularities are the singularities of the L-factors and the zeroes of $P(u)P'(u')$. Thus they are hyperplanes of the form:

$$js+u_{i_1}+u_{i_2}+\cdots+u_{i_j}+u'_{i'_1}+u'_{i'_2}+\cdots+u'_{i'_j}=z_0,$$
$$l(u)=z_1,\ l(u')=z_2$$

with $z_*\in\mathbb{Z}_F$. Consider a hyperplane of the form $l(u)=z_1$ which intersects $\mathbb{C}\times\mathfrak{S}$. Then it also intersects $\mathfrak{A}\times\mathfrak{S}$. However the function $\Psi(s,W_u,W'_{u'},\Phi)$ is holomorphic in this region because the integral is convergent, thus this is not actually a singular hyperplane. Likewise for a hyperplane $l(u')=z_2$. We conclude that the only singularities of $\Psi(s,W_u,W'_{u'},\Phi)$ are those of the L-factors and we are done. □

PROPOSITION 12. *Let u_0 and u'_0 be fixed elements of $\mathbb{C}^n$. Then the meromorphic function*

$$\Psi(s,W_{u_0},W'_{u'_0},\Phi)$$

is bounded at infinity in vertical strips.

Proof. Fix a multi strip $\mathfrak{S}$ of the form (32) in $\mathbb{C}^{2n+1}$. Only finitely many singular hyperplanes of $\Psi(s,W_u,W'_{u'})$ intersect $\mathfrak{S}$. Let $Q(s,u,u')$ be the product of the corresponding (non-homogeneous) linear forms, with the appropriate multiplicity, repeated according to their multiplicity. The product of $Q(s,u,u')$ and any of the Tate integrals considered in the previous proposition is actually bounded in $\mathfrak{S}$. Thus the product

$$k(s,u,u'):=Q(s,u,u')P(u)P'(u')\Psi(s,W_u,W'_{u'})$$

is holomorphic and bounded in $\mathfrak{S}$. Now we choose slightly smaller multi strip $\mathfrak{S}'$ and $\mathfrak{S}$" such that

$$\mathfrak{S}'\subset\text{Closure}(\mathfrak{S}')\subset\mathfrak{S}''\subset\text{Closure}(\mathfrak{S}'')\subset\mathfrak{S}.$$

By the Cauchy integral formula, k and all its derivatives are bounded in $\mathfrak{S}$". On the other hand, by the previous proposition

$$k(s,u,u')=P(u)P'(u')h(s,u,u')$$

where h is holomorphic in $\mathfrak{S}$. We claim that h and all its derivatives are bounded in Closure($\mathfrak{S}'$) which will prove the proposition. We think of $\mathfrak{S}$" as the product $\mathfrak{a}\times\mathfrak{b}$ where

$$\mathfrak{a}=\{s|-a<\Re s<a\}$$

and

$$\mathfrak{b} = \{(u, u')|\forall i \ -a < \Re u_i < a, \ \forall j \ -a \Re u'_j < a\}.$$

Clearly, it will be enough to show that for every $(u_0, u'_0) \in \mathfrak{b}$, there is a tubular neighborhood $\mathfrak{u}$ of (u_0, u'_0) in $\mathfrak{b}$ such that h and all its derivatives are bounded in $\mathfrak{a} \times \mathfrak{u}$. Indeed, a finite number of the sets $\mathfrak{a} \times \mathfrak{u}$ cover Closure($\mathfrak{S}'$). Thus it will suffice to apply repeatedly the following lemma. □

LEMMA 7. *Let $\mathfrak{a} = \{s \in \mathbb{C}| - a < \Re s < a\}$ and let $\mathfrak{u}$ be a tubular open set in $\mathbb{C}^n$. Suppose that k is a holomorphic function on $\mathfrak{a} \times \mathfrak{u}$ uniformly bounded as well as its derivatives. Suppose that*

$$h(s, u) = (l(u) - z)k(s, u)$$

where k is holomorphic and $l(u) = \sum_{i=1}^{n} u_i v_i$ with (v_i) non-zero in $\mathbb{R}^n$ and z is real. Given $u_0 \in \mathfrak{u}$, there is a tubular neighborhood $\mathfrak{u}_0$ of u_0 in $\mathfrak{u}$ such that h and all its derivatives are bounded in $\mathfrak{a} \times \mathfrak{u}_0$.

