On the Gross-Prasad conjecture for unitary groups

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This paper is dedicated to Freydoon Shahidi.

ABSTRACT. We propose a new approach to the Gross-Prasad conjecture for unitary groups. It is based on a relative trace formula. As evidence for the soundness of this approach, we prove the infinitesimal form of the relevant fundamental lemma in the case of unitary groups in three variables.

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1. Introduction

Consider a quadratic extension of number fields E/F. Let η be the corresponding quadratic idele-class character of F. Denote by σ the non trivial element of $\mathrm{Gal}(E/F)$. We often write $\sigma(z) = \overline{z}$ and $N_r(z) = z\overline{z}$. Let U_n be a unitary group in n variables and U_{n-1} a unitary group in $(n-1)$ variables. Suppose that $\iota : U_{n-1} \to U_n$ is an embedding. In a precise way, let β be an Hermitian non-degenerate form on an E vector space V_n and let $e_n \in V_n$ be a vector such that $\beta(e_n, e_n) = 1$. Let V_{n-1} be the orthogonal complement of e_n. Then let U_n be the automorphism group of β_n and let U_{n-1} be the automorphism group of $\beta|_{V_{n-1}}$. Then ι is defined by the conditions $\iota(h)e_n = e_n$ and $\iota(h)v = hv$ for $v \in V_{n-1}$.

Let π be an automorphic cuspidal representation of U_n and σ an automorphic cuspidal representation of U_{n-1}. For ϕ_π in the space of π and ϕ_σ in the space of σ set

$$A_U(\phi_\pi, \phi_\sigma) := \int_{U_{n-1}(F)\backslash U_{n-1}(F_{\mathbb{A}})} \phi_\pi(\iota(h))\phi_\sigma(h)dh\,. \tag{1}$$

Suppose that this bilinear form does not vanish identically. Let Π be the standard base change of π to $Gl_n(E)$ and let Σ be the standard base change of σ to $Gl_{n-1}(E)$. For simplicity, assume that Π and Σ are themselves cuspidal. The conjecture of Gross-Prasad for orthogonal groups extends to the present set up of unitary groups and predict that the central value of the $L-$function $L(s, \Pi \times \Sigma)$ does not vanish. Cases of this conjecture have been proved by Jiang, Ginzburg and Rallis, at least in the context of orthogonal groups ([**15**] and [**16**]). The conjecture has to be made much more precise. One must ask to which extent the converse is true. One must specify which forms of the unitary group and which element of the packets corresponding to Π and Σ are to be used in the formulation of the converse. Finally, the relation between A_U (or rather $A_U\overline{A_U}$) and the $L-$value should be made more precise.

We will not discuss the general case, where there is no restriction on the representations. We remark however that the case where σ is trivial or one dimensional is already very interesting even in the case $n = 2$ (See [**10**]) and $n = 3$ (See [**18**], [**19**], [**20**], also [**3**], [**4**]).

In this note we propose an approach based on a relative trace formula. The results of this note are quite modest. We only prove the infinitesimal form of the fundamental lemma for the case $n = 3$. We do not claim this implies the fundamental lemma itself or the smooth matching of functions. We hope, however, this will interest other mathematicians. In particular, we feel the fundamental lemma itself is an interesting problem.

We now describe in rough form the relative trace formula at hand. Let f_n and f_{n-1} be smooth functions of compact support on $U_n(F_{\mathbb{A}})$ and $U_{n-1}(F_{\mathbb{A}})$ respectively. We introduce the distribution

$$A_{\pi,\sigma}(f_n \otimes f_{n-1}) := \sum A_U(\pi(f_n)\phi_\pi, \sigma(f_{n-1})\phi_\sigma)\overline{A_U(\phi_\pi, \phi_\sigma)}\,, \tag{2}$$

where the sum is over orthonormal bases for each representation.

Let $\iota : Gl_{n-1} \to Gl_n$ be the obvious embedding. For ϕ_Π in the space of Π and ϕ_Σ in the space of Σ, we define

$$A_G(\phi_\Pi, \phi_\Sigma) := \int_{Gl_{n-1}(E)\backslash Gl_{n-1}(E_{\mathbb{A}})} \phi_\Pi(\iota(g))\phi_\Sigma(g)dg \tag{3}$$

Thus the bilinear form A_G is non-zero if and only if $L(\frac{1}{2}, \Pi \times \Sigma) \neq 0$. In fact we understand completely the relation between the special value and the bilinear form A_G.

Say that n is odd. Let us also set

$$P_n(\phi_\Pi) = \int_{Gl_n(F)\backslash Gl_n(F_{\mathbb{A}})} \phi_\Pi(g_0) dg_0 \tag{4}$$

$$P_{n-1}(\phi_\Sigma) = \int_{Gl_{n-1}(F)\backslash Gl_{n-1}(F_{\mathbb{A}})} \eta(\det g_0)\phi_\Sigma(g_0) dg_0 \tag{5}$$

Strictly speaking, the first integral should be over the quotient of

$$\{g \in Gl_n(F_{\mathbb{A}}) : |\det g| = 1\}$$

by $Gl_n(F)$. Similarly for the other integral. The study of the poles of the Asai $L-$function and its integral representation (see [**2**] and [**3**], also [**9**]) predict that P_n and P_{n-1} are not identically 0. If n is even, then η must appear in the definition of P_n and not appear in the definition of P_{n-1}. This will change somewhat the following discussion but will lead to the same infinitesimal analog.

Let f'_n and f'_{n-1} be smooth functions of compact support on $Gl_n(E_{\mathbb{A}})$ and $Gl_{n-1}(E_{\mathbb{A}})$ respectively. Consider the distribution

$$A_{\Pi,\Sigma}(f'_n \otimes f'_{n-1}) := \tag{6}$$

$$\sum A_G(\Pi(f'_n)\phi_\Pi, \sigma(f'_{n-1})\phi_\Sigma)\overline{P_n(\phi_\Pi)P_{n-1}(\phi_\Sigma)}\,,$$

where the sum is over an orthonormal basis of the representations.

One should have an equality

$$A_{\pi,\sigma}(f_n \otimes f_{n-1}) = A_{\Pi,\Sigma}(f'_n \otimes f'_{n-1})\,, \tag{7}$$

for pairs (f_n, f_{n-1}) and (f'_n, f'_{n-1}) satisfying an appropriate condition of **matching orbital integrals**. In turn, the equality should be used to understand the precise relation between the L value and the bilinear form A_U.

To continue, we associate to the function $f_n \otimes f_{n-1}$ in the usual way a kernel $K_{f_n \otimes f_{n-1}}(g_1 : g_2, h_1 : h_2)$ on

$$(U_n(F_{\mathbb{A}}) \times U_{n-1}(F_{\mathbb{A}})) \times (U_n(F_{\mathbb{A}}) \times U_{n-1}(F_{\mathbb{A}}))\ .$$

The kernel is invariant on the left by the group of rational points. We consider the (regularized) integral

$$\int_{\left(U_{n-1}(F)\backslash U_{n-1}(F_{\mathbb{A}})\right)^2} K_{f_n \otimes f_{n-1}}(\iota(g_2) : g_2, \iota(h_2) : h_2) dg_2 dh_2\,. \tag{8}$$

Likewise, we associate to the function $f'_n \otimes f'_{n-1}$ a kernel $K'_{f'_n \otimes f'_{n-1}}(g_1 : g_2, h_1 : h_2)$ on

$$(Gl_n(E_{\mathbb{A}}) \times Gl_{n-1}(E_{\mathbb{A}})) \times (Gl_n(E_{\mathbb{A}}) \times Gl_{n-1}(E_{\mathbb{A}}))$$

and we consider the (regularized) integral

$$\int K'_{f'_n \otimes f'_{n-1}}(\iota(g_2) : g_2, h_1 : h_2) dg_2 dh_1 \eta(\det h_2) dh_2 \tag{9}$$

where

$$g_2 \in Gl_{n-1}(E)\backslash Gl_{n-1}(E_{\mathbb{A}})\,,\ h_1 \in Gl_n(F)\backslash Gl_n(F_{\mathbb{A}})\,,\ h_2 \in Gl_{n-1}(F)\backslash Gl_{n-1}(F_{\mathbb{A}})\,.$$

The conditions of matching orbital integrals should guarantee that (8) and (9) are equal. In turn this should imply (7).

In more detail, (8) is equal to

$$\int \sum_{\gamma\in U_n(F)} f_n\left(\iota(g_2)^{-1}\gamma\,\iota(h_2)\right)) \sum_{\xi\in U_{n-1}(F)} f_{n-1}\left(g_2^{-1}\xi h_2\right))dg_2dh_2$$

or

$$\int \sum_{\gamma\in U_n(F)} f_n\left(\iota(g_2)\gamma\,\iota(h_2)\right)) \sum_{\xi\in U_{n-1}(F)} f_{n-1}\left(g_2\xi h_2\right))dg_2dh_2\,.$$

In the sum over γ we may replace γ by $\iota(\xi)\gamma$. Then $\iota(g_2\xi)$ appears. Now we combine the sum over ξ and the integral over $g_2\in U_{n-1}(F)\backslash U_{n-1}(E_{\mathbb{A}})$ into an integral for $g_2\in U_{n-1}(E_{\mathbb{A}})$ to get

$$\int\sum_\gamma f_n\left(\iota(g_2)\gamma\iota(h_2)\right))f_{n-1}\left(g_2h_2\right))dg_2dh_2\,.$$

After a change of variables, this becomes

$$\int\sum_\gamma f_n\left(\iota(g_2)\iota(h_2)^{-1}\gamma\,\iota(h_2)\right) f_{n-1}\left(g_2\right) dg_2dh_2\,.$$

At this point, we introduce a new function $f_{n,n-1}$ on $U_n(F_{\mathbb{A}})$ defined by

$$f_{n,n-1}(g) := \int_{U_{n-1}(F_{\mathbb{A}})} f_n(\iota(g_2)g)f_{n-1}(g_2)dg_2\,. \tag{10}$$

Then we can rewrite the previous expression as

$$\int_{U_{n-1}(F)\backslash U_{n-1}(F_{\mathbb{A}})}\sum_\gamma f_{n,n-1}\left(\iota(h_2)^{-1}\gamma\,\iota(h_2)\right)dh_2\,.$$

The group U_{n-1} operate on U_n by conjugation:

$$\gamma\mapsto\iota(h)^{-1}\gamma\iota(h)$$

For **regular** elements of $U_n(F)$ the stabilizer is trivial. Thus, ignoring terms which are not regular, the above expression can be rewritten

$$\sum_\gamma\int_{U_{n-1}(F_{\mathbb{A}})} f_n\left(\iota(h)^{-1}\gamma\iota(h)\right)dh\,, \tag{11}$$

where the sum is now over a set of representatives for the regular orbits of $U_{n-1}(F)$ in $U_n(F)$.

Likewise, we can write (9) in the form

$$\int \sum_{\gamma\in Gl_n(E)} f'_n(\iota(g_2)^{-1}\gamma h_1) \sum_{\xi\in Gl_{n-1}(F)} f'_{n-1}(g_2^{-1}\xi h_2)\eta(\det h_2)dg_2dh_1dh_2\,.$$

The same kind of manipulation as before gives

$$=\int \sum_{\gamma\in Gl_n(E)} f'_n(\iota(g_2)\gamma h_1)f'_{n-1}(g_2h_2)dg_2dh_1\eta(\det h_2)dh_2$$

where now g_2 is in $Gl_{n-1}(E_{\mathbb{A}})$. If we change variables, this becomes

$$=\int \sum_{\gamma\in Gl_n(E)} f'_n(\iota(g_2)\iota(h_2)^{-1}\gamma h_1)f'_{n-1}(g_2)dg_2dh_1\eta(\det h_2)dh_2\,.$$

We introduce a new function $f'_{n,n-1}$ on $Gl_n(E_{\mathbb{A}})$ defined by

$$f'_{n,n-1}(g) := \int_{Gl_{n-1}(E_{\mathbb{A}})} f'_n(\iota(g_2)g) f'_{n-1}(g_2) dg_2\,.$$

The above expression can be rewritten

$$\int \sum_{\gamma \in Gl_n(E)} f'_{n,n-1}(\iota(h_2)^{-1}\gamma h_1) dh_1 \eta(\det h_2) dh_2\,,$$

where h_1 is in $Gl_n(F)\backslash Gl_n(F_{\mathbb{A}})$ and h_2 is in $Gl_{n-1}(F)\backslash Gl_{n-1}(F_{\mathbb{A}})$. We also write this as

$$\int \sum_{\gamma \in Gl_n(E)/Gl_n(F)} \left(\int f'_{n,n-1}(\iota(h_2)^{-1}\gamma h_1) dh_1 \right) \eta(\det h_2) dh_2 \tag{12}$$

with $h_1 \in Gl_n(F_{\mathbb{A}})$.

At this point we introduce the symmetric space S_n defined by the equation $ss^\sigma = 1$. Thus

$$S_n(F) := \{ s \in Gl_n(E) : s\overline{s} = 1\,.\} \tag{13}$$

Let $\Phi_{n,n-1}$ be the function on $S_n(F_{\mathbb{A}})$ defined by

$$\Phi_{n,n-1}(g\overline{g}^{-1}) = \int_{Gl_n(F_{\mathbb{A}})} f'_{n,n-1}(gh_1) dh_1\,.$$

The expression (12) can be written as

$$\int_{Gl_{n-1}(F_{\mathbb{A}})/Gl_{n-1}(F)} \sum_{\xi \in S_n(F)} \Phi_{n,n-1}\left[\iota(h_2)^{-1}\xi\iota(h_2)\right] \eta(\det h_2) dh_2\,.$$

The group $Gl_n(F)$ operates on $S_n(F)$ by

$$s \mapsto \iota(g)^{-1} s \iota(g)\,.$$

Again, for **regular** elements of $S_n(F)$ the stabilizer under $Gl_{n-1}(F)$ is trivial. Thus, at the cost of ignoring non regular elements, we get

$$\sum_{\xi} \int_{Gl_{n-1}(F_{\mathbb{A}})} \Phi_{n,n-1}\left(\iota(h)^{-1}\xi\iota(h)\right) \eta(\det h) dh\,, \tag{14}$$

where the sum is over a set of representatives for the regular orbits of $Gl_{n-1}(F)$ in $S_n(F)$.

To carry through our trace formula we need to find a way to match regular orbits of $U_{n-1}(F)$ in $U_n(F)$ with regular orbits of $Gl_{n-1}(F)$ in $S_n(F)$. We will use the notation $\xi \to \xi'$ for such a matching. The global condition of **matching orbital integrals** is then

$$\int_{U_{n-1}(F_{\mathbb{A}})} f_{n,n-1}(\iota(h)^{-1}\xi\iota(h)) dh =$$

$$\int_{Gl_{n-1}(F_{\mathbb{A}})} \Phi_{n,n-1}(\iota(h)^{-1}\xi'\iota(h))\eta(\det h) dh$$

if $\xi \to \xi'$. If ξ' does not correspond to any ξ then

$$\int \Phi_{n,n-1}(\iota(h)^{-1}\xi'\iota(h)\eta(\det h) dh = 0\,.$$

A formula of this type is discussed in [**6**], [**7**], [**8**] for $n = 2$. Or rather, the results of these papers could be modified to recover a trace formula of the above type.

As a first step, we consider the infinitesimal analog of the above trave formula. Now n needs not be odd. We set $\mathfrak{G}_n = M(n \times n, E)$. We often drop the index n if this does not create confusion. We let $\mathfrak{U}_n \subset \mathfrak{G}_n$ be the Lie algebra of the group U_n. Then U_{n-1} operates on $\mathfrak{U}_n$ by conjugation. Likewise, we consider the vector space $\mathfrak{S}_n$ tangent to S_n at the origin. This is the vector space of matrices $X \in \mathfrak{G}_n$ such that $X + \overline{X} = 0$. Again the group $Gl_{n-1}(F)$ operates by conjugation on $\mathfrak{S}_n$. The trace formula we have in mind is

$$\int_{U_{n-1}(F)\backslash U_{n-1}(F_{\mathbb{A}})} \sum_{\xi \in \mathfrak{U}_n(F)} f\left(\iota(h)^{-1}\xi\iota(h)\right) dh = \tag{15}$$

$$\int_{Gl_{n-1}(F)\backslash Gl_{n-1}(F_{\mathbb{A}})} \sum_{\xi' \in \mathfrak{S}_n(F)} \Phi\left(\iota(h)^{-1}\xi'\iota(h)\right) \eta(\det h) dh\,,$$

where f is a smooth function of compact support on $\mathfrak{U}_n(F_{\mathbb{A}})$ and Φ a smooth function of compact support on $\mathfrak{S}_n(F_{\mathbb{A}})$. Once more, the integrals on both sides are not convergent and need to be regularized. The equality takes place if the functions satisfy a certain matching orbital integral condition. We will define a notion of strongly regular elements and a condition of matching of strongly regular elements noted

$$\xi \to \xi'\,.$$

Then the global condition of matching between functions is as before: if $\xi \to \xi'$ then

$$\int_{U_{n-1}(F_{\mathbb{A}})} f\left(\iota(h)\xi\iota(h)^{-1}\right) dh$$

$$= \int_{Gl_{n-1}(F_{\mathbb{A}})} \Phi\left(\iota(h)\xi'\iota(h)^{-1}\right) \eta(\det h) dh\,;$$

if ξ' does not correspond to a ξ then

$$\int_{Gl_{n-1}(F_{\mathbb{A}})} \Phi\left(\iota(h)\xi\iota(h)^{-1}\right) \eta(\det h) dh = 0\,.$$

We now investigate in detail the matching of orbits announced above.

2. Orbits of $Gl_{n-1}(E)$

Let E be an arbitrary field. We first introduce a convenient definition. Let P_n, P_{n-1} be two polynomials of degree n and $n-1$ respectively in $E[X]$. We will say that they are **strongly relatively prime** if the following condition is satisfied. There exists a sequence of polynomials P_i of degree i, $n \geq i \geq 0$, where P_n and P_{n-1} are the given polynomials, and the P_i are defined inductively by the relation

$$P_{i+2} = Q_i P_{i+1} + P_i\,.$$

In particular, P_0 is a non-zero constant. In other words, we demand that the P_n and P_{n-1} be relatively prime and the Euclidean algorithm which gives the (constant) G.C.D. of P_n and P_{n-1} have exactly $n-1$ steps. Of course the sequence, if it exists, is unique. Moreover, for each i, the polynomials P_{i+1}, P_i are strongly relatively prime.

Let V_n be a vector space of dimension n over the field E. We often write $V_n(E)$ for V_n. We set $\mathfrak{G} = \mathrm{Hom}_E(V_n, V_n)$. Let $e_n \in V_n$ and $e_n^* \in V_n^*$ (dual vector space). Assume $\langle e_n^*, e_n\rangle \neq 0$. Let V_{n-1} be the kernel of e_n^*. Thus

$$V_n = V_{n-1} \oplus Ee_n\,.$$

We define an embedding $\iota : Gl(V_{n-1}(E)) \to Gl(V_n(E))$ by

$$\begin{aligned} \iota(g)v_{n-1} &= gv_{n-1} \text{ for } v_{n-1} \in V_{n-1}\,, \\ \iota(g)e_n &= e_n\,. \end{aligned}$$

We let $Gl(V_n(E))$ acts on V_n^* on the right by

$$\langle v^*g, v\rangle = \langle v^*, gv\rangle\,.$$

Then $\iota(Gl(V_{n-1}(E)))$ is the subgroup of $Gl((V_n)(E))$ which fixes e_n^* and e_n.

Suppose $A_n \in \mathfrak{G}$. We can represent A_n by a matrix

$$\begin{pmatrix} A_{n-1} & e_{n-1} \\ e_{n-1}^* & a_n \end{pmatrix},$$

with $A_{n-1} \in \mathrm{Hom}(V_{n-1}, V_{n-1})$, $e_{n-1} \in V_{n-1}$, $e_{n-1}^* \in V_{n-1}^*$, $a_n \in E$. This means that, for all $v_{n-1} \in V_{n-1}(E)$,

$$A_n(v_{n-1}) = A_{n-1}(v_{n-1}) + \langle e_{n-1}^*, v_{n-1}\rangle e_n$$

and

$$A_n(e_n) = e_{n-1} + a_n e_n\,.$$

In particular

$$A_n(e_{n-1}) = A_{n-1}(e_{n-1}) + \langle e_{n-1}^*, e_{n-1}\rangle e_n\,.$$

The group $Gl(V_{n-1}(E))$ acts on $\mathfrak{G}$ by

$$A \mapsto \iota(g)A\iota(g)^{-1}\,.$$

The operator $\iota(g)A\iota(g)^{-1}$ is represented by the matrix

$$\begin{pmatrix} gA_{n-1}g^{-1} & ge_{n-1} \\ e_{n-1}^*g^{-1} & a_n \end{pmatrix}.$$

Thus the scalar product $\langle e_{n-1}^*, e_{n-1}\rangle$ is an invariant of this action. We oft3en call it the first invariant of this action. Moreover, if we replace e_n and e_n^* by scalar multiples, the spaces V_{n-1}, Ee_n and the scalar product $\langle e_{n-1}^*, e_{n-1}\rangle$ do not change. We will say that A_n is **strongly regular with respect to the pair** (e_n, e_n^*) (or with respect to the pair (V_{n-1}, e_n)) if the polynomials

$$\det(A_n - \lambda) \text{ and } \det(A_{n-1} - \lambda)$$

are strongly relatively prime.

Now assume that A_n is strongly regular with respect to (e_n, e_n^*). We have

$$\det(A_n - \lambda) = (a_n - \lambda)\det(A_{n-1} - \lambda) + R(\lambda)$$

with R of degree $n-2$. The leading term of R is $-\langle e_{n-1}^*, e_n\rangle(-\lambda)^{n-2}$. Thus $\langle e_{n-1}^*, e_n\rangle$ is non-zero. Thus we can write

$$V_{n-1} = V_{n-2} \oplus Ee_{n-1}$$

where V_{n-2} is the kernel of e_{n-1}^* and represent A_{n-1} by a matrix

$$\begin{pmatrix} A_{n-2} & e_{n-2} \\ e_{n-2}^* & a_{n-1} \end{pmatrix},$$

with $A_{n-2} \in \mathrm{Hom}(V_{n-2}, V_{n-2})$, $e_{n-2} \in V_{n-2}$, $e^*_{n-1} \in V^*_{n-2}$, $a_{n-1} \in E$. As before, this means that

$$A_{n-1}(v_{n-2}) = A_{n-2}(v_{n-2}) + \langle e^*_{n-2}, v_{n-2}\rangle e_{n-1}$$

$$A_{n-1}(e_{n-1}) = e_{n-2} + a_{n-1}e_{n-1}\,.$$

Choose a basis ϵ_i, $1 \le i \le n-2$ of V_{n-2}. Since $\langle e^*_{n-1}, \epsilon_i\rangle = 0$ we have

$$A_n(\epsilon_i) = A_{n-1}(\epsilon_i) + \langle e^*_{n-1}, \epsilon_i\rangle e_n = A_{n-1}(\epsilon_i) = A_{n-2}(\epsilon_i) + \langle e^*_{n-2}, \epsilon_i\rangle e_{n-1}\,.$$

On the other hand,

$$A_n(e_{n-1}) = e_{n-2} + a_{n-1}e_{n-1} + \langle e^*_{n-1}, e_{n-1}\rangle e_n\,.$$

Thus the matrix of A_n with respect to the basis

$$(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-2}, e_{n-1}, e_n)$$

has the form

$$\begin{pmatrix} \mathrm{Mat}(A_{n-2}) & *_{n-2} & 0_{n-2} \\ *^{n-2} & a_{n-1} & 1 \\ 0^{n-2} & \langle e^*_{n-1}, e_{n-1}\rangle e_n & a_n \end{pmatrix} \tag{16}$$

where $\mathrm{Mat}(A_{n-2})$ is the matrix of A_{n-2} with respect to the basis $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-2})$. The index $n-2$ indicates a column of size $n-2$ and the exponent $n-2$ a row of size $n-2$. Likewise the matrix of A_{n-1} with respect to the basis $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-2}, e_{n-1})$ has the form

$$\begin{pmatrix} \mathrm{Mat}(A_{n-2}) & *_{n-2} \\ *^{n-2} & a_{n-1} \end{pmatrix}.$$

It follows that

$$\det(A_n - \lambda) = \det(A_{n-1} - \lambda)(a_n - \lambda) - \langle e^*_{n-1}, e_{n-1}\rangle \det(A_{n-2} - \lambda)\,.$$

Thus the polynomials $\det(A_{n-1} - \lambda)$ and $\det(A_{n-2} - \lambda)$ are strongly relatively prime and the operator A_{n-1} is strongly regular with respect to (e_{n-1}, e^*_{n-1}). At this point we proceed inductively. We construct a sequence of subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n$$

with $\dim(V_i) = i$, vectors $e_i \in V_i$, and linear forms $e^*_i \in V^*_i$ such that V_{i-1} is the kernel of e^*_i. The matrix of A_n with respect to the basis

$$(e_1, e_2, \ldots, e_{n-1}, e_n)$$

is the tridiagonal matrix

$$\begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_1 & a_2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_2 & a_3 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & c_{n-3} & a_{n-2} & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{n-2} & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_{n-1} & a_n \end{pmatrix} \tag{17}$$

where $c_i = \langle e^*_i, e_i\rangle \neq 0$. We note the relations

$$\det(A_i - \lambda) = \det(A_{i-1} - \lambda) - c_{i-1} \det(A_{i-2} - \lambda)\,,\ i \ge 2\,.$$

Now suppoe

$$(e'_1, e'_2, \dots, e'_{n-1})$$

is a basis of V_{n-1} and the matrix of A_n with respect to the basis

$$(e'_1, e'_2, \dots, e'_{n-1}, e_n)$$

has the form

$$\begin{pmatrix} a'_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c'_1 & a'_2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c'_2 & a'_3 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & c'_{n-3} & a'_{n-2} & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c'_{n-2} & a'_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c'_{n-1} & a'_n \end{pmatrix}.$$

Thus, for $i \geq 1$

$$A_n e'_i = e'_{i-1} + a'_i e'_i + c_{i-1} e_{i+1}$$

(where $e'_n = e_n$, $e_{-1} = 0$ and $e'_{n+1} = 0$) Call A'_i the sub square matrix obtained by deleting the last $n-i$ rows and the last $n-i$ columns. Then we have

$$\det(A'_i - \lambda) = \det(A'_{i-1} - \lambda) - c'_{i-1} \det(A'_{i-2} - \lambda)\,,\; i \geq 2\,.$$

Also

$$\det(A_n - \lambda) = \det(A'_n - \lambda)\,,\; \det(A_{n-1} - \lambda) = \det(A'_{n-1} - \lambda)\,.$$

It follows inductively that $a_i = a'_i$, $c_j = c'_j$, $e'_i = e_i$.

We have proved the following Proposition.

Proposition 1. *If A is strongly regular with respect to the pair (V_{n-1}, e_n) there is a unique basis*

$$(e_1, e_2, \dots e_{n-1})$$

of V_{n-1} such that the matrix of A with respect to the basis

$$(e_1, e_2, \dots e_{n-1}, e_n)$$

has the form (17). In particular, the a_i, $1 \leq i \leq n$, and the c_j, $1 \leq j \leq n-1$, are uniquely determined.

Remark. If we demand that the matrix have the form

$$\begin{pmatrix} a'_1 & b'_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c'_1 & a'_2 & b'_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c'_2 & a'_3 & b'_3 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & c'_{n-3} & a'_{n-2} & b'_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c'_{n-2} & a'_{n-1} & b'_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c'_{n-1} & a'_n \end{pmatrix},$$

with respect to a basis of the form

$$(e'_1, e'_2, \dots e'_{n-1}, e_n)\,,$$

where $(e'_1, e'_2, \dots e'_{n-1})$ is a basis of V_{n-1}, then $a'_i = a_i$, $1 \leq i \leq n$, $b'_j c'_j = c_j$, $1 \leq i \leq n-1$ and the e'_i are scalar multiple of the e_i.

According to [**21**], an element $A_n \in \mathfrak{G}$ is **regular** if the vectors

$$A_{n-1}^i e_{n-1}\,,\ 0 \le i \le n-2$$

are linearly independent and the linear forms

$$e_i^* A_{n-1}^i\,,\ 0 \le i \le n-2$$

are linearly independent. This is equivalent to the condition that the stabilizer of A_n in $Gl(V_n(E))$ be trivial and the orbit of A_n under $Gl(V_n(\overline{E}))$ be Zariski closed. A strongly regular element is regular. The above and forthcoming discussion concerning strongly regular elements should apply to regular elements as well. However, we have verified it is so only in the case $n = 2, 3$.

3. Orbits of $Gl_{n-1}(F)$

Now suppose that E is a quadratic extension of F. Let σ be the non trivial element of the Galois group of E/F.

Suppose that V_n is given an F form. For clarity we often write $V_n(E)$ for V_n and $V_n(F)$ for the $F-$form. We denote by $v \mapsto v^\sigma$ the corresponding action of σ on $V_n(E)$. Then $V_n(F)$ is the space of $v \in V_n(E)$ such that $v^\sigma = v$. We assume $e_n^\sigma = e_n$ and $V_{n-1}^\sigma = V_{n-1}$. We have an action of σ on $\mathrm{Hon}_E(V_n, V_n)$ noted $A \mapsto A^\sigma$ and defined by

$$A^\sigma(v) = A(v^\sigma)^\sigma\,.$$

We denote by $\mathfrak{S}$ the space of $A \in \mathrm{Hon}_E(V_n, V_n)$ such that

$$A^\sigma = -A\,.$$

The group $Gl(V_{n-1}(F))$ can be identified with the group of $g \in Gl(V_{n-1}(E))$ fixed by σ. It operates on $\mathfrak{S}$.

We say that an element of $\mathfrak{S}_n$ is **strongly regular** if it is strongly regular as an element of $\mathrm{Hon}_E(V_n, V_n)$. We study the orbits of $Gl(V_n(F))$ in the set of strongly regular elements of $\mathfrak{S}$.

We fix $\sqrt{\tau}$ such that $E = F(\sqrt{\tau})$. If A is strongly regular, there is a unique basis $(e_1, e_2, \dots, e_{n-1})$ of $V_n(F)$ such that the matrix of A with respect to the basis

$$(e_1, e_2, \dots, e_{n-1}, e_n)$$

has the form

$$\begin{pmatrix}
a_1 & \sqrt{\tau} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\frac{c_1}{\sqrt{\tau}} & a_2 & \sqrt{\tau} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \frac{c_2}{\sqrt{\tau}} & a_3 & \sqrt{\tau} & \cdots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \cdots & \frac{c_{n-3}}{\sqrt{\tau}} & a_{n-2} & \sqrt{\tau} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{c_{n-2}}{\sqrt{\tau}} & a_{n-1} & \sqrt{\tau} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{c_{n-1}}{\sqrt{\tau}} & a_n
\end{pmatrix}. \tag{18}$$

Then the a_i and the c_j are the invariants of A. Furthermore, $a_i \in F\sqrt{\tau}$ and $c_j \in F^\times$. Two strongly regular elements A and A' of $\mathfrak{S}_n$ are conjugate under $Gl(V_{n-1}(F))$ if and only they are conjugate under $Gl(V_{n-1}(E))$, or, equivalently, if and only if they have the same invariants. Finally, given $a_i \in F\sqrt{\tau}$, $1 \le i \le n$, and $c_j \in F^\times$, $1 \le j \le n-1$, there is a strongly regular element of $\mathfrak{S}_n$ with those invariants.