Proof of the Lemma. Perhaps we should recall that a tubular neighborhood of u_0 in $\mathbb{C}^n$ is a set of the form $\{u|\Re u \in \Omega\}$ where Ω is a neighborhood of $\Re u_0$ in $\mathbb{R}^n$. There is no harm in assuming u_0 real. Our assertion is trivial if $l(u_0) - z \neq 0$. Thus we may assume $l(u_0) = z$. After a real change of coordinates in the variables u, we may assume $u_0 = 0$, $z = 0$, and

$$k(s, z_1, z_2, \ldots, z_n) = z_1 h(s, z_1, z_2, \ldots, z_n)$$

and $\mathfrak{u}$ contains the set

$$\mathfrak{u}_0 = \{z|\forall i \ -b < \Re z_i < b\}.$$

Then

$$h(s, z_1, z_2, \ldots, z_n) = \int_0^1 \frac{\partial}{\partial z_1} k(s, t z_1, z_2, \ldots, z_n) dt$$

and this formula shows that h and all its derivatives are bounded on $\mathfrak{u}_0$. □

If we fix u_0 and u'_0 then the functions $\Psi(s, W_{u_0}, W'_{u_0}, \Phi)$ are holomorphic multiples of $L(s, \sigma_{u_0} \times \sigma_{u'_0})$. One can improve the previous result as follows. Let $\mathcal{H}$ be the space of holomorphic multiples $h(s)$ of $L(s, \sigma_{u_0} \times \sigma_{u'_0})$ such that, for any strip $-a < \Re s < a$, and any polynomial $P(s)$ such that $P(s)L(s, \sigma_{u_0} \times \sigma_{u'_0})$ has no pole on $-a \leq \Re s \leq a$, the product $P(s)h(s)$ is bounded in the open strip. Then $\Psi(W_{u_0}, W'_{u_0}, \Phi)$ is in $\mathcal{H}$. In fact, we can prove even more. Suppose that f and f' are elements of $V(\mu)$ and $V(\mu')$, each in some bonded set $\mathcal{B}$ and $\mathcal{B}'$ respectively. Let

$$W(g) = \mathcal{W}_{u_0}(\Xi_{u_0}(g)), \ W'(g) = \mathcal{W}_{u_0}(\Xi_{u_0}(g)).$$

Suppose that Φ is in some bounded set $\mathcal{C}$ of $\mathcal{S}(F^n)$. Then we claim

$$\Psi(s, W, W', \Phi)$$

is in a bounded set of $\mathcal{H}$. Since the spaces $V(\mu)$, $V(\mu')$ and $\mathcal{S}(F^n)$ are bornological spaces, this proves that the trilinear map

$$(f, f', \Phi) \mapsto \Psi(\bullet, W, W', \Phi)$$

from $V(\mu) \times V(\mu') \times \mathcal{S}(F^n)$ to $\mathcal{H}$ is continuous. To verify our claim we use the Dixmier Malliavin Lemma to write

$$f = \sum_i \Xi_{u_0}(\phi_i) f_i$$

with ϕ_i in a bounded set of $\mathcal{D}(G_n)$ and f_i in a bounded set of $V(\mu)$. We consider the standard sections $f_{i,u}$ attached to the f_i, the convolution sections $f_{i,\phi_i,u}$, and the corresponding Whittaker functions $W_{i,u}$. Likewise we write

$$f' = \sum_j \Xi_{u'_0}(\phi'_j) f'_j$$

and define $W'_{j,u'}$. Then

$$\Psi(s, W, W') = \sum_{i,j} \Psi(s, W_{i,u_0}, W'_{j,u'_0}, \Phi).$$

Our construction shows that each term is in a bounded set of $\mathcal{H}$.

We now are ready to state our main theorem.

THEOREM 3. *The ratio*

$$\frac{\Psi(s, W_u, W'_{u'}, \Phi)}{L(s, \sigma_u \otimes \sigma'_{u'})} \tag{38}$$

is a holomorphic function of (s, u, u') *in* $\mathbb{C}^{2n+1}$.