4. Orbits of U_{n-1}

Let V_n be a $E-$vector space of dimension n and β a non-degenerate Hermitian form on V_n. Let e_n be an anisotropic vector, that is,

$$\beta(e_n, e_n) \neq 0\,.$$

Usually, we will scale β by demanding that $\beta(e_n, e_n) = 1$.

Let V_{n-1} be the subspace orthogonal to e_n. Thus

$$V_n = V_{n-1} \oplus E e_n\,.$$

Let $U(\beta)$ be the unitary group of β. Let θ be the restriction of β to V_{n-1}. and $U(\theta)$ the unitary group of θ. Thus we have an injection $\iota : U(\theta) \to U(\beta)$. We have the adjoint action of $U(\beta)$ on $\mathrm{Lie}(U(\beta))$ and thus an action of $U(\theta)$ on $\mathrm{Lie}(U(\beta))$. We have an embedding of $\mathrm{Lie}(U(\beta))$ into $\mathrm{Hom}(V_n, V_n)$. We say that an element of $\mathrm{Lie}(U(\beta))$ is **strongly regular** if it is strongly regular as an element of $\mathrm{Hom}_E(V_n, V_n)$. As before to $A_n \in \mathrm{Hom}_E(V_n, V_n)$ we associate a matrix

$$\begin{pmatrix} A_{n-1} & e_{n-1} \\ e^*_{n-1} & a_n \end{pmatrix}.$$

The condition that A_n be in $\mathrm{Lie}(U(\beta))$ is

$$A_{n-1} \in \mathrm{Lie}(U(\theta)), a_n + \overline{a_n} = 0$$

and

$$\langle e^*_{n-1}, v\rangle = -\frac{\beta(v, e_{n-1})}{\beta(e_n, e_n)}\,,$$

for all $v \in V_{n-1}$. Thus the first invariant of the matrix is

$$\langle e^*_n, e_n\rangle = -\frac{\beta(e_{n-1}, e_{n-1})}{\beta(e_n, e_n)}\,.$$

Assume that A_n is strongly regular. Then $\beta(e_{n-1}, e_{n-1}) \neq 0$ and V_{n-1} is an orthogonal direct sum

$$V_{n-1} = V_{n-2} \oplus E e_{n-1}\,.$$

We can then repeat the process and obtain in this way an orthogonal basis

$$(e_1, e_2, \dots, e_{n-1}, e_{n-1})$$

such that $\beta(e_i, e_i) \neq 0$ and the matrix of A_n with respect to the basis

$$(e_1, e_2, \dots, e_{n-1}, e_n)$$

has the form (17). Moreover, it is the only orthogonal basis with this property. In addition, for $1 \leq i \leq n-1$,

$$c_i = -\frac{\beta(e_i, e_i)}{\beta(e_{i+1}, e_{i+1})}\,.$$

Finally, $a_i \in F\sqrt{\tau}$ for $1 \leq i \leq n$ and $c_j \in F^\times$ for $1 \leq j \leq n-1$. Two strongly regular elements of $\mathrm{Lie}(U(\beta))$ are conjugate under $U(\theta)$ if and only if they are conjugate under $Gl(V_{n-1})$, or, what amounts to the same, have the same invariants.

From now on let us scale β by demanding that $\beta(e_n, e_n) = 1$. Then θ determine β and we write $\beta = \theta^e$.

Given $a_i \in F\sqrt{\tau}$, $1 \leq i \leq n$, $c_j \in F^\times$, $1 \leq j \leq n-1$ there is a non degenerate Hermitian form θ on V_{n-1}, a strongly regular element A of $\mathrm{Lie}(U(\theta^e))$ whose invariants are the a_i and the c_j. The isomorphism class of θ is uniquely determined and for any choice of θ the conjugacy class of A under $U(\theta)$ is uniquely determined.

The determinant of θ is equal to

$$(-1)^{\frac{(n-1)n}{2}} c_1 c_2^2 \cdots c_{n-1}^{n-1}.$$

5. Comparison of the orbits, the fundamental lemma

We now consider a $E-$vector space V_n and a vector $e_n \neq 0$, a linear complement V_{n-1} of e_n. We are also given a $F-$form of V_n or what amounts to the same an action of σ on V_n. We assume that $e_n^\sigma = e_n$ and $V_{n-1}^\sigma = V_{n-1}$. For an Hermitian form θ on V_{n-1} we denote by θ^e the Hermitian form on V_n such that V_{n-1} and E_n are orthogonal, $\theta^e|V_{n-1} = \theta$, $\theta^e(e_n, e_n) = 1$. Then $U(\theta) \subset Gl(V_{n-1}(E))$ and $Gl(V_{n-1}(F)) \subset Gl(V_{n-1}(E))$. Let ξ be a strongly regular element of $\mathrm{Lie}(U(\theta^e))$ and ξ' a strongly regular element of $\mathfrak{S}$ we say that ξ' matches ξ and we write

$$\xi \to \xi'$$

if ξ and ξ' have the same invariants, or, what amounts to the same, are conjugate under $Gl(V_n(E))$. Every ξ matches a ξ'. The converse is not true. However, given ξ' there is a θ and a strongly regular element ξ of $\mathrm{Lie}(U(\theta^e))$ such that $\xi \to \xi'$. The form θ is unique, within equivalence, and the element ξ is unique, within conjugation by $U(\theta)$.

For instance, suppose that E is a quadratic extension of F, a local, non-Archimedean fields. Up to equivalence, there are only two choices for θ. Let θ_0 be a form whose determinant is a norm and θ_1 a form whose determinant is not a norm. Let ξ' be a strongly regular element of $\mathfrak{S}(F)$ and c_i, $1 \leq i \leq n-1$ the corresponding invariants. If

$$(-1)^{\frac{(n-1)n}{2}} c_1 c_2^2 \cdots c_{n-1}^{n-1}$$

is a norm then ξ' matches an element $\mathrm{Lie}(U(\theta_0^e))$. Otherwise it matches an element of $\mathrm{Lie}(U(\theta_1^e))$.

We have a conjecture of **smooth matching**. If Φ is a smooth function of compact support on $\mathfrak{S}(F)$ and ξ' is strongly regular, we define the orbital integral

$$\Omega_G(\xi', \Phi) = \int_{Gl(V_{n-1}(F))} \Phi\left(\iota(g)\xi'\iota(g)^{-1}\right) \eta(\det g) dg.$$

Likewise, if f_i, $i = 0, 1$, is a smooth function of compact support on $\mathrm{Lie}(U(\theta_i^e)(F)$, ξ_i a strongly regular element, we define the orbital integral

$$\Omega_{U_i}(\xi_i, f_i) = \int_{U(\theta_i^e)(F)} f_i(\iota(g)\xi_i\iota(g)^{-1}) dg.$$

Conjecture 1 (Smooth matching). *There is a factor $\tau(\xi')$, defined for ξ' strongly regular with the following property. Given Φ there is a pair (f_0, f_1) and conversely such that*

$$\Omega_G(\xi', \Phi) = \tau(\xi')\Omega_{U_i}(\xi_i, f_i)$$

if $\xi_i \to \xi'$.

We have a conjectural **fundamental lemma**. Assume that E/F is an unramified quadratic extension and the residual characteristic is odd. Thus -1 is a norm in E. To be specific let us take $V_n = E^n$, $V_n(F) = F^n$,

$$e_n = \begin{pmatrix} 0 \\ 0 \\ * \\ 0 \\ 1 \end{pmatrix},$$

$V_{n-1}(E) \simeq E^{n-1}$ the space of column vectors whose last entry is 0. Finally let θ_0 be the form whose matrix is the identity matrix. Thus $\mathrm{Lie}(U(\theta_0^e))$ is the space of matrices $A \in M(n \times n, E)$ such that $A + {}^t\overline{A} = 0$. On the other hand $\mathfrak{S}(F)$ is the space of matrices A such that $A + \overline{A} = 0$.

Let f_0 (resp. Φ_0) be the characteristic function of the matrices with integral entries in $\mathrm{Lie}(U(\theta_0^e))$ (resp. $\mathfrak{S}(F)$). Choose the Haar measures so that the standard maximal compact subgroups have mass 1.

Conjecture 2 (fundamental lemma). *Let ξ' be a strongly regular element of $\mathfrak{S}(F)$ and a_i, c_j the corresponding invariants. If*

$$c_1 c_2^2 \cdots c_{n-1}^{n-1}$$

has even valuation, then

$$\Omega_G(\xi', \Phi_0) = \tau(\xi')\Omega_{U_0}(\xi, f_0)\,,$$

where $\xi \in \mathrm{Lie}(U(\theta_0^e))$ matches ξ' and $\tau(\xi') = \pm 1$. Otherwise

$$\Omega_G(\xi', \Phi_0) = 0\,.$$

Before we proceed we remark that in the general setting the linear forms

$$A_n \mapsto \mathrm{Tr}(A_n)\,, \mapsto \mathrm{Tr}(A_{n-1})$$

are invariant under $Gl(V_{n-1}(E))$. Thus in the above discussion and conjectures we may replace $\mathfrak{G} := \mathrm{Hom}(V_n, V_n)$ by the space

$$\mathfrak{g} := \{A_n : \mathrm{Tr}(A_n) = 0\,,\ \mathrm{Tr}(A_{n-1}) = 0\}\,.$$

Then $\mathrm{Lie}(U(\theta_0)^e)$ is replaced by

$$\mathfrak{u}_{\theta_0} := \mathrm{Lie}(U(\theta_0^e)) \cap \mathfrak{g}$$

and $\mathfrak{S}$ by

$$\mathfrak{s} := \mathfrak{S} \cap \mathfrak{g}\,.$$

6. Smooth matching and the fundamental Lemma for $n = 2$

Let E/F be an arbitrary quadratic extension. We choose τ such that $E = F\sqrt{\tau}$. For $n = 2$ we take $V_2 = E^2$ and $V_1 = E$. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in E \right\}.$$

The only invariant is the determinant. There is no difference between between regular and strongly regular. The above element is regular if and only if $bc \neq 0$.

Similarly,

$$\mathfrak{s} = \left\{ \begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix} : b' + \overline{b'} = 0\,,\ c' + \overline{c'} = 0 \right\}.$$

The matrix of β has the form

$$\begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$$

with $\theta \in F^\times$. The isomorphism class of β depends on the class of θ modulo the subgroup $N_r(E^\times)$ of norms. The corresponding vector space $\mathfrak{u}_\theta(F)$ is the space of matrices of the form

$$\begin{pmatrix} 0 & b \\ -\bar{b}\theta & 0 \end{pmatrix}.$$

Such an element is regular if $b \neq 0$. The group $U_1(F) = \{t : t\bar{t} = 1\}$ operates by conjugation. The action of t is given by:

$$\begin{pmatrix} 0 & b \\ -\bar{b}\theta & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & bt \\ -\overline{bt}\theta & 0 \end{pmatrix}.$$

The only invariant of this action is the determinant. Two regular elements

$$\begin{pmatrix} 0 & b_1 \\ -\overline{b_1}\theta & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ -\overline{b_2}\theta & 0 \end{pmatrix}$$

are in the same orbit if and only if $b_1\overline{b_1} = b_2\overline{b_2}$. The only non-regular element is the 0 matrix.

On the other hand $\mathfrak{s}(F)$ is the space of matrices of the form

$$\begin{pmatrix} 0 & b\sqrt{\tau} \\ \frac{c}{\sqrt{\tau}} & 0 \end{pmatrix}, \ b, c \in F.$$

Such an element is regular if and only if $bc \neq 0$. The group $F^\times$ operates by conjugation. The action of $t \in F^\times$ is given by

$$\begin{pmatrix} 0 & b\sqrt{\tau} \\ \frac{c}{\sqrt{\tau}} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & bt\sqrt{\tau} \\ \frac{t^{-1}c}{\sqrt{\tau}} & 0 \end{pmatrix}.$$

The orbits of non-regular elements are the 0 matrix and the orbit of the following elements

$$\begin{pmatrix} 0 & \sqrt{\tau} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{\tau}} & 0 \end{pmatrix}.$$

The only invariant of this action is the determinant. Two regular elements

$$\begin{pmatrix} 0 & b_1\sqrt{\tau} \\ \frac{c_1}{\sqrt{\tau}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2\sqrt{\tau} \\ \frac{c_2}{\sqrt{\tau}} & 0 \end{pmatrix}$$

are conjugate if and only if $b_1c_1 = b_2c_2$.

The correspondence between regular elements is as follows:

$$\begin{pmatrix} 0 & b \\ -\bar{b}\theta & 0 \end{pmatrix} \to \begin{pmatrix} 0 & b'\sqrt{\tau} \\ \frac{c'}{\sqrt{\tau}} & 0 \end{pmatrix}$$

if $b\bar{b}\theta = -b'c'$. Thus we have a bijection between the disjoint union of the regular orbits of the spaces $\mathfrak{u}_\theta(F)$, $\theta \in E^\times/N_rF^\times)$, and the regular orbits in $\mathfrak{s}(F)$.

Now suppose that E/F is a local extension. Modulo the group of norms we have two choices θ_0 and θ_1 for θ. For f_i smooth of compact support on $\mathfrak{u}_i := \mathfrak{u}_{\theta_i}$ the orbital integral evaluated on

$$\xi_i = \begin{pmatrix} 0 & b \\ -\theta_i\bar{b} & 0 \end{pmatrix}$$

has the form

$$\Omega_U(f_i,\xi_i)=\int_{U_1} f_i\begin{pmatrix}0 & bu\\ -\theta_i\overline{bu} & 0\end{pmatrix}du\,.$$

The integral depends only on $b\bar b$ and can be written as

$$\Omega_U(f_i,-\theta_i b\bar b)\,.$$

For Φ smooth of compact support on $\int$ the orbital integral evaluated on

$$\xi'=\begin{pmatrix}0 & a\sqrt{\tau}\\ \frac{1}{\sqrt{\tau}} & 0\end{pmatrix}$$

takes the form

$$\Omega(\Phi,a):=\Omega_G(f,\xi')=\int_{F^\times}\begin{pmatrix}0 & a\sqrt{\tau}t\\ \frac{1}{\sqrt{t}\tau} & 0\end{pmatrix}\eta(t)d^\times t\,.$$

We appeal to the following Lemma

Lemma 1. *Let E/F be a quadratic extension of local fields and η the corresponding quadratic character. Given a smooth function of compact support ϕ on F^2, there are two smooth functions of compact support on F ϕ_1,ϕ_2 such that*

$$\int\phi(t^{-1},at)\eta(t)d^\times t=\phi_1(a)+\eta(a)\phi_2(a)$$

and

$$\phi_1(0)=\int\phi(x,0)\eta(x)d^\times x\,,\ \phi_2(0)=\int\phi(0,x)\eta(x)d^\times x\,.$$

Conversely, given ϕ_1,ϕ_2 there is ϕ such that the above conditions are satisfied.

Here we recall that the local Tate integral

$$\int\phi(x)\eta(x)|x|^s d^\times x$$

converges absolutely for $\Re s>0$ and extends to a meromorphic function of s which is holomorphic at $s=0$. The improper integral

$$\int\phi(x)\eta(x)d^\times x$$

is the value at $s=0$.