Proof. We recall the functional equation satisfied by the functions Ψ ([JS]). We introduce, as before,

$$\widetilde{W}_u(g) := W_u(w\, {}^t g^{-1}),\ \widetilde{W}'_{u'}(g) := W'_{u'}(w\, {}^t g^{-1}).$$

The function $\widetilde{W}_u(g)$ has the same properties as the function W_u, except that the representation σ_u is replaced by the contragredient representation $\widetilde{\sigma_u}$. The equation in question reads:

$$\frac{\Psi(\widetilde{W}_u, \widetilde{W}'_{u'}, \Phi)}{L(1 - s, \widetilde{\sigma_u} \otimes \widetilde{\sigma'_{u'}})} = \epsilon(s, \sigma_u \otimes \sigma'_{u'}, \psi) \frac{\Psi(s, W_u, W'_{u'}; \Phi)}{L(s, \sigma \otimes \sigma')}.$$

Fix u and u' purely imaginary. Then the integral defining $\Psi(s, W_u, W'_{u'}; \Phi)$ converges for $\Re s > 0$. Thus the right-hand side, which is a priori defined for $\Re s >> 0$

extends to a function holomorphic in the half plane $\Re s > 0$. Likewise, the left-hand size extends to a holomorphic function in the half plane $\Re s < 1$. We conclude that the right-hand side, for u, u' fixed and imaginary, is actually an entire function of s.

Now by the previous proposition, the ratio (38) has the form

$$h(s,u,u')\prod_{i=2}^{i=n} L\left(s, \bigwedge^{i} \sigma_u \otimes \bigwedge^{i} \sigma'_{u'}\right) \tag{39}$$

where h is holomorphic. The singularities of $L(s, \bigwedge^i \sigma_u \otimes \bigwedge^i \sigma'_{u'})$ are hyperplanes of the form:

$$is + u_{j_1} + u_{j_2} + \cdots + u_{j_i} + u'_{k_1} + u'_{k_2} + \cdots u'_{k_i} = -z$$

with $z \geq 0, z \in \mathbb{Z}_F, j_1 < j_2 < \cdots < j_i, k_1 < k_2 < \cdots < k_i$. Fix u and u' imaginary in such a way that the values of s determined by these equations are all distinct. Then, for $\Re s >> 0$, the expression (39) is equal to the integral divided by $L(s, \sigma_u \otimes \sigma'_{u'})$. Thus for fixed imaginary (u, u') it extends to an entire function of s. This forces $h(s, u, u') = 0$ for any s satisfying one (and only one) of these equations. It follows that h vanishes identically on any of these hyperplanes. Thus it is divisible by each one of the linear forms $is + u_{j_1} + u_{j_2} + \cdots + u_{j_i} + u'_{k_1} + u'_{k_2} + \cdots u'_{k_i} + z$. If these singularities have multiplicity this argument can be repeated. Our assertion follows. □

9. Other series for the real case. It remains to prove the analog of the previous theorem for representations which are (non-unitarily) induced by discrete series of $GL(2)$ and characters of $GL(1)$. Thus we assume from now on that F is real.

We first recall the following result on the convergence of the integrals.

PROPOSITION 13. *Let π and π' be unitary irreducible tempered representations of $GL(n)$. For W (resp. W') in the Whittaker model of π (resp. π') and any $\Phi \in \mathcal{S}(F^n)$, the integral $\Psi(s, W, W', \Phi)$ converges absolutely for $\Re s > 0$, uniformly for $\Re s$ in a compact set.*

Proof. We think of the smooth vectors of π as elements of the Whittaker model. Let λ be the evaluation at e. Thus

$$W(g) = \lambda(\pi(g)W).$$

According to [W] (Lemma 15.2.3 and Theorem 15.2.5, page 375), the linear form λ is tame for the pair (B, A); thus there is $\Lambda_\pi \in \mathfrak{a}^*$ and a continuous semi-norm q such that, for $a \in A^-$ and any smooth vector v,

$$\lambda(\pi(a)v) \leq e^{\langle \Lambda_\pi + \rho, H(a)\rangle} (1+ \| \log a \|)^d\, q(v).$$

Since π is tempered, $\Lambda_\pi = \sum_{1\le i\le n-1} x_i\alpha_i$ with $x_i \ge 0$. We also recall that there is an integer N such that $q(\pi(g)v) \le \| g \|^N q(v)$.

A similar result applies to π' with a linear form $\Lambda_{\pi'}$, a semi-norm q' and a number N'.

Now

$$\begin{aligned}
&\Psi(s, W, W', \Phi)\\
&\quad = \int_{A_{n-1}\times F^\times \times K_n} W\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} k\right] W'\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} k\right] \Phi[(0,0,\ldots,0,a_n)k]\\
&\qquad | a_n |^{ns}\ \omega\omega'(a_n) e^{\langle H(a), -2\rho\rangle} \mid \det a \mid^s\ da d^\times a_n dk,
\end{aligned}$$

where ω and ω' are the central characters. We write as before

$$a = \prod_{i=1}^{i=n-1} b_i,\ b_i = \check{\varpi}_i(a_i) = \operatorname{diag}(\overbrace{a_i, a_i, \ldots, a_i}^{i}, 1, \ldots, 1)\,.$$

By the Dixmier-Malliavin Lemma we may assume that

$$W(g) = \int W_0(gx)\phi(x)dx$$

where ϕ is smooth of compact support. Then

$$\begin{aligned}
&W\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} k\right]\\
&\quad = \int W_0\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} n_1 m_1 k_1\right]\phi(n_1 m_1 k_1 k^{-1}) dn_1 dm_1 e^{H(\langle m_1\rangle, -2\rho\rangle} dk_1\\
&\quad = \int W_0\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} m_1 k_1\right]\phi(n_1 m_1 k_1 k^{-1})\theta(a n_1 a^{-1}) dn_1 dm_1 e^{H(\langle m_1\rangle, -2\rho\rangle} dk_1.
\end{aligned}$$

After integrating over n_1 we see that this has the form:

$$\int W_0\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} m_1 k_1\right]\phi_1(a_1, a_2, \ldots a_{n-1}; m_1, k_1, k) dm_1 dk_1,$$

where ϕ_1 is a smooth function on $\mathbb{F}^n \times A_n \times K_n \times K_n$ whose support has a compact projection on A_n and which is rapidly decreasing with respect to the first $n-1$ variables.

If T is a subset (possibly empty) of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ of simple roots (or, equivalently, a subset of $[1, n-1]$), we denote by $A(T)$ the set of diagonal matrices a such that $| \alpha_i(a) | \le 1$ for $\alpha_i \in T$ and $| \alpha_i(a) | > 1$ for $\alpha_i \notin T$. Thus A is the union of the set $A(T)$. It will suffice to prove that for each T the integral

$$\int_{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in A(T)} W\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right] W'\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right] \Phi[(0, 0, \ldots, 0, a_n)k]$$
$$| a_n |^{ns} \omega\omega'(z) e^{\langle H(a), -2\rho\rangle} | \det a |^s \, da d^\times a_n dk$$

converges absolutely. We write

$$a = b_T b^T$$

where $b_T = \prod_{i \in t} b_i$. We have thus

$$\left| W_0 \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} m_1 k_1\right]\right| \leq e^{\langle H(b_T), \Lambda_\pi + \rho\rangle} (1 + \| \log b_T \|)^d q\left(\pi(b^T m_1 k_1) W_0\right)$$

or

$$\left| W_0 \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} m_1 k_1\right]\right| \leq e^{\langle H(b_T), \Lambda_\pi + \rho\rangle} (1 + \| \log b_T \|)^d \| b^T \|^N \| m_1 \|^N \pi(W_0).$$

Thus

$$\left| W \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right]\right| \leq e^{\langle H(b_T), \Lambda_\pi + \rho\rangle} (1 + \| \log b_T \|)^d$$
$$\times \| b^T \|^N \int | \phi | (a_1, a_2, \cdots, a_{n-1}; m_1, k_1, k) \| m_1 \|^N dm_1 dk_1.$$

Thus after integrating over m_1 and k_1, we find that

$$\left| W \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right]\right| \leq e^{\langle H(b_T), \Lambda_\pi + \rho\rangle} (1 + \| \log b_T \|)^d \| b^T \|^N \phi_0(a_1, a_2, \cdots, a_{n-1})$$

where $\phi_0 \geq 0$ is a Schwartz function. In turn ϕ_0 is majorized by a Schwartz function $\phi_T(a_i)$ depending only on the a_i, $i \notin T$. There is a similar majorization for W'.