The lemma implies that

$$\Omega_G(\Phi,a)=\phi_1(a)+\eta(a)\phi_2(a)$$

where ϕ_1,ϕ_2 are smooth functions of compact support on F. Then the condition that the pair (f_0,f_1) matches Φ becomes

$$\Omega_U(f_i,-b\bar b\theta_i)=\phi_1(-b\bar b\theta_i)+\eta(-\theta_i)\phi_2(-b\bar b\theta_i)\,.$$

It is then clear that given Φ there is a matching pair (f_0,f_1) and conversely.

We pass to the fundamental lemma. We assume the field are non Archimedean, the residual characteristic is odd, and the extension is unramified. We take τ to be a unit. We also take $\theta_0=1$. On the other hand θ_1 is any element with odd valuation. Let f_0 be the characteristic function of the integral elements of $\mathfrak{u}_0$. Then, with the previous notations,

$$\Omega(f_0,-b\bar b)=\Omega(f_0,\xi_0)=f_0\begin{pmatrix}0 & b\\ -\bar b & 0\end{pmatrix}\,.$$

This is zero unless $|b\bar{b}| \leq 1$ in which case it is 1. On the other hand, let Φ_0 be the characteristic function of thee integrals elements of $\mathfrak{s}$. Then

$$\Omega_G(\Phi_0, a) = \int_{1 \leq |t| \leq |a|^{-1}} \eta(t) d^\times t \,.$$

This is zero unless $|a| \leq 1$. Then it is zero unless a is a norm in which case it is one.

Thus if $\xi_0 \to \xi'$, that is, $a = -b\bar{b}$, we get

$$\Omega(f_0, \xi) = \Omega(\Phi_0, \xi') \,.$$

Otherwise, we get

$$\Omega(\Phi_0, \xi') = 0 \,.$$

The fundamental lemma is established.

7. The trace formula for $n = 2$

In general, it will be convenient to consider all pairs (U_n, U_{n-1}) simultaneously. We illustrate this idea for the case $n = 2$. Let E/F a quadratic extension of number fields.

The trace formula we want to consider has the following shape:

$$\sum_{\theta \in E^\times / N_r E^\times)} \int_{U_1(F) \backslash U_1(F_{\mathbb{A}})} \sum_{\xi \in \mathfrak{U}_\theta(F)} f_\theta \left(\iota(h)^{-1} \xi \iota(h) \right) dh = \tag{19}$$

$$\int_{Gl_2(F) \backslash Gl_2(F_{\mathbb{A}})} \sum_{\xi' \in \mathfrak{s}(F)} \Phi \left(\iota(h)^{-1} \xi' \iota(h) \right) \eta(\det h) dh \,.$$

The left hand side converges and is equal to

$$\sum_\theta \left[f_\theta(0) \mathrm{Vol}(U_1(F) \backslash U_1(F_{\mathbb{A}})) + \sum_{\beta \in E^\times / N_r E^\times)} \int_{U_1(F_{\mathbb{A}})} f_\theta \begin{pmatrix} 0 & t\beta \\ -\overline{\beta t} \theta & 0 \end{pmatrix} dt \right] .$$

The right hand side msut be interpreted as an improper integral. It is equal to

$$\int_{F^\times} \Phi \begin{pmatrix} 0 & t\sqrt{\tau} \\ 0 & 0 \end{pmatrix} \eta(t) d^\times t + \int_{F^\times_{\mathbb{A}}} \Phi \begin{pmatrix} 0 & 0 \\ \frac{t}{\sqrt{\tau}} & 0 \end{pmatrix} \eta(t) d^\times t$$

$$+ \sum_{\alpha \in F^\times} \int \Phi \begin{pmatrix} 0 & \alpha t \sqrt{\tau} \\ \frac{1}{t\sqrt{\tau}} & 0 \end{pmatrix} \eta(t) d^\times t \,.$$

For the two first terms, we recall that if ϕ is a Schwartz-Bruhat function on $F_{\mathbb{A}}$ then the global Tate integral

$$\int \phi(t) |t|^s \eta(t) d^\times t$$

converges for $\Re s > 1$ and has analytic continuation to an entire function of s. The improper integral

$$\int \phi(t) \eta(t) d^\times t$$

is the value of this function at $s = 0$. The remaining terms are absolutely convergent.

The matching condition is between a family (f_θ) and a function Φ. The global matching condition has the following form:

$$\int_{U_1(F_{\mathbb{A}})} f_\theta \begin{pmatrix} 0 & t\beta \\ -\overline{\beta t}\theta & 0 \end{pmatrix} dt = \int_{F_{\mathbb{A}}^\times} \Phi \begin{pmatrix} 0 & \alpha t\sqrt{\tau} \\ \frac{1}{t\sqrt{\tau}} & 0 \end{pmatrix} \eta(t) d^\times t$$

if $-\beta\overline{\beta}\theta = \alpha$. At a place of F inert in E, the corresponding local matching condition is described in the previous section. At a place which splits in E, it is elementary. The local matching conditions imply

$$\sum_\theta f_\theta(0) \mathrm{Vol}(U_1(F)\backslash U_1(F_{\mathbb{A}})) =$$

$$\int_{F^\times} \Phi \begin{pmatrix} 0 & t\sqrt{\tau} \\ 0 & 0 \end{pmatrix} \eta(t) d^\times t + \int_{F_{\mathbb{A}}^\times} \Phi \begin{pmatrix} 0 & 0 \\ \frac{t}{\sqrt{\tau}} & 0 \end{pmatrix} \eta(t) d^\times t \,.$$

We will not give the proof. It can be derived from [**8**].

8. Orbits of $Gl_2(E)$

We take $V_3(E) = E^3$ (column vectors). We set

$$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

We identify V_3^* to the space of row vectors with 3 entries. We take $e_3^* = (0, 0, 1)$. Then $V_2(E) = E^2$ is the space of row vectors whose last component is 0. We denote by $\mathfrak{G}$ the space $\mathrm{Hom}_E(V_3, V_3)$ and by $\mathfrak{g}$ the subspace of A such that $\mathrm{Tr}(A) = 0$ and $\mathrm{Tr}(A|V_2) = 0$. Thus $\mathfrak{g}(E)$ is the space of 3×3 matrices X with entries in E of the form:

$$X = \begin{pmatrix} a & b & x_1 \\ c & -a & x_2 \\ y_1 & y_2 & 0 \end{pmatrix}$$

The group $Gl_2(E)$ operates on $\mathfrak{g}(E)$. We introduce several invariants of this action:

$$A_1(X) = \det \begin{pmatrix} a & b \\ c & -a \end{pmatrix} , \tag{20}$$

$$A_2(X) = (y_1, y_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \tag{21}$$

$$B_1(X) = \det X \,. \tag{22}$$

We denote by $R(X)$ the resultant of the following polynomials in λ:

$$\det \left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix} - \lambda \right] , -\det[X - \lambda] \,.$$

It is also an invariant. More explicitly,

$$A_1(X) = -a^2 - bc \tag{23}$$

$$A_2(X) = x_1 y_1 + x_2 y_2 \tag{24}$$

$$B_1(X) = (x_1 y_1 - x_2 y_2)a + x_1 y_2 c + x_2 y_1 b \tag{25}$$

$$R(X) = A_1(X) A_2(X)^2 + B_1(X)^2 \tag{26}$$

Clearly, X is strongly regular if and only if $A_2(X) \neq 0$ and $R(X) \neq 0$. If X is strongly regular the invariants c_1, c_2 and a_1, a_2, a_3 introduced earlier can be computed in terms of the new invariants as follows:

$$c_2 = A_2(X) \tag{27}$$
$$-c_1 c_2^2 = R(X) \tag{28}$$
$$a_1 = -B_1(X) A_2^{-1}(X) \tag{29}$$
$$a_2 = -a_1 \tag{30}$$
$$a_3 = 0 \tag{31}$$

We also introduce

$$B_2(X) := \begin{pmatrix} -x_2 & x_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{32}$$

$$B_3(X) := \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \tag{33}$$

Explicitly,

$$\begin{aligned} B_2(X) &= -2x_1x_2a + x_1^2 c - x_2^2 b \\ B_3(X) &= -2y_1y_2a + y_1^2 b - y_2^2 c \end{aligned}$$

We remark that if we replace $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ by $h \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $h \in Sl(2, F)$ then $(-x_2, x_1)$ is replaced by $(-x_2, x_1)h^{-1}$. It follows that B_2 is $Sl_2(E)$ invariant. Likewise for B_3.

We let $\mathfrak{g}(E)'$ be the set of X such that $A_2(X) \neq 0$ and $\mathfrak{g}(E)^s$ the set of $X \in \mathfrak{g}(E)'$ such that $R(X) \neq 0$. Thus $\mathfrak{g}(E)^s$ is the set of strongly regular elements.

Lemma 2. *Every $Sl_2(E)$ orbit in $\mathfrak{g}(E)'$ contains a unique element of the form*

$$X = \begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ 0 & t & 0 \end{pmatrix}$$

and then $A_1(X) = -a^2 - bc$, $A_2(X) = t \neq 0$, $B_1(X) = -at$, $B_2(X) = -b$, $B_3(X) = -t^2 c$, $R(X) = -t^2 bc$. In particular, A_2, B_1, B_2, B_3 form a complete set of invariants for the orbits of $Sl_2(E)$ in $\mathfrak{g}(E)'$.

Proof: If $A_2(X) \neq 0$ then a fortiori $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$. Since $Sl_2(F)$ is transitive on the space of non-zero vectors in F^2, we may as well assume

$$X = \begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ y_1 & y_2 & 0 \end{pmatrix}$$

Then $y_2 = A_2(X) \neq 0$. We now conjugate X by

$$\iota \begin{pmatrix} 1 & 0 \\ -\frac{y_1}{y_2} & 1 \end{pmatrix}$$

and obtain a matrix like the one in the lemma. In $Gl_2(E)$ the stabilizer of the column $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the row $\begin{pmatrix} 0 & t \end{pmatrix}$ (where $t \neq 0$) is the group

$$H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \alpha \in E^\times \right\}$$

Thus the stabilizer in $Sl_2(E)$ of a matrix like the one in the lemma is indeed trivial. The remaining assertions of the lemma are easy. □

Lemma 3. *If X is in $\mathfrak{g}(E)'$ then X is strongly regular if and only if it is regular.*

Proof: We may assume that

$$X = \begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ 0 & t & 0 \end{pmatrix},$$

with $t \neq 0$. Then X is strongly regular if and only $R(X) = -t^2bc \neq 0$. On the other hand, it is regular if and only if the column vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ -a \end{pmatrix}$$

are linearly independent and the row vectors

$$(0, t), (ct, -ta)$$

are linearly independent. It is so if and only if $b \neq 0$ and $c \neq 0$. Our assertion follows. □

Lemma 4. *Every orbit of $Gl_2(E)$ in $\mathfrak{g}(E)^s$ contains a unique element of the form*

$$X = \begin{pmatrix} a & b & 0 \\ 1 & -a & 1 \\ 0 & t & 0 \end{pmatrix},$$

where $b \neq 0$ and $t \neq 0$. Then

$$\begin{aligned} A_1(X) &= -a^2 - b \\ A_2(X) &= t \\ B_1(X) &= -at \\ R(X) &= -bt^2 \end{aligned}$$

If the invariants A_1, A_2, B_1 take the same values on two matrices in $\mathfrak{g}(E)^s$, then they are in the same orbit of $Gl_2(E)$. Finally, given a_1, a_2, b_1 in E with $a_2 \neq 0$ and $a_1a_2^2 + b_1^2 \neq 0$ there is $X \in \mathfrak{g}(E)^s$ such that $A_1(X) = a_1$, $A_2(X) = a_2$ and $B_1(X) = b_1$.

Proof: The first assertion follows from the general case, or more simply, from the previous Lemma. Indeed, by the previous lemma, every orbit contains an element of the form

$$X = \begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ 0 & t & 0 \end{pmatrix}$$

and then $-bct^2 = R(X)$. Thus $bc \neq 0$. Conjugating by

$$\iota \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$$

we obtain an element of the required form. The stabilizer of this element in $Gl_2(E)$ is trivial. The remaining assertions are obvious. □

9. Orbits of $Gl_2(F)$

Now we consider the orbits of $Gl_2(F)$ in $\mathfrak{s}$. Of course, $\mathfrak{s} = \sqrt{\tau}\mathfrak{g}(F)$. We define $\mathfrak{s}' = \mathfrak{s} \cap \mathfrak{g}(E)'$ and $\mathfrak{s}^s = \mathfrak{s} \cap \mathfrak{g}(E)^s$. For $Y \in \mathfrak{g}(F)$, we have

$$\begin{aligned} A_1(\sqrt{\tau}Y) &= \tau A_1(Y) \\ A_2(\sqrt{\tau}Y) &= \tau A_2(Y) \\ B_1(\sqrt{\tau}Y) &= \tau\sqrt{\tau}B_1(Y)\,. \end{aligned}$$

Also

$$R(\sqrt{\tau}Y) = \tau^3 R(Y)\,.$$

Thus, on $\mathfrak{s}^s$, the functions A_1, A_2 (with values in F) together with the function B_1 (with values in $F\sqrt{\tau}$) form a complete set of invariants for the action of $Gl_2(F)$. Conversely, given $a_1 \in F$, $a_2 \in F^\times$ and $b_1 \in F\sqrt{\tau}$ such that $a_1 a_2^2 + b_1^2 \neq 0$ there is $X \in \mathfrak{s}^s$ with those numbers for invariants.

10. Orbits of the unitary group

We formulate the fundamental lemma in terms of the Hermitian matrix

$$\theta_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

rather in terms of the Hermitian unit matrix. Then

$$\theta_0^e = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We let $U_{2,1}$ be the unitary group for the Hermitian matrix θ_0^e. Thus the Lie algebra of $U_{2,1}$ is the space $\mathfrak{U}(F)$ of matrices Ξ of the form

$$\Xi = \begin{pmatrix} a & b & z_1 \\ c & d & z_2 \\ -\overline{z_2} & \overline{z_1} & e \end{pmatrix}$$

with $a + \overline{d} = 0$, $b \in F\sqrt{\tau}$, $c \in F\sqrt{\tau}$, $e \in F\sqrt{\tau}$. We let $U_{1,1}$ be the unitary group for the Hermitian matrix θ_0. The corresponding Hermitian form is

$$Q(z_1, z_2) = z_1\overline{z_2} + z_2\overline{z_1}$$

We embeds $U_{1,1}$ into $U_{2,1}$ by

$$\iota(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

We obtain an action of $U_{1,1}(F)$ by conjugation. As before, we set $\mathfrak{u} = \mathfrak{U} \cap \mathfrak{g}$. Thus $\mathfrak{u}$ is the space of matrices Ξ of the form

$$\Xi = \begin{pmatrix} a & b & z_1 \\ c & -a & z_2 \\ -\overline{z_2} & -\overline{z_1} & 0 \end{pmatrix}, \; a \in F, b \in F\sqrt{\tau}\,,\; c \in F\sqrt{\tau}\,. \tag{34}$$

Then

$$\begin{aligned} A_1(\Xi) &= -a^2 - bc \\ A_2(\Xi) &= -Q(z_1, z_2) \\ B_1(\Xi) &= a(\overline{z_1} z_2 - \overline{z_2} z_1) - b z_2 \overline{z_2} - c z_1 \overline{z_1} \end{aligned}$$

We set $\mathfrak{u}' = \mathfrak{u} \cap \mathfrak{g}'$ and $\mathfrak{u}^s = \mathfrak{u} \cap \mathfrak{g}^s$. We study directly the orbits of $U_{1,1}$ on $\mathfrak{u}^s$.