We see then that we have to check the convergence of the three following integrals, for $s > 0$:

$$\int e^{\langle H(b_T), \Lambda_\pi + \Lambda_{\pi'}} > (1 + \| \log b_T \|)^{d+d'} | \det b_T |^s \, db_T$$

$$\int \| b^T \|^{N+N'} | \det b^T |^s \, e^{\langle b^T, -2\rho\rangle} \phi_T(a_i) db^T$$

$$\int \phi_1(a_n) | a_n |^{ns} \, da_n.$$

The first one is in fact

$$\int_{|a_i| \leq 1, i \in T} \prod_{i \in T} | a_i |^{is + x_i + x_i'} (1 + \| (\log a_i) \|)^{d+d'} \bigotimes_{i \in T} d^\times a_i$$

thus converges for $\Re s > 0$ since $x_i \geq 0$, $x_i' \geq 0$. The convergence of the other integrals is trivial. $\square$

We now consider a general induced representation. In a precise way, we consider unitary irreducible representations τ_i, $1 \leq i \leq r$ of degree $r_i = 1, 2$ of the Weil group of $\mathbb{R}$; we assume that $\sum r_i = n$. To each τ_i is attached an irreducible unitary representation (quasi-square integrable if $r_i = 2$) π_i of $GL(r_i, \mathbb{R})$. We choose a non-zero linear form $\mathcal{W}_i$ on the space of smooth vectors of π_i such that

$$\mathcal{W}_i\left[\pi_i\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}v\right] = \psi_i(x)\mathcal{W}_i(v).$$

If the degree is 1 then $\mathcal{W}_i$ is the linear form taking the value 1 on $1 \in \mathbb{C}$.

If v is a r-tuple of complex numbers we consider the representation

$$\Pi_v := I(\pi_1 \otimes \alpha^{v_1} \times \pi_2 \otimes \alpha^{v_2} \times \cdots \times \pi_r \otimes \alpha^{v_r})$$

induced from the lower parabolic subgroup of type $(r_1, r_2, \ldots, r_r)$. Let π be the tensor product of the representations π_i and $\mathcal{W}_\pi$ be the tensor product of the linear forms $\mathcal{W}_i$. We define a linear form $\mathcal{W}_v$ by the integral:

$$\mathcal{W}_v(f) = \int_{N_P} \mathcal{W}_\pi f(n) e^{\langle H'(n), v+\rho_P\rangle}\theta(n)dn$$

The integral converges when $\Re v_r > \Re v_{r-1} > \cdots \Re v_1$. It has analytic continuation to an entire function of u.

We set

$$\tau_0 := \bigoplus_{1\leq i\leq r} \tau_i,\ \tau_v := \bigoplus_{1\leq i\leq r} \tau_i \otimes \alpha^{v_i}.$$

If the degree of τ_i is 2 then the representation π_i is in the discrete series of $GL(2, \mathbb{R})$ and is a subrepresentation of an induced representation of the form

$$I(\xi_i \otimes \alpha^{p_i}, \eta_i \otimes \alpha^{p_2})$$

where ξ_i, η_i are normalized characters (of module 1), p_i, q_i are real numbers with $q_i - p_i > 0$. Then we take

$$\mathcal{W}_i(f) = \int f\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]\psi_i(x)dx.$$

We may view Π_v as a subrepresentation of an induced representation of the form

$$\Xi_u = I(\mu_1 \otimes \alpha^{u_1}, \mu_2 \otimes \alpha^{u_2}, \ldots, \mu_n \otimes \alpha^{u_n})$$

where the μ_i are characters of module 1 and the u_i complex numbers obtained in the following way. If the degree of τ_i is one then we associate with τ_i a character $\mu_k = \pi_i$ and a complex number $u_k = v_i$. If the degree of τ_i is 2 then we associate to τ_i characters $\mu_k = \xi_i$, $\mu_{k+1} = \eta_i$, and complex numbers $u_k = p_i + v_i$, $u_{k+1} = q_i + v_i$. We remark that if the differences $\Re v_{i+1} - \Re v_i$ are positive and large enough then

$$\Re u_n > \Re u_{n-1} > \cdots > \Re u_1.$$

We write $u = T(v)$, $u_i = T_i(v)$. Thus T is an affine linear transformation.

As before, for arbitrary U, the representations Ξ_u operate on the same vector space $\mathcal{V}(\Xi_0)$ of scalar valued functions on K. Likewise, all the representations Π_v operate on the same vector space $\mathcal{V}(\Pi_0)$ which is a subspace of $\mathcal{V}(\Xi_0)$. This being so, if f is in $\mathcal{V}(\Pi_0)$ then we have the standard section f_v of Π_v and the standard section $f_{T(v)}$ of Π_u and, in fact,

$$f_v = f_{T(v)}.$$

Moreover

$$\mathcal{W}_v(f_v) = \mathcal{W}_{T(u)}(f_{T(u)})$$

when $\Re v_{i+1} - \Re v_i >> 0$. This relation remains true for all values of v.

We now consider similar data for the group $GL(n')$. Suppose that

$$W_v(g) = \mathcal{W}_v(\Pi_v(g) f_{\phi,v})$$

where ϕ is the convolution of $(n-1)$ elements of $\mathcal{D}(G_n)$. We note that we may define, more generally,

$$W_u(g) = \mathcal{W}_u(\Xi_u(g) f_{\phi,u})$$

for $u \in \mathbb{C}^n$. Then

$$W_v(g) = W_{T(v)}(g).$$

Similarly suppose that

$$W'_{v'}(g) = \mathcal{W}'_{v'}(\Pi'_{u'} f'_{\phi',v'}).$$

THEOREM 4. *If $\Re v$ and $\Re v'$ are in compact sets, then there is $s_0 \in \mathbb{R}$ such that the integral*

$$\Psi(s, W_v, W'_{v'})$$

converges for $\Re s > s_0$, uniformly for $\Re s$ in a compact set. If u and u' are purely imaginary then the integral converges for $\Re s > 0$. Finally, the integral has analytic continuation to a meromorphic function of (s, v, v') which is of the form

$$L(s, \tau_v \otimes \tau'_{v'}) \times H(s, v, v')$$

where H is a holomorphic function on $\mathbb{C}^{r+r'+1}$.

We first remark that the vectors $(s, T(u), T'(u'))$ are in an affine subspace of $\mathbb{C}^{n+n'+1}$ defined by real equations. Moreover, that affine subspace is not contained in any singular hyperplane for the function

$$\Psi(s, W_u, W'_u).$$

We may therefore restrict this function to the affine subspace at hand. The result is clearly a meromorphic function of (s, v, v') which is a holomorphic multiple of

$$L(s, \sigma_{T(v)} \otimes \sigma'_{T'(v')}).$$

In fact this product has the form

$$L(s, \sigma_{T(v)} \otimes \sigma'_{T'(v')}) = \prod_{1\le i\le r, 1\le j\le r'} P_{i,j}(s + v_i + v'_j) L(s, \tau_v \otimes \tau'_{v'}) \times$$

where each $P_{i,j}$ is a polynomial. Furthermore

$$\epsilon(s, \sigma_{T(v)} \otimes \sigma'_{T'(v')}, \psi) \frac{L(s + \sigma_{T(v)} \otimes \sigma'_{T'(v')})}{L(1 - s, (\sigma_{T(v)} \otimes \sigma'_{T'(v')})^{\widetilde{}}}$$
$$= \epsilon(s, \tau_v \otimes \tau'_{v'}, \psi) \frac{L(s, \tau_v \otimes \tau'_{v'})}{L(1 - s, \widetilde{\tau_v} \otimes \widetilde{\tau'_{v'}})}.$$

It follows that the functional equation can be written

$$\frac{\Psi(1 - s, \widetilde{W}_u, \widetilde{W'}_{u'} \hat{\Phi})}{L(1 - s, \widetilde{\tau_u} \otimes \widetilde{\tau'_{u'}})} = \epsilon(s, \tau_u \otimes \tau'_{u'}, \psi) \frac{\Psi(s, W_u, W'_{u'} \Phi)}{L(s, \tau_u \otimes \tau'_{u'})}.$$

One then finishes the proof as in the previous section.

10. Complements. Consider now a standard section $f_{u,u'}$ of the tensor product representation $\Xi_u \otimes \Xi_{u'}$ (or $\Pi_u \otimes \Pi u''$) and a corresponding convolution section $f_{\phi,u,u'}$ where now ϕ is a smooth function of compact support on $G_n \times G_n$. We then consider the Whittaker integral

$$\mathcal{W}_{u,u'}(f_{u,u'}) = \int_{N_n \times N_n} f_{u,u'}(v, v') \theta(v) \overline{\theta}(v') dv dv'.$$

It is then easy to obtain directly an integral representation for the functions

$$W_{u,u'}(g, g') = \mathcal{W}_{u,u'}(\Xi_u \otimes \Xi_{u'}(g, g') f_{\phi,u,u'})$$

and obtain the analytic properties of the integral

$$\Psi(s, W_{u,u'}, \Phi) := \int_{N_n \backslash G_n} W_{u,u'}(g) \Phi(\epsilon_n g) \mid \det g \mid^s dg.$$