LEMMA 5. *For $t \in F^\times$ choose $(z_{1,0}, z_{2,0})$ such that $Q(z_{1,0}, z_{2,0}) = -t$. Any orbit of $SU_{1,1}$ in $\mathfrak{u}'$ on which A_1 takes the value t contains a unique element of the form*

$$\begin{pmatrix} a & b & z_{1,0} \\ c & -a & z_{2,0} \\ -\overline{z_{2,0}} & -\overline{z_{1,0}} & 0 \end{pmatrix}$$

PROOF: Since $SU_{1,1}$ acting on E^2 is transitive on the sphere $S_{-t} = \{v \in E^2 | Q(v) = -t\}$ and each point of the sphere has a trivial stabilizer in $SU_{1,1}$, our assertion is trivial. □

LEMMA 6. *For $t \in F^\times$ choose $(z_{1,0}, z_{2,0})$ such that $Q(z_{1,0}, z_{2,0}) = -t$. Any orbit of $U_{1,1}$ in $\mathfrak{u}^s$ on which A_1 takes the value t contains an element of the form*

$$\Xi = \begin{pmatrix} a & b & z_{1,0} \\ c & -a & z_{2,0} \\ -\overline{z_{2,0}} & -\overline{z_{1,0}} & 0 \end{pmatrix}$$

The stabilizer in $U_{1,1}$ of such an element is trivial. Moreover $A_1(\Xi) \in F$, $A_2(\Xi) \in F$, $B_1(\Xi) \in F\sqrt{\tau}$ and $-R(\Xi)$ is a non-zero norm. $A_1(\Xi)$, $A_2(\Xi)$, $B_1(\Xi)$ completely determine the orbit of Ξ. Finally, if $a_1 \in F$, $a_2 \in F$ and $b_1 \in F\sqrt{\tau}$ are such that $a_2 \neq 0$, $a_1 a_2^2 + b_1^2 \neq 0$ and $-(a_1 a_2^2 + b_1^2)$ is a norm, then there is Ξ in $\mathfrak{u}^s$ such that $A_1(\Xi) = a_1$, $A_2(\Xi) = a_2$ and $B_1(\Xi) = b_1$.

PROOF: As before, the orbit in question contains a least one element of this type, say Ξ_0. To prove the remaining assertions we introduce the matrix

$$M = \begin{pmatrix} -\overline{z_{1,0}} t^{-1} & z_{1,0} \\ \overline{z_{2,0}} t^{-1} & z_{2,0} \end{pmatrix} \in Sl_2(E)\,.$$

Then

$${}^t\overline{M} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M = \begin{pmatrix} t^{-1} & 0 \\ 0 & -t \end{pmatrix}.$$

It follows that $\iota(M)^{-1} \mathfrak{U}\, \iota(M)$ is the Lie algebra of the unitary group for the Hermitian matrix

$$\begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $\iota(M)^{-1}\mathfrak{u}\iota(M)$ becomes the space of matrices of the form

$$\begin{pmatrix} \alpha & \beta & z_1 \\ \overline{\beta}t^{-2} & -\alpha & z_2 \\ -\overline{z_1}t^{-1} & \overline{z_2}t & 0 \end{pmatrix}, \ \alpha \in F\sqrt{\tau}\,.$$

and $\Xi_1 = \iota(M)^{-1}\Xi_0\iota(M)$ is a matrix of the form

$$\Xi_1 = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ \overline{\beta_1}t^{-2} & -\alpha_1 & 1 \\ 0 & t & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} A_1(\Xi_0) = A_1(\Xi_1) &= -\alpha_1^2 - \beta_1\overline{\beta_1}t^{-2} \\ A_2(\Xi_0) = A_2(\Xi_1) &= t \\ B_1(\Xi_0) = B_1(\Xi_1) &= -\alpha_1 t \\ R(\Xi_0) = R(\Xi_1) &= -\beta_1\overline{\beta_1} \end{aligned}$$

The stabilizer H of the column $(0,1)$ and the row $(0,t)$ in the group $\iota(M)^{-1}U_{1,1}\iota(M)$ is the group

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \ u \in U_1\,.$$

Since Ξ_1 is in $\mathfrak{g}(E)^s$ we have $\beta_1 \neq 0$. Thus the stabilizer of Ξ_1 of Ξ_1 in H or in $\iota(M)^{-1}U_{1,1}\iota(M)$ is trivial. If the invariants A_1, A_2, B_1 take the same value on two such elements Ξ_1 and Ξ_2 of $\iota(M)^{-1}\mathfrak{u}\iota(M)$, then we have $t_1 = t_2$, $\alpha_1 = \alpha_2$ and $\beta_1\overline{\beta_1} = \beta_2\overline{\beta_2}$. Then $\beta_1 = \beta_2 u$ with $u \in U_1$. Then Ξ_1 and Ξ_2 are conjugate by an element of H. □

11. Comparison of orbits

In accordance with our general discussion, we match the orbit of $\Xi \in \mathfrak{u}^s$ with the orbit of $X \in \mathfrak{s}^s$ and we write $\Xi \to X$ if the matrices are conjugate by $Gl_2(E)$, or, what amounts to the same, if they have the same invariants A_1, A_2, B_1. In particular, we have the following Proposition.

PROPOSITION 2. *Given $X \in \mathfrak{s}^s$, there is a matrix Ξ in $\mathfrak{u}^s$ which matches X if and only if $-R(X)$ is a (non-zero) norm.*

12. The fundamental lemma for $n = 3$

We now let E/F be an unramified quadratic extensions of non-Archimedean fields. We assume the residual characteristic is not 2. We let $f_{\mathfrak{u}}$ be the characteristic function of the matrices with integral entries in $\mathfrak{u}$ and $\Phi_{\mathfrak{s}}$ be similarly the characteristic function of the set of matrices with integral entries in $\mathfrak{s}$. For $\Xi \in \mathfrak{u}^s$ we set

$$\Omega_U(\Xi) = \int_{U_{1,1}} f_{\mathfrak{u}}(u\Xi u^{-1})du \tag{35}$$

Likewise, for $X \in \mathfrak{s}^s$ we set

$$\Omega_G(X) = \int_{Gl_2(F)} \Phi_0(gXg^{-1})\eta(\det g)dg \tag{36}$$

The **fundamental lemma** asserts that if Ξ matches X then

$$\Omega_U(\Xi) = \tau(X)\Omega_G(X) \tag{37}$$

where $\tau(X) = \pm 1$ is the transfer factor. If, on the contrary, X matches no Ξ then

$$\Omega_G(X) = 0\,.$$

To prove the fundamental lemma we exploit the isomorphism between $U_{1,1}$ and $Sl(2,F)$. Now $U_{1,1}$ is the product of the normal subgroup $SU_{1,1}$ and the torus

$$T = \left\{ t = \begin{pmatrix} z & 0 \\ 0 & \overline{z}^{-1} \end{pmatrix}, z \in E^\times \right\}.$$

with intersection

$$T \cap SU_{1,1} = \left\{ t = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in F^\times \right\}.$$

Let T_0 be the subgroup of $t \in T$ with $|z| = 1$. Then $U_{1,1} = SU_{1,1}T_0$.

The function $f_{\mathfrak{u}}$ is invariant under T_0. Thus, in fact,

$$\Omega_U(\Xi) = \int_{SU_{1,1}} f_{\mathfrak{u}}(u\Xi u^{-1})du\,.$$

To establish the fundamental lemma we will use the isomorphism $\theta : SU_{1,1} \to Sl_2(F)$ defined by

$$\theta(g) = \begin{pmatrix} \sqrt{\tau} & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 \\ 0 & 1 \end{pmatrix} \tag{38}$$

and a compatible $F-$linear bijective map $\Theta : \mathfrak{u} \to \mathfrak{g}(F)$ defined as follows. If

$$\Xi = \begin{pmatrix} \alpha & \beta & z_1 \\ \gamma & -\alpha & z_2 \\ -\overline{z_2} & -\overline{z_1} & 0 \end{pmatrix}, \ \alpha \in F, \beta \in \sqrt{\tau}F, \gamma \in \sqrt{\tau}F$$

then

$$\Theta(\Xi) = X\,,\ X = \begin{pmatrix} a & b & x_1 \\ c & -a & x_2 \\ y_1 & y_2 & 0 \end{pmatrix} \tag{39}$$

where

$$\begin{array}{lll} a = \alpha & b = \beta\sqrt{\tau} & c = \frac{\gamma}{\sqrt{\tau}} \\ x_1 = \frac{z_1 + \overline{z_1}}{2} & y_1 = \frac{z_2 + \overline{z_2}}{2} & \\ x_2 = \frac{z_2 - \overline{z_2}}{2\sqrt{\tau}} & y_2 = \frac{-\sqrt{\tau}(z_1 - \overline{z_1})}{2} & \end{array}$$

The inverse formulas for z_1, z_2 read

$$z_1 = x_1 - \frac{y_2}{\sqrt{\tau}}\,,\ z_2 = y_1 + x_2\sqrt{\tau}\,.$$

Note that

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \sqrt{\tau} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 \\ 0 & 1 \end{pmatrix}.$$

The linear bijection Θ has the following property of compatibility with the isomorphism θ:

$$\Theta(\iota(g)\Xi\iota(g)^{-1}) = \iota(\theta(g))\Theta(\Xi)\iota(\theta(g))^{-1}$$

for $g \in SU(1,1)$.

We can use Θ to define an action μ of T on $\mathfrak{g}$. It is defined by

$$\Theta\left(\iota(t)\Xi\iota(t)^{-1}\right) = \mu(t)\left(\Theta(\Xi)\right) .$$

Explicitly if $t = \mathrm{diag}(z, \overline{z}^{-1})$, $z = p + \sqrt{\tau}$, then

$$\mu(t)\left[\begin{pmatrix} a & b & x_1 \\ c & -a & x_2 \\ y_1 & y_2 & 0 \end{pmatrix}\right] =$$

$$\begin{pmatrix} a & bz\overline{z} & px_1 - qy_2 \\ c(z\overline{z})^{-1} & -a & \frac{px_2+qy_1}{p^2-q^2\tau} \\ \frac{py_1+q\tau x_2}{p^2-q^2\tau} & py_2 - q\tau x_1 & 0 \end{pmatrix}$$

For $t \in T \cap SU_{1,1} = T \cap Sl_2(F)$ $\mu(t)$t is the conjugation by $\iota(t)$. Again $T = T_0(T \cap Sl_2(F))$.

We compare the invariants of Ξ and $X = \Theta(\Xi)$. From

$$-\overline{z_2}z_1 - \overline{z_1}z_2 = -2(x_1y_1 + x_2y_2)$$

and

$$\alpha(\overline{z_1}z_2 - \overline{z_2}z_1) - \beta z_2\overline{z_2} - \gamma z_1\overline{z_1} =$$

$$\sqrt{\tau}(2ax_1x_2 + bx_2^2 - cx_1^2) + \frac{1}{\sqrt{\tau}}(2ay_1y_2 - by_1^2 + cy_2^2)$$

we get

$$A_1(\Xi) = A_1(\Theta(\Xi)) \tag{40}$$

$$A_2(\Xi) = -2A_2(\Theta(\Xi)) \tag{41}$$

$$B_1(\Xi) = -\sqrt{\tau}B_2(\Theta(\Xi)) - \frac{1}{\sqrt{\tau}}B_3(\Theta(\Xi)) \tag{42}$$

Also

$$R(\Xi) = 4A_1(X)A_2(X)^2 + \tau B_2(X)^2 + \frac{1}{\tau}B_3(X)^2 + 2B_2(X)B_3(X) .$$

We let $\widetilde{\mathfrak{g}}(F)$ be the image of $\mathfrak{u}^s$ under Θ. Thus $\tilde{\mathfrak{g}}(F)$ is contained in $\mathfrak{g}(F)'$. The functions A_1, A_2 and $-\sqrt{\tau}B_2 - \frac{1}{\sqrt{\tau}}B_3$ form a complete set of invariants for the action of $Sl_2(F)$ and T_0 on $\tilde{\mathfrak{g}}$.

We will let Φ_0 be the characteristic function of the set of integers in $\mathfrak{g}(F)$. For $X \in \mathfrak{g}'$ we set

$$\Omega_{Sl_2}(X) = \int_{Sl_2(F)} \Phi_0(\iota(g)X\iota(g)^{-1})dg . \tag{43}$$

Thus $\Omega_U(\Xi) = \Omega_{Sl_2}(\Theta(\Xi))$.

We match the orbits of $U_{2,1}$ in $\mathfrak{u}^s$ with the orbits of $Gl_2(F)$ in $\mathfrak{s}^s$ by matching the invariants: for Ξ in $\mathfrak{u}^s$ and $Y \in \mathfrak{g}(F)^s$, $\Xi \to \sqrt{\tau}Y$ if

$$\begin{aligned} A_1(\Xi) &= A_1(\sqrt{\tau}Y) \\ A_2(\Xi) &= A_2(\sqrt{\tau}Y) \\ B_1(\Xi) &= B_1(\sqrt{\tau}Y) \end{aligned}$$

This leads to the following relation in terms of $X = \Theta(\Xi)$ and Y:

$$A_1(X) = \tau A_1(Y)$$

$$-2A_2(X) = \tau A_2(Y)$$

$$-\sqrt{\tau}B_2(X) - \frac{1}{\sqrt{\tau}}B_3(X) = \tau\sqrt{\tau}B_1(Y)$$

The last relation can be simplified:

$$-\tau B_2(X) - B_3(X) = \tau^2 B_1(Y)$$

To make this relation explicit, we may replace $X \in \widetilde{\mathfrak{g}}(F)$ by a conjugate under $Sl_2(F)$ and thus assume:

$$X = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & -a_1 & 1 \\ 0 & t_1 & 0 \end{pmatrix} \tag{44}$$

The condition that X be in $\tilde{\mathfrak{g}}(F)$ reads

$$t_1 \neq 0\,,\; \tau b_1^2 + \frac{t_1^4 c_1^2}{\tau} - 2b_1c_1t_1^2 - 4a_1^2t_1^2 \neq 0\,.$$

The second condition can also be written as

$$(\sqrt{\tau}b_1 - \frac{t_1^2 c_1}{\sqrt{\tau}})^2 - 4a_1^2t_1^2 \neq 0\,.$$

As a matter of fact, assuming $t_1 \neq 0$, the second condition fails only if $a_1 = 0$ and $\tau b_1 = t_1^2 c_1$.

Likewise, we may assume:

$$Y = \begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ 0 & t & 0 \end{pmatrix} \tag{45}$$

Then

$$\begin{aligned} A_1(Y) &= -a^2 - bc \\ A_2(Y) &= t \\ B_1(Y) &= -ta \end{aligned}$$

Moreover $R(\sqrt{Y}) = \tau^3 R(Y) = -bc\tau^3 t^2$. This matrix is in $\mathfrak{g}(F)^s$ if and only if $t \neq 0$ and $bc \neq 0$. It matches some X if and only if $-R(\sqrt{Y})$ is a norm. Since $-\tau$ is a norm this is equivalent to $-bc$ being a norm.

The condition of matching of orbits becomes: $X \to Y$ if

$$a_1^2 + b_1c_1 = \tau(a^2 + bc) \tag{46}$$

$$-2t_1 = \tau t \tag{47}$$

$$\tau b_1 + t_1^2 c_1 = -\tau^2 ta \tag{48}$$

In a precise way, this system of equations for (a_1, b_1, c_1, t_1) has a solution if and only if $-bc$ is a norm. If we write

$$-\tau^2 bc = y^2 - \tau a_1^2 \tag{49}$$

then we can take a_1 for the first entry of X, and then take $t_1 = -\frac{\tau t}{2}$,

$$b_1 = -\frac{t}{2}(y + \tau a)\,,\; c_1 = \frac{2}{t\tau}(y - \tau a)\,. \tag{50}$$

Note that $a_1 = 0$ and $\tau b_1 = t_1^2 c_1$ would imply $y = 0$ and thus $bc = 0$. Thus X is indeed in $\tilde{\mathfrak{g}}(F)$.

The fundamental lemma takes then the following form.

THEOREM 1 (The fundamental lemma for $n = 3$). *For $Y \in \mathfrak{g}(F)^s$ of the form (45) define*

$$\Omega_{Gl_2}(Y) = \int_{Gl_2(F)} \Phi_0(gYg^{-1})\eta(\det g)dg\,. \tag{51}$$

If $-bc$ is not a norm then $\Omega_{Gl_2}(Y) = 0$. If $-bc$ is a norm, let (a_1, b_1, c_1, t_1) satisfying the conditions (46) and let X be the element of $\tilde{\mathfrak{g}}(F)$ defined by (44). Then

$$\Omega_{Gl_2}(Y) = \eta(c)\Omega_{Sl_2}(X)$$

We now prove the fundamental lemma.

13. Orbital integrals for $Sl_2(F)$

In this section we compute the orbital integral $\Omega_{Sl_2}(X)$ where

$$X = \begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ 0 & t & 0 \end{pmatrix}. \tag{52}$$

Suppose $\Omega_{Sl_2}(X) \neq 0$. This implies that the orbit of X intersects the support of Φ_0 we get that the invariants of X are integral. In particular $a^2 + bc, t, at, b, t^2c$ are all integers.

We set

$$g = k \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ k \in Gl_2(\mathcal{O}_F)\,,$$

$$dg = dk|m|^2 d^\times m dk$$

The integration over k is superfluous. Thus we get

$$\Omega_{Sl_2}(X) =$$

$$\int\int \Phi_0 \left[\begin{pmatrix} a + cu & m^2(b - 2au - u^2c) & mu \\ cm^{-2} & -a - cu & m^{-1} \\ 0 & tm & 0 \end{pmatrix} \right] du|m|^2 d^\times m\,.$$

LEMMA 7. *The integral converges absolutely, provided $t \neq 0$.*

PROOF: Indeed the range of u and m are limited by

$$|u| \leq |m|^{-1}\,,\ 1 \leq |m| \leq |t|^{-1}\,.$$

Thus the integral is less than the integral

$$\int\int_{|u|\leq|m|^{-1}, 1\leq|m|\leq|t|^{-1}} du|m|^2 d^\times m$$

$$= \int_{1\leq|m|\leq|t|^{-1}} |m|d^\times m$$

which is finite. □

Explicitly, the integral is equal to

$$\int\int du|m|^2 d^\times m$$

over

$$\begin{cases} |a + cu| \leq 1 & |u| \leq |m|^{-1} \\ |c| \leq |m|^2 & 1 \leq |m| \leq |t|^{-1} \\ |b - 2au - u^2c| \leq |m|^{-2} & \end{cases}$$

We first compute the integral for $c \neq 0$. We may change u to uc^{-1} to get

$$|c|^{-1} \int\int du|m|^2 d^\times m$$

$$\begin{cases} |a+u| \leq 1 & |u| \leq |cm^{-1}| \\ |c| \leq |m|^2 & 1 \leq |m| \leq |t|^{-1} \\ |a^2+bc-(a+u)^2| \leq |cm^{-2}| \end{cases}$$

Since $|a^2+bc| \leq 1$ and $|cm^{-2}| \leq 1$ we see that the condition $|a+u| \leq 1$ is superfluous. We may then change u to $u-a$ to obtain

$$\Omega_{Sl_2}(X) = |c|^{-1} \int\int du|m|^2 d^\times m \tag{53}$$

$$\begin{cases} |u-a| \leq |cm^{-1}| & |a^2+bc-u^2| \leq |cm^{-2}| \\ |c| \leq |m|^2 & 1 \leq |m| \leq |t^{-1}| \end{cases}$$

Before embarking on the computation, we prove a lemma which will show that the orbital integral Ω_{Gl_2} converges absolutely.

LEMMA 8. *Let ω be a compact set of $F^\times$. Then, with the previous notations, the relations $A_2(X) \in \omega$, $R(X) \in \omega$ and $\Omega_{Sl_2}(X) \neq 0$ imply that c is in a compact set of $F^\times$.*

PROOF: Indeed, both t and bc are then in compact sets of $F^\times$. If $\Omega_{Sl_2}(X) \neq 0$ then there are m and u satisfying the above conditions. We have then $|c| \leq |t^{-2}|$ so that $|c|$ is bounded above. If $|bc| \leq |cm^{-2}|$ then, since $|m^{-1}| \leq 1$ we have $|c| \geq |bc|$ and $|c|$ is bounded below. If $|cm^{-2}| < |bc|$ then $|a^2-u^2| = |bc|$. Now $|a^2+bc| \leq 1$ so $|a|$ is bounded above. Thus $|u|$ is also bounded above. Hence $|a+u|$ is bounded above by A say. Then $|bc| \leq A|a-u| \leq |cm^{-1}|A \leq |c|A$. Hence $|c| \geq |bc|A^{-1}$. Thus $|c|$ is bounded below, away from zero, in all cases. □

We have now to distinguish various cases depending on the square class of $-A_1(X) = a^2+bc$.

13.1. Some notations. To formulate the result of our computations in a convenient way, we will introduce some notations.

For $A \in F^\times$ we set

$$\mu(A) := \int_{1 \leq |m| \leq |A|} |m| d^\times m \tag{54}$$

Thus $\mu(A) = 0$ if $|A| < 1$. Otherwise $\mu(A) = \frac{|A|-q^{-1}}{1-q^{-1}}$. In particular, if $|A| = 1$, then $\mu(A) = 1$. Note that the above integral can be written as a sum

$$\sum_{1 \leq |m| \leq |A|} |m|$$

where the sum is over powers of a uniformizer satisfying the required inequalities.

If $A, B, C, \dots,$ are given then we set

$$\mu(A, B, C, \dots) := \mu(D) \text{ where } |D| = \inf(|A|, |B|, |C|, \dots) \tag{55}$$

We also define

$$\mu(A : B) := \int_{|B| \leq |m| \leq |A|} |m| d^\times m\,.$$

Thus $\mu(A:1) = \mu(A)$. We also define

$$\mu(A, B, C, \dots : P, Q, R, \dots) = \mu(D : S)$$

where $|D| = \inf(|A|, |B|, |C|, \dots)$ while $|S| = \sup(|P|, |Q|, |R|, \dots)$. Then

$$\mu(A, B, C \cdots : D) = |D| \mu(AD^{-1}, BD^{-1}, CD^{-1} \cdots) .$$

Clearly, if $1 \leq |C| \leq \inf(|A|, |B|)$, then

$$\mu(A, B : C\varpi^{-1}) + \mu(C) = \mu(A, B) . \tag{56}$$

We will use frequently the following elementary lemma.

Lemma 9. *The difference*

$$\mu(A, B, C) - \mu(A\varpi, B, C)$$

is 0 *unless* $1 \leq |A| \leq \inf(|B|, |C|)$, *in which case, the difference is* $|A|$.

For $A \in F^{\times}$ we set

$$\nu(A) := \int_{1 \leq |m| \leq |A|} d^{\times} m \tag{57}$$

Thus $\nu(A) = 0$ if $|A| < 1$. Otherwise $\nu(A) = 1 - v(A)$. In particular, if $|A| = 1$, then $\nu(A) = 1$. If $A, B, C, \dots,$ are given then we set

$$\nu(A, B, C, \dots) = \nu(D) \, , \, |D| = \inf(|A|, |B|, |C|, \dots) \tag{58}$$

We also define

$$\nu(A : B) = \int_{|B| \leq |m| \leq |A|} d^{\times} m$$

Thus $\nu(A : 1) = \nu(A)$. We define also

$$\nu(A, B, C, \dots : P, Q, R, \dots) = \nu(D : S)$$

where

$$|D| = \inf(|A|, |B|, |C|, \dots) \; , \; |S| = \sup(|P|, |Q|, |R|, \dots) \; .$$

Clearly,

$$\nu(A, B, C \cdots : D) = \nu(AD^{-1}, BD^{-1}, CD^{-1} \cdots) . \tag{59}$$

We will use frequently the following elementary lemma:

Lemma 10. *The difference*

$$\nu(A, B, C) - \nu(A\varpi, B, C)$$

is zero unless $1 \leq |A| \leq \inf(|B|, |C|)$ *in which case it is* 1.

If $x \in F^{\times}$ is an element of even valuation, then we denote by $\sqrt[v]{x}$ **any** element of $F^{\times}$ whose valuation is one-half the valuation of x. If x has odd valuation then $\sqrt[v]{x\varpi}$ is defined but not $\sqrt[v]{x}$. With this convention, the condition

$$|a| \leq |x^2| \leq |b|$$

is equivalent to

$$\left| \left\{ \begin{array}{c} \sqrt[v]{a} \\ \sqrt[v]{a\varpi^{-1}} \end{array} \right\} \right| \leq |x| \leq \left| \left\{ \begin{array}{c} \sqrt[v]{b} \\ \sqrt[v]{b\varpi} \end{array} \right\} \right| . \tag{60}$$

If $|a| \leq |b|$ then

$$|a| \leq \left| \left\{ \begin{array}{c} \sqrt[v]{ab} \\ \sqrt[v]{ab\varpi} \end{array} \right\} \right| \leq \left| \left\{ \begin{array}{c} \sqrt[v]{ab} \\ \sqrt[v]{ab\varpi^{-1}} \end{array} \right\} \right| \leq |b| . \tag{61}$$

13.2. Case where a^2+bc is odd. Suppose first a^2+bc has odd valuation, or, as we shall say, is **odd**. Then there is a uniformizer ϖ such that $a^2+bc=\delta^2\varpi$. In the range (53) for the integral the quadratic condition becomes $|\delta^2\varpi-u^2|\leq|cm^{-2}|$ and, in turn, this is equivalent to $|\delta^2\varpi|\leq|cm^{-2}|$ and $|u^2|\leq|cm^{-2}|$. Thus the integral is equal to

$$|c|^{-1}\int\int du|m|^2d^\times m \tag{62}$$

over

$$\left\{\begin{array}{ll} |u|\leq\left|\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right||m^{-1}| & |u-a|\leq|cm^{-1}|\\ 1\leq|m|\leq|t^{-1}| & |c|\leq|m^2|\leq|c\delta^{-2}\varpi^{-1}|\end{array}\right.$$

If $|c|\leq 1$ then the condition $|c|\leq|m^2|$ is superfluous. Moreover

$$|c|\leq\left|\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right|.$$

Thus the two conditions on u can be rewritten

$$|u-a|\leq|cm^{-1}|\,,\ |a|\leq\left|\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right||m^{-1}|$$

The integral over u is then equal to $|cm^{-1}|$ and so we are left with

$$\int|m|d^\times m \tag{63}$$

over the domain

$$1\leq|m|$$

$$|m|\leq|t^{-1}|\,,\ |m|\leq\left|\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right||a^{-1}|\,,\ |m|\leq\left|\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi^{-1}}\end{array}\right\}\right||\delta^{-1}|\,.$$

With the notation (55) we have, for $|c|\leq 1$,

$$\Omega_{Sl_2}(X)=\mu\left(t^{-1},\delta^{-1}\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi^{-1}}\end{array}\right\},a^{-1}\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right).$$

We pass to the case $|c|>1$. Then the condition $|c|\leq|m^2|$ implies the condition $1\leq|m|$. On the other hand, since

$$\left|\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right|\leq|c|\,.$$

the conditions on u become

$$|u|\leq\left|\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right||m^{-1}|\,,\ |a|\leq cm^{-1}|\,.$$

The integral over u is then equal to

$$\left|\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right||m^{-1}|$$

and so we are left with

$$\left|\begin{array}{c}\frac{1}{\sqrt[v]{c}}\\ \frac{1}{\sqrt[v]{c\varpi^{-1}}}\end{array}\right|\int|m|d^\times m \tag{64}$$

over

$$\left|\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi^{-1}}\end{array}\right|\leq|m|$$

$$|m| \leq |ca^{-1}|\,,\ |m| \leq |t^{-1}|\,,\ |m| \leq \left| \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi^{-1}} \end{array} \right| |\delta^{-1}|$$

We change m to

$$m \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi^{-1}} \end{array} \right\}$$

and we get

$$\int |m| d^{\times} m$$

over

$$1 \leq |m|$$

$$|m| \leq \left| \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right| |a^{-1}|\,,\ |m| \leq \left| \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right| |t^{-1}|\,,\ |m| \leq |\delta^{-1}|$$

Thus, for $|c| > 1$, we find

$$\Omega_{Sl_2}(X) = \mu \left(t^{-1} \left\{ \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right\}, \delta^{-1}, a^{-1} \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\} \right)$$

PROPOSITION 3. *In summary, if $a^2 + bc = \delta^2 \varpi$, (or more generally if $a^2 + bc = \delta^2 \varpi \epsilon$ where ϵ is a unit and ϖ a uniformizer), then*

$$\Omega_{Sl_2}(X) = \left\{ \begin{array}{ll} \mu \left(t^{-1}\,,\ \delta^{-1} \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi^{-1}} \end{array} \right\}\,,\ a^{-1} \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\} \right) & \text{if } |c| \leq 1 \\ \mu \left(t^{-1} \left\{ \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right\}\,,\ \delta^{-1}\,,\ a^{-1} \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\} \right) & \text{if } |c| > 1 \end{array} \right. . \tag{65}$$

We note that if $a = 0$ the identity is to be interpreted as

$$\Omega_{Sl_2}(X) = \left\{ \begin{array}{ll} \mu \left(t^{-1}\,,\ \delta^{-1} \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi^{-1}} \end{array} \right\} \right) & \text{if } |c| \leq 1 \\ \mu \left(t^{-1} \left\{ \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right\}\,,\ \delta^{-1} \right) & \text{if } |c| > 1 \end{array} \right. .$$

13.3. Case where $a^2 + bc$ is even but not a square. We now assume that $a^2 + bc$ has even valuation but is not a square. T hus $a^2 + bc = \delta^2 \tau$ where τ is a unit and a non-square. In the range for the integral (53) the quadratic condition on u becomes $|\delta^2 \tau - u^2| \leq |cm^{-2}|$. In turn this is equivalent to $|\delta^2| \leq |cm^{-2}|$ and $|u^2| \leq |cm^{-2}|$. Thus the integral is equal to

$$|c|^{-1} \int du |m|^2 d^{\times} m \tag{66}$$

over

$$\left\{ \begin{array}{ll} |u| \leq \left| \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\} \right| |m^{-1}| & |u - a| \leq |cm^{-1}| \\ 1 \leq |m| \leq |t^{-1}| & |c| \leq |m^2| \leq |c\delta^{-2}| \end{array} \right.$$

If $|c| \leq 1$ then the condition $|c| \leq |m^2|$ is superfluous. The conditions on u can be rewritten

$$|u - a| \leq |cm^{-1}|\,,\ |a| \leq \left| \left\{ \left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\} \right\} \right| |m^{-1}|$$

After integrating over u we find

$$\int |m| d^{\times} m \tag{67}$$

over

$$1 \leq |m|$$

$$|m| \leq |t^{-1}|\,,\ |m| \leq \left|\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right| |a^{-1}|\,,\ |m| \leq \left|\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right| |\delta^{-1}|$$

Thus, for $|c| \leq 1$,

$$\Omega_{Sl_2}(X) = \mu\left(t^{-1}\,,\ \delta^{-1}\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\,,\ a^{-1}\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right)$$

If $|c| > 1$, then the condition $1 \leq |m|$ is superfluous. On the other hand, the conditions on u become

$$|u| \leq \left|\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right| |m^{-1}|\,,\ |a| \leq |cm^{-1}|$$

After integrating over u we find

$$\left|\left\{ \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right\}\right| \int |m| d^{\times} m$$

over

$$\left|\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi^{-1}} \end{array} \right\}\right| \leq |m|$$

$$|m| \leq |t^{-1}|\,,\ |m| \leq |ca^{-1}|\,,\ |m| \leq |\delta^{-1}| \left|\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right|$$

We change m to

$$m\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi^{-1}} \end{array} \right\}$$

to get

$$\int |m| d^{\times} m$$

over

$$1 \leq |m|$$

$$|m| \leq |a^{-1}| \left|\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right|\,,\ |m| \leq |\delta^{-1}| \left|\left\{ \begin{array}{c} 1 \\ \varpi \end{array} \right\}\right|\,,\ |m| \leq |t^{-1}| \left| \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right|$$

Thus, for $|c| > 1$ we get

$$\Omega_{sL_2}(X) = \mu\left(t^{-1}\left\{ \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right\}\,,\ \delta^{-1}\left\{ \begin{array}{c} 1 \\ \varpi \end{array} \right\}\,,\ a^{-1}\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right).$$

We have proved the following Proposition.