In the previous construction we pass from a standard section f_u to a convolution section $f_{\phi,u}$ but in fact the construction is more general. Let us say that a section f_u is **well behaved** if, for u in any multi strip, the restriction of f_u to K_n is uniformly bounded as well as all its K_n-derivatives. At this point, we recall the space $\mathcal{S}(G_n)$ ([W] 7.1); it is the space of smooth functions Φ on G_n such that for every X and Y in the enveloping algebra of G_n and for every integer N,

$$\sup_{g \in G_n} \parallel g \parallel^N |\lambda(X) \rho(Y) \Phi(g)| < +\infty.$$

If f_u is a standard section and $\Phi \in \mathcal{S}(G_n)$ then

$$f_{\Phi,u}(g) := \int f_u(gx)\Phi(x)dx$$

is a well behaved section. In addition, the group G_n (and any Lie subgroup of G_n) operates by right and left translations on G and the Dixmier-Malliavin Lemma applies to these representations. Thus if Φ is in $\mathcal{S}(G_n)$ then it can be written as a finite sum of convolution products

$$\Phi(g) = \sum_i \phi_i * \Phi_i,$$

with $\phi_i \in \mathcal{D}(G_n)$ and $\Phi_i \in \mathcal{S}(G_n)$. Suppose that f_u is a standard section. Then the section $f_{\Phi,u}$ verifies

$$f_{\Phi,u}(g) = \sum_i \int f_{\Phi_i,u}(gx)\phi_i(x)dx.$$

It follows that our previous results apply to Whittaker functions defined by

$$W_u(g) = \mathcal{W}_u(\Xi_u(g)f_{\Phi,u})$$

with f_u a standard section and $\Phi \in \mathcal{S}(G_n)$.

11. Other Rankin-Selberg integrals. In [JS] we associate to every pair of integers (n, n') with $n' \le n$ a family of Rankin-Selberg integrals with similar analytic properties. To be specific, let us consider only the integrals attached to principal series representations. Thus we consider a n-tuple of characters $(\mu_1, \mu_2, \ldots, \mu_n)$ and a n'-tuple $(\mu'_1, \mu'_2, \ldots, \mu'_{n'})$ and $u \in \mathbb{C}^n$, $u' \in \mathbb{C}^{n'}$. If $n' = n - 1$ the integrals can be treated as in the case $n = n'$. However, for $n' \le n - 2$, an additional complication is that, in order to state the functions equation, one has to consider integrals $\Psi(s, W_u, W'_{u'}; j)$ which involve an auxiliary integration over the space $M(j \times n')$ of matrices with j rows and n' columns, for $0 \le j \le n - n' - 1$, namely:

(40)

$$\Psi(s, W, W'; j) = \int W\left[\begin{pmatrix} g & 0 & 0 \\ X & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix}\right] W'(g) \mid \det g \mid^{s-\frac{n-n'}{2}} dg dX.$$

The following lemma will allow us to reduce the study of the integrals $\Psi(s, W_u, W'_{u'}; j)$ to the study of the integrals $\Psi(s, W_u, W'_{u'}; 0)$.

Lemma 8. *Suppose $n' \le n - 2$ and $0 < j \le n - n' - 1$. Let f_u be a standard section. Let $\Phi \in \mathcal{S}(G_n)$, and W_u the Whittaker function attached to the section $f_{\Phi,u}$. Then there is $\Phi_1 \in \mathcal{S}(G_n)$ such that the Whittaker function $W_{1,u}$ attached to the* *section* $f_{\Phi_1,u}$ *verifies*

$$\int W_u\left[\begin{pmatrix} g & 0 & 0 \\ X & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix}\right] dX = W_{1,u}\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix}\right]$$

for all $g \in G_{n'}$.

Proof of the Lemma. The proof is by descending induction on j. We show the induction step. By the Dixmier Malliavin Lemma we can write

$$\Phi(g) = \sum_i \int \Phi_i\left[\begin{pmatrix} 1_{n'} & 0 & y & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-n'-j-1} \end{pmatrix} g\right] \phi_i(-y)dy,$$

with $\phi_i \in \mathcal{D}(F)$ and $\Phi \in \mathcal{S}(G_n)$. Then

$$W_u(g) = \sum_i \int W_{i,u}\left[g\begin{pmatrix} 1_{n'} & 0 & y & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-n'-j-1} \end{pmatrix}\right] \phi_i(y)dy$$

where $W_{i,u}$ is the Whittaker function corresponding to the section $f_{\Phi_i,u}$. Then