Proposition 4. *If $a^2 + bc = \delta^2\tau$ where τ is a non square unit and $\delta \neq 0$, then*

(68)

$$\Omega_{Sl_2}(X) = \begin{cases} \mu\left(t^{-1}\,,\ \delta^{-1}\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\,,\ a^{-1}\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right) & \text{if } |c| \leq 1 \\ \mu\left(t^{-1}\left\{ \begin{array}{c} \frac{1}{\sqrt[v]{c}} \\ \frac{1}{\sqrt[v]{c\varpi^{-1}}} \end{array} \right\}\,,\ \delta^{-1}\left\{ \begin{array}{c} 1 \\ \varpi \end{array} \right\}\,,\ a^{-1}\left\{ \begin{array}{c} \sqrt[v]{c} \\ \sqrt[v]{c\varpi} \end{array} \right\}\right) & \text{if } |c| > 1 \end{cases}$$

The meaning of the notations is that if c is even, then the formula is true with the top element of each column $\left\{ \begin{matrix} \bullet \\ \bullet \end{matrix} \right\}$. On the contrary, if c is odd, then the formula is true with the bottom element of each column $\left\{ \begin{matrix} \bullet \\ \bullet \end{matrix} \right\}$.

13.4. Case where $a^2 + bc$ is a square and $c \neq 0$. We now assume that $a^2 + bc = \delta^2$ with $\delta \in F^\times$ and $c \neq 0$. Then $a \pm \delta \neq 0$. In (53), the quadratic condition on u becomes $|\delta^2 - u^2| \leq |cm^{-2}|$. This condition is satisfied if and only if one of the three following conditions is satisfied:

$$\begin{array}{lll} I & |\delta^2| \leq |cm^{-2}| & |u^2| \leq |cm^{-2}| \\ II & |cm^{-2}| < |\delta^2| & |u - \delta| \leq |cm^{-2}\delta^{-1}| \\ III & |cm^{-2}| < |\delta^2| & |u + \delta| \leq |cm^{-2}\delta^{-1}| \end{array} \tag{69}$$

Accordingly, we write the integral as a sum of three terms $\Omega^I_{Sl_2}$, $\Omega^{II}_{Sl_2}$, $\Omega^{III}_{Sl_2}$.

The term $\Omega^I_{Sl_2}$ is given by the same expression as before namely (68).

It clear that the term $\Omega^{III}_{Sl_2}$ is obtained from the term $\Omega^{II}_{Sl_2}$ by exchanging δ and $-\delta$. Thus we have only to compute $\Omega^{II}_{Sl_2}$:

$$\Omega^{II}_{Sl_2} = |c|^{-1} \int |m|^2 d^\times m \tag{70}$$

over

$$\left\{ \begin{array}{ll} |u - a| \leq |cm^{-1}| & |u - \delta| \leq |cm^{-2}\delta^{-1}| \\ |c\delta^{-2}| < |m^2| & |c| \leq |m^2| \\ 1 \leq |m| & |m| \leq |t^{-1}| \end{array} \right.$$

We remark that $|a^2 + bc| \leq 1$ implies $|\delta| \leq 1$ and so the condition $|c\delta^{-2}| < |m^2|$ implies $|c| \leq |m^2|$. We further divide the domain of integration into two sub domains defined by $|m| \leq |\delta^{-1}|$ and $|\delta^{-1}| < |m|$ respectively. The last condition implies $1 \leq |m|$. Correspondingly, we write $\Omega^{II}_{Sl_2}$ as the sum of two terms $\Omega^{II.1}_{Sl_2}$ and $\Omega^{II.2}_{Sl_2}$ defined respectively by

$$\Omega^{II.1}_{Sl_2} = |c|^{-1} \int |m|^2 d^\times m \tag{71}$$

over

$$\left\{ \begin{array}{ll} |u - a| \leq |cm^{-1}| & |u - \delta| \leq |cm^{-2}\delta^{-1}| \\ |c\delta^{-2}| < |m^2| & 1 \leq |m| \\ |m| \leq |\delta^{-1}| & |m| \leq |t^{-1}| \end{array} \right.$$

and

$$\Omega^{II.2}_{Sl_2} = |c|^{-1} \int |m|^2 d^\times m \tag{72}$$

over

$$\left\{ \begin{array}{ll} |u - a| \leq |cm^{-1}| & |u - \delta| \leq |cm^{-2}\delta^{-1}| \\ |c\delta^{-2}| < |m^2| & |\delta^{-1}| < |m| \\ |m| \leq |t^{-1}| & \end{array} \right.$$

In $\Omega^{II.1}_{Sl_2}$ the conditions on u are equivalent to

$$|u - a| \leq |cm^{-1}|\,,\ |a - \delta| \leq |cm^{-2}\delta^{-1}|$$

The second condition can be written

$$|m| \leq \left| \left\{ \begin{array}{c} \delta^{-1} \sqrt[v]{c\delta(a-\delta)^{-1}} \\ \delta^{-1} \sqrt[v]{c\delta(a-\delta)^{-1}\varpi} \end{array} \right\} \right| .$$

After integrating over u, we find:

$$\Omega_{Sl_2}^{II.1} = \int |m| d^\times m \tag{73}$$

over

$$\left\{ \begin{array}{ll} |m| \le |\delta^{-1}| & |m| \le |t^{-1}| \\ 1 \le |m| & |c\delta^{-2}| < |m|^2 \\ |m| \le \left| \left\{ \begin{array}{l} \delta^{-1} \sqrt[v]{c\delta(a-\delta)^{-1}} \\ \delta^{-1} \sqrt[v]{c\delta(a-\delta)^{-1}\varpi} \end{array} \right\} \right| & \end{array} \right.$$

If $|c\delta^{-2}| < 1$ then the condition $|c\delta^{-2}| < |m^2|$ is implied by $1 \le |m|$. Thus we find, for $|c\delta^{-2}| < 1$,

$$\Omega_{Sl_2}^{II.1} = \mu \left(t^{-1}, \delta^{-1}, \delta^{-1} \left\{ \begin{array}{l} \sqrt[v]{c\delta(a-\delta)^{-1}} \\ \sqrt[v]{c\delta(a-\delta)^{-1}\varpi} \end{array} \right\} \right)$$

If $|c\delta^{-2}| \ge 1$ then the condition $|c\delta^{-2}| < |m^2|$ implies the condition $1 \le |m|$. On the other hand, the conditions $|c\delta^{-2}| < |m^2|$ is equivalent to

$$\left| \left\{ \begin{array}{l} \delta^{-1}\varpi^{-1} \sqrt[v]{c} \\ \delta^{-1} \sqrt[v]{c\varpi^{-1}} \end{array} \right\} \right| \le |m| \, .$$

Thus we find, for $|c\delta^{-2}| \ge 1$,

$$\Omega_{Sl_2}^{II.1} = \mu \left(t^{-1} \, , \, \delta^{-1} \, , \, \delta^{-1} \left\{ \begin{array}{l} \sqrt[v]{c\delta(a-\delta)^{-1}} \\ \sqrt[v]{c\delta(a-\delta)^{-1}\varpi} \end{array} \right\} : \left\{ \begin{array}{l} \delta^{-1}\varpi^{-1} \sqrt[v]{c} \\ \delta^{-1} \sqrt[v]{c\varpi^{-1}} \end{array} \right\} \right)$$

We pass to the computation of $\Omega_{Sl_2}^{II.2}$. The conditions on u read:

$$|u - \delta| \le |cm^{-2}\delta^{-2}| \, , |a - \delta| \le |cm^{-1}| \, .$$

Thus, after integrating over u, we find

$$\Omega_{Sl_2}^{II.2} = |\delta^{-1}| \int d^\times m \tag{74}$$

over

$$\left\{ \begin{array}{ll} |\delta^{-1}| < |m| & |c\delta^{-2}| < |m^2| \\ |m| \le |t^{-1}| & |m| \le |c(a-\delta)^{-1}| \end{array} \right.$$

If $|c| \le 1$ then the condition $|c\delta^{-2}| < |m^2|$ is already implied by $|\delta^{-1}| < |m|$. Thus we find the domain of integration is

$$|\delta^{-1}\varpi^{-1}| \le |m| \, , \, |m| \le |t^{-1}| \, , \, |m| \le |c(a-\delta)^{-1}| \, .$$

Thus after a change of variables, we get

$$|\delta^{-1}| \int d^\times m$$

over

$$1 \le |m| \, , \, |m| \le \delta\varpi |t^{-1}| \, , \, |m| \le |\delta\varpi c(a-\delta)^{-1}|$$

or

$$|\delta^{-1}| \nu \left(c\delta\varpi(a-\delta)^{-1}, \delta\varpi t^{-1} \right) \, .$$

If $|c| > 1$ then the relation $|\delta^{-1}| < |m|$ is implied by $|c\delta^{-2}| < |m^2|$. This relation is equivalent to

$$\left| \left\{ \begin{array}{l} \sqrt[v]{c}\delta^{-1}\varpi^{-1} \\ \sqrt[v]{c\varpi}\varpi^{-1}\delta^{-1} \end{array} \right\} \right| \, .$$

After a change of variables, we find, for $|c| > 1$,

$$\Omega^{II.2}_{Sl_2} = |\delta^{-1}|\nu\left(\left\{\begin{array}{c}\sqrt[v]{c}\,\delta(a-\delta)^{-1}\varpi\\ \sqrt[v]{c\varpi}\,\delta(a-\delta)^{-1}\end{array}\right\},\left\{\begin{array}{c}\frac{\delta\varpi t^{-1}}{\sqrt[v]{c}}\\ \frac{\delta\varpi t^{-1}}{\sqrt[v]{c\varpi}}\end{array}\right\}\right). \tag{75}$$

In summary, we have proved:

Proposition 5. *If $a^2 + bc = \delta^2$ with $\delta \neq 0$ and $c \neq 0$ then $\Omega_{Sl_2}(X)$ is the sum of*

(76)

$$\Omega^{I}_{Sl_2}(X) = \left\{\begin{array}{ll}\mu\left(t^{-1},\ \delta^{-1}\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\},\ a^{-1}\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right) & |c| \leq 1\\ \mu\left(t^{-1}\left\{\begin{array}{c}\frac{1}{\sqrt[v]{c}}\\ \frac{1}{\sqrt[v]{c\varpi^{-1}}}\end{array}\right\},\ \delta^{-1}\left\{\begin{array}{c}1\\ \varpi\end{array}\right\},\ a^{-1}\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right) & |c| > 1\end{array}\right.$$

(77)

$$\Omega^{II.1}_{Sl_2} =$$

$$\left\{\begin{array}{ll}\mu\left(t^{-1},\delta^{-1},\delta^{-1}\left\{\begin{array}{c}\sqrt[v]{c\delta(a-\delta)^{-1}}\\ \sqrt[v]{c\delta(a-\delta)^{-1}\varpi}\end{array}\right\}\right) & |c\delta^{-2}| < 1\\ \mu\left(t^{-1},\ \delta^{-1},\ \delta^{-1}\left\{\begin{array}{c}\sqrt[v]{c\delta(a-\delta)^{-1}}\\ \sqrt[v]{c\delta(a-\delta)^{-1}\varpi}\end{array}\right\} : \left\{\begin{array}{c}\delta^{-1}\varpi^{-1}\sqrt[v]{c}\\ \delta^{-1}\sqrt[v]{c\varpi^{-1}}\end{array}\right\}\right) & |c\delta^{-2}| \geq 1\end{array}\right.$$

$$\Omega^{II.2}_{Sl_2} = \left\{\begin{array}{ll}|\delta^{-1}|\nu\left(c\delta\varpi(a-\delta)^{-1}, \delta\varpi t^{-1}\right) & |c| \leq 1\\ |\delta^{-1}|\nu\left(\left\{\begin{array}{c}\sqrt[v]{c}\,\delta(a-\delta)^{-1}\varpi\\ \sqrt[v]{c\varpi}\,\delta(a-\delta)^{-1}\end{array}\right\},\left\{\begin{array}{c}\frac{\delta\varpi t^{-1}}{\sqrt[v]{c}}\\ \frac{\delta\varpi t^{-1}}{\sqrt[v]{c\varpi}}\end{array}\right\}\right) & |c| > 1\end{array}\right. \tag{78}$$

plus the terms $\Omega^{III.1}_{Sl_2}$ and $\Omega^{III.2}_{Sl_2}$ obtained by changing δ into $-\delta$.

We also note that if $\delta = 0$ but $c \neq 0$ then the conditions (69) become $|u^2| \leq |cm^{-2}|$ so that $\Omega_{Sl_2} = \Omega^{I}_{Sl_2}$ with $|\delta^{-1}| = \infty$. We record this as a Proposition.

Proposition 6. *If $a^2 + bc = 0$ but $c \neq 0$ then*

$$\Omega_{Sl_2}(X) = \left\{\begin{array}{ll}\mu\left(t^{-1},\ a^{-1}\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right) & \text{if } |c| \leq 1\\ \mu\left(t^{-1}\left\{\begin{array}{c}\sqrt[v]{c^{-1}}\\ \sqrt[v]{c^{-1}\varpi}\end{array}\right\},\ a^{-1}\left\{\begin{array}{c}\sqrt[v]{c}\\ \sqrt[v]{c\varpi}\end{array}\right\}\right) & \text{if } |c| > 1\end{array}\right. \tag{79}$$

In particular if $a = 0$, $b = 0$ but $c \neq 0$ then

$$\Omega_{Sl_2}(X) = \left\{\begin{array}{ll}\mu\left(t^{-1}\right) & \text{if } |c| \leq 1\\ \mu\left(t^{-1}\left\{\begin{array}{c}\sqrt[v]{c^{-1}}\\ \sqrt[v]{c^{-1}\varpi}\end{array}\right\}\right) & \text{if } |c| > 1\end{array}\right. \tag{80}$$

13.5. Case where $c = 0$. We will need the corresponding result when $c = 0$ (and $a = \delta$).

Proposition 7. *If $c = 0$ then*

$$\Omega_{Sl_2}(X) =$$

$$\mu\left(t^{-1}, a^{-1}, \left\{\begin{array}{c}\frac{1}{\sqrt[v]{b}}\\ \frac{1}{\sqrt[v]{b\varpi^{-1}}}\end{array}\right\}\right) + |a^{-1}|\nu(at^{-1}\varpi, a^2\varpi b^{-1})$$

PROOF:

$$\Omega_{Sl_2}(X) = \int\int du|m|^2 d^\times m$$

over

$$|u|, \leq |m^{-1}| \quad , \quad |\frac{b}{2a} - u| \leq |m^{-2}a^{-1}|$$
$$1 \leq |m| \quad , \quad |m| \leq |t^{-1}|$$

Since $A_1(X)$ is a integer we have $|a| \leq 1$.