$$\int W_u\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix}\begin{pmatrix} 1_{n'} & 0 & 0 & 0 \\ X_1 & 1_{j-1} & 0 & 0 \\ X_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-n'-j} \end{pmatrix}\right] dX_2$$

$$= \sum_i \int W_{i,u}\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix}\begin{pmatrix} 1_{n'} & 0 & 0 & 0 \\ X_1 & 1_{j-1} & 0 & 0 \\ X_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-n'-j} \end{pmatrix}\right]$$

$$dX_2 \psi(X_2 Y)\phi_i(Y)dY$$

$$= \sum_i \int W_{i,u}\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix}\begin{pmatrix} 1_{n'} & 0 & 0 & 0 \\ X_1 & 1_{j-1} & 0 & 0 \\ X_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-n'-j} \end{pmatrix}\right]$$

$$dX_2 \widehat{\phi_i}(-X_2)$$

$$= W_{0,u}\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix}\begin{pmatrix} 1_{n'} & 0 & 0 & 0 \\ X_1 & 1_{j-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-n'-j} \end{pmatrix}\right]$$

where $W_{0,u}$ is the Whittaker function attached to the section $f_{\Phi_0,u}$ and Φ_0 is the element of $\mathcal{S}(G)$ defined by:

$$\Phi_0(g) = \sum_i \int \Phi_i \left[\begin{pmatrix} 1_{n'} & 0 & 0 & 0 \\ 0 & 1_{j-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-n'-j} \end{pmatrix} g \right] dX_2 \widehat{\phi_i}(X_2).$$

The lemma follows. □

As for the integrals $\Psi(s, W_u, W'_{u'}; 0)$, they can be studied in the same way as before.

REFERENCES

[CKPSS] J. W. Cogdell, H., Kim, I. I., Piatetski-Shapiro F. Shahidi, On lifting from classical groups to GL_N, Publ. Math. I.H.E.S. (to appear).

[CPS1] J. W. Cogdell, I. I. Piatetski-Shapiro, Converse theorems for GL_n. *Inst. Hautes tudes Sci. Publ. Math. No.* **79** (1994), 157–214.

[CPS2] J. W. Cogdell, I. I. Piatetski-Shapiro, Converse theorems for GL, II, J. reine angew. *Math.* **507** (1999), 165–188.

[D] A. Deitmar, Mellin transforms of Whittaker functions, preprint.

[DM] J. Dixmier P. Malliavin, Factorisations de fonctions et de vecteurs indéfiniment différentiables, *Bull. Sci. Math.* (2), **102** (1978), 307–330.

[GJR] S. Gelbart, H. Jacquet, J. Rogawski Generic representations for the unitary group in three variables, to appear in Israel J. Math.

[GS] S. Gelbart, F. Shahidi, Boundedness of automorphic L-functions in vertical strips. *J. Amer. Math. Soc.* **14** (2001), no. 1, 79–107 (electronic).

[H] L. Hörmander, The Analysis of Linear partial differential operators I, Springer-Verlag, 1983.

[JS] H. Jacquet, J. Shalika, Rankin Selberg convolutions: Archimedean theory. Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), 125–207, Israel Math. Conf. Proc., 2, Weizmann, Jerusalem, 1990.

[KS] H. Kim, F. Shahidi, Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for GL_2, *C. R. Acad. Sci. Paris Sr. I Math.* **331** (2000), no. 8, 599–604.

[RS] Z. Rudnick, P. Sarnak, The pair correlation function of fractional parts of polynomials, *Comm. Math. Phys.* **194** (1998), no. 1, 61–70.

[S] D. Soudry, On the Archimedean theory of Rankin-Selberg convolutions for $SO_{2l+1} \times GL_n$, *Ann. Scient. École. Norm. Sup.* **28** (1995), 161–224.

[St1] E. Stade, On explicit integral formulas for $\mathrm{GL}(n, \mathbb{R})$-Whittaker functions. [With an appendix by Daniel Bump, Solomon Friedberg and Jeffrey Hoffstein.] *Duke Math. J.* **60** (1990), no. 2, 313–362.

[St2] E. Stade, Mellin transforms of $\mathrm{GL}(n, \mathbb{R})$ Whittaker functions, *Amer. J. Math.* **123** (2001), no. 1, 121–161.

[W] N. R. Wallach, *Real Reductive Groups I & II,* Academic Press, Pure and Applied Mathematics Vol. 132 & 132 II.