We first consider the contribution of the terms for which $|m| \leq |a^{-1}|$. Then the condition on u become

$$|u| \leq |m^{-1}|\,, \, |\frac{b}{2a}| \leq |m^{-2}a^{-1}|\,.$$

After integrating over u we find

$$\int |m| d^\times m$$

over

$$1 \leq |m| \; |m| \leq |t^{-1}|\,, \, |m^2| \leq |b^{-1}|$$

that is,

$$\mu\left(t^{-1}, a^{-1}, \left\{ \begin{array}{c} \frac{1}{\sqrt[v]{b}} \\ \frac{1}{\sqrt[v]{b\varpi^{-1}}} \end{array} \right\}\right).$$

Next, we consider the contributions of the terms for which $|a^{-1}\varpi^{-1}| \leq |m|$. Then the conditions on u become:

$$|u| \leq |m^{-2}a^{-1}|\,, \, |\frac{b}{2a}| \leq |m^{-1}|\,.$$

After integrating over u we find

$$|a^{-1}| \int d^\times m$$

over

$$1 \leq |m|\,, \, |a^{-1}\varpi^{-1}| \leq |m|\,,$$
$$|m| \leq |t^{-1}|\,, \, |m| \leq |ab^{-1}|\,.$$

However, $|a| = \leq 1$. Thus the condition $1 \leq |m|$ is superfluous. Thus this is

$$\nu(t^{-1}, ab^{-1} : a^{-1}\varpi^{-1}) = \nu(at^{-1}\varpi, a^2\varpi b^{-1})\,.$$

The Proposition follows. □

14. Proof of the fundamental lemma for $n = 3$

We let

$$Y = \begin{pmatrix} a & b & 0 \\ 1 & -a & 1 \\ 0 & t & 0 \end{pmatrix} \tag{81}$$

with $t \neq 0$ and $b \neq 0$. Then:

$$\Omega_{Gl_2}(Y) = \int_{F^\times} \Omega_{Sl_2} \begin{pmatrix} a & bs^{-1} & 0 \\ s & -a & 1 \\ 0 & t & 0 \end{pmatrix} \eta(s) d^\times s$$

Since the integrand depends only on the absolute value of s, this integral can be computed as a sum:

$$\sum_s \Omega_{Sl_2}\begin{pmatrix} a & bs^{-1} & 0 \\ s & -a & 1 \\ 0 & t & 0 \end{pmatrix}\eta(s)\,,$$

where s is summed over the powers of a uniformizer ϖ. It follows from lemma (8) that the sum is finite, that is, the integral converges absolutely, provided Y is in $\mathfrak{g}(F)^s$. In the two next sections, we compute this integral and check Theorem (1). That is, if $-b$ is not a norm we show that $\Omega_{Gl_2}(Y) = 0$. Otherwise we solve the equations (46), define X by (44) and check that

$$\Omega_{Sl_2}(X) = \Omega_{Gl_2}(Y)\,. \tag{82}$$

Before we proceed we remark that $\Omega_{Gl_2}(Y) \neq 0$ implies $|A_1(Y)| \leq 1$ and $|A_2(Y)| \leq 1$. Likewise, if X is defined, $\Omega_{Sl_2}(X) \neq 0$ implies $|A_1(X)| \leq 1$ and $|A_2(X)| \leq 1$. Finally, if X is defined then $|A_1(X)| = |A_1(Y)|$ and $|A_2(X)| = |A_2(Y)|$. Thus if $|A_1(Y)| > 1$ or $|A_2(Y)| > 1$ our assertions are trivially true. Thus we may assume $|A_1(Y)| \leq 1$ and $|A_2(Y)| \leq 1$, that is, $|a^2+b| \leq 1$ and $|t| \leq 1$.

As before, the discussion depends on the square class of $a^2+b = -A_1(Y)$.

15. Proof of the fundamental Lemma: a^2+b is not a square

15.1. Case where a^2+b is odd. We consider the case where $a^2+b = -A_1(Y)$ is odd (that is has odd valuation) and we write $a^2+b = \delta^2\varpi$ where ϖ is a uniformizer. The integral Ω_{Gl_2} is then the sum of two terms $\Omega^A_{Gl_2}$ and $\Omega^B_{Gl_2}$ corresponding to the contributions of $|s| \leq 1$ and $|s| > 1$ respectively. If $|s| \leq 1$ we write $s = r^2$ or $s = r^2\varpi$ with $|r| \leq 1$. Then

$$\Omega^A_{Gl_2} = \sum_{|r|\leq 1}\left[\mu(t^{-1}, \delta^{-1}r, a^{-1}r) - \mu(t^{-1}, \delta^{-1}r, a^{-1}r\varpi)\right]. \tag{83}$$

By Lemma 9, expression $\Omega^A_{Gl_2}$ is equal to

$$\sum |a^{-1}r|$$

over

$$|r| \leq 1\,,\ 1 \leq |a^{-1}r| \leq \inf(|t^{-1}|, |\delta^{-1}r|)\,.$$

This is zero unless $|\delta| \leq |a|$. If $|\delta| \leq |a|$, after changing r to ra, we find

$$\sum_{1\leq |r|\leq \inf(|a^{-1}|,|t^{-1}|)} |r|\,.$$

In other words, we find:

$$\Omega^A_{Gl_2} = \begin{cases} \mu(a^{-1}, t^{-1}) & \text{if } |\delta| \leq |a| \\ 0 & \text{if } |\delta| > |a| \end{cases} \tag{84}$$

We pass to the contribution of $|s| > 1$. We write $s = r^2$ or $s = r^2\varpi$ with $|r| > 1$. Then

$$\Omega^B_{Gl_2} = \sum_{1<|r|}\left[\mu(t^{-1}r^{-1}, \delta^{-1}, a^{-1}r) - \mu(t^{-1}r^{-1}, \delta^{-1}, a^{-1}r\varpi)\right]. \tag{85}$$

Applying lemma (9) we get

$$\sum |a^{-1}r|$$

over

$$1 < |r|\,,\, 1 \le |a^{-1}r| \le \inf(|\delta^{-1}|, |t^{-1}r^{-1}|)$$

This is zero unless $|\delta| < |a|$. If $|\delta| < |a|$, after changing r to ra, we find this is

$$\sum |r|$$

over

$$\sup(|a^{-1}\varpi^{-1}|, 1) \le |r|\,,\, |r| \le |\delta^{-1}|\,,\, |r^2| \le |t^{-1}a^{-1}|$$

Thus we find

$$\Omega^B_{Gl_2} = \begin{cases} \mu\left(\delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\} : 1, a^{-1}\varpi^{-1}\right) & \text{if } |\delta| < |a| \\ 0 & \text{if } |\delta| \ge |a| \end{cases} \tag{86}$$

We can combine both results to obtain

PROPOSITION 8. *If* $a^2 + b = \delta^2\varpi$ *then*

$$\Omega_{Gl_2}(Y) = \begin{cases} \mu\left(t^{-1}, \delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right) & \text{if } |\delta| \le |a| \\ 0 & \text{if } |\delta| > |a| \end{cases}$$

PROOF: Clearly, our integral is 0 if $|\delta| > |a|$. If $|\delta| = |a|$ then the integral reduces to $\mu(t^{-1}, \delta^{-1})$. However,

$$\left|\left\{ \begin{array}{c} \sqrt[v]{t^{-1}\delta^{-1}} \\ \sqrt[v]{t^{-1}\delta^{-1}\varpi} \end{array} \right\}\right|$$

belongs to the interval determined by $|t^{-1}|$ and $|\delta^{-1}|$ and so the integral can be written in the stated form.

Assume now $|\delta| < |a|$. If $|a| > 1$ then $\mu(a^{-1}, t^{-1}) = 0$ and $|a^{-1}\varpi^{-1}| \le 1$. Thus $\Omega^A_{Gl_2} = 0$ and $\Omega^B_{Gl_2}$ reduces to

$$\mu\left(\delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right).$$

Since $|t| \le 1$ we have $|at| > |t^2|$ or

$$|t^{-1}| > \left|\left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right|$$

so that the result can again being written in the required form.

Finally, assume $|\delta| < |a| \le 1$. Then $|a^{-1}\omega^{-1}| > 1$ and

$$\Omega_{Gl_2} = \mu\left(\delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\} : a^{-1}\varpi^{-1}\right) + \mu(a^{-1}, t^{-1})\,.$$

Suppose first $|t| \le |a|$. Then $\mu(a^{-1}, t^{-1}) = \mu(a^{-1})$. Then $|a^{-1}\varpi^{-1}| \le |\delta^{-1}|$ and

$$|a^{-1}| \le \left|\left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right|$$

The sum for Ω_{Gl_2} is then by (56) equal to

$$\mu\left(\delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right)$$

Since

$$\left|\left\{\begin{array}{c}\sqrt[v]{t^{-1}a^{-1}}\\ \sqrt[v]{t^{-1}a^{-1}\varpi}\end{array}\right\}\right| \leq |t^{-1}|$$

this can be written in the required form.

Suppose now $|t| > |a|$. Then $\mu(a^{-1}, t^{-1}) = \mu(t^{-1})$. On the other hand,

$$\left|\left\{\begin{array}{c}\sqrt[v]{t^{-1}a^{-1}}\\ \sqrt[v]{t^{-1}a^{-1}\varpi}\end{array}\right\}\right| < |a^{-1}\varpi^{-1}|$$

so that $\Omega^B_{Gl_2}$ vanishes. On the other hand, since $|\delta|^{-1} \geq |t^{-1}|$ and

$$\left|\left\{\begin{array}{c}\sqrt[v]{t^{-1}a^{-1}}\\ \sqrt[v]{t^{-1}a^{-1}\varpi}\end{array}\right\}\right| \geq |t^{-1}|$$

the expression given in the Proposition is indeed equal to $\mu(t^{-1})$. □.

We now check the fundamental lemma in the case at hand. If $-b = a^2 - \delta^2\varpi$ is not a norm, then the valuation of b is odd and $|\delta| > |a|$. Then $\Omega_{Gl_2}(Y) = 0$. Now suppose that $-b$ is a norm, that is, $|a| \geq |\delta|$. Then $-b$ is in fact a square. Thus we may solve the equations of matching (46) in the following way. If $|u| < 1$ we denote by $\sqrt{1+u}$ the square root of $1+u$ which is congruent to one modulo $\varpi\mathcal{O}_F$. Recall τ is a non-square unit. Then we write

$$-\tau^2 b = y^2\,,\ y = -\tau a\sqrt{1 - \delta^2 a^{-2}\varpi}\,;$$

Then we take

$$a_1 = 0\,,\ b_1 = -\frac{t}{2}(y + \tau a)\,,\ c_1 = \frac{2}{\tau t}(y - \tau a)\,,\ t_1 = -\frac{\tau t}{2}\,.$$

We have then $a_1^2 + b_1c_1 = \tau(a^2 + b) = \delta^2\varpi\tau$. Thus $a_1^2 + b_1c_1$ is odd. We have also $|c_1| = |at^{-1}|$ and $|t_1| = |t|$. Let X be as in (44). We then have by Proposition 3,

$$\Omega_{Sl_2}(X) = \begin{cases} \mu\left(t^{-1}, \delta^{-1}\left\{\begin{array}{c}\sqrt[v]{at^{-1}}\\ \sqrt[v]{at^{-1}\varpi^{-1}}\end{array}\right\}\right) & \text{if } |a| \leq |t| \\ \mu\left(t^{-1}\left\{\begin{array}{c}\frac{1}{\sqrt[v]{at^{-1}}}\\ \frac{1}{\sqrt[v]{at^{-1}\varpi^{-1}}}\end{array}\right\}, \delta^{-1}\right) & \text{if } |a| > |t| \end{cases}$$

Suppose first $|a| \leq |t|$. Since $|\delta| \leq |a|$ we easily get

$$|t^{-1}| \leq \left|\delta^{-1}\left\{\begin{array}{c}\sqrt[v]{at^{-1}}\\ \sqrt[v]{at^{-1}\varpi^{-1}}\end{array}\right\}\right|$$

and so the expression for $\Omega_{Sl_2}(X)$ reduces to $\mu(t^{-1})$. But the same is true of the expression for $\Omega_{Gl_2}(Y)$.

Now suppose $|a| > |t|$. Then the expression for $\Omega_{SL_2}(X)$ becomes

$$\mu\left(\left\{\begin{array}{c}\sqrt[v]{t^{-1}a^{-1}}\\ \sqrt[v]{t^{-1}a^{-1}\varpi}\end{array}\right\}, \delta^{-1}\right).$$

Since

$$|t^{-1}| \geq \left|\left\{\begin{array}{c}\sqrt[v]{t^{-1}a^{-1}}\\ \sqrt[v]{t^{-1}a^{-1}\varpi}\end{array}\right\}\right|$$

this is also the expression for $\Omega_{Gl_2}(Y)$ and we are done. □

15.2. Case where a^2+b is even and not a square. Suppose now that $a^2+b=\delta^2\tau$ where τ is, as before, a non-square unit.

PROPOSITION 9. *Suppose $a^2+b=\delta^2\tau$. Then $\Omega_{Gl_2}(Y)$ is the sum of*

$$|\delta^{-1}|\nu\left(\delta t^{-1},\varpi\delta^2 t^{-1}a^{-1}\right)$$

and

$$\begin{cases} \mu\left(\delta^{-1},\begin{Bmatrix}\sqrt[v]{t^{-1}a^{-1}}\\ \sqrt[v]{t^{-1}a^{-1}\varpi}\end{Bmatrix},\right) & \text{if } |a|\geq \sup(|\delta|,|t|)\\ \mu(t^{-1},\delta^{-1}\varpi) & \text{if } |a|<\sup(|\delta|,|t|)\end{cases}$$

PROOF: We proceed as before and write $\Omega_{Gl_2}(Y)$ as the sum of $\Omega^A_{Gl_2}$ and $\Omega^B_{Gl_2}$, these being respectively the contributions of the terms corresponding to $|s|\leq 1$ and $|s|>1$. For $|s|\leq 1$, we set aside the term $|s|=1$ and we write $s=r^2\varpi^2$ or $s=r^2\varpi$ with $|r|\leq 1$. We find

$$\begin{aligned}&\Omega^A_{Gl_2}\\ =\;&\mu(t^{-1},\delta^{-1},a^{-1})\\ +\;&\sum_{|r|\leq 1}\left[\mu(t^{-1},\delta^{-1}r\varpi,a^{-1}r\varpi)-\mu(t^{-1},\delta^{-1}r\varpi,a^{-1}r\varpi)\right]\\ =\;&\mu(t^{-1},\delta^{-1},a^{-1})\end{aligned}$$

For $|s|>1$ we write $s=r^2$ or $s=r^2\varpi$ with $|r|>1$. We find

$$\Omega^B_{Gl_2}=\sum_{|r|>1}\left[\mu(t^{-1}r^{-1},\delta^{-1},a^{-1}r)-\mu(t^{-1}r^{-1},\delta^{-1}\varpi,a^{-1}r\varpi)\right] \tag{87}$$

If we add to this $\Omega^A_{Gl_2}$ we find

$$\Omega_{Gl_2}$$

$$= \mu(t^{-1},\delta^{-1}\varpi,a^{-1}\varpi) \tag{88}$$

$$+ \sum_{|r|\geq 1}\left[\mu(t^{-1}r^{-1},\delta^{-1},a^{-1}r)-\mu(t^{-1}r^{-1},\delta^{-1}\varpi,a^{-1}r\varpi)\right] \tag{89}$$

Applying lemma (9), the second sum can be computed as

$$\sum \inf\left(|\delta^{-1}|,|a^{-1}r|\right) \tag{90}$$

the sum over

$$|r|\geq 1\,,\ 1\leq \inf\left(|\delta^{-1}|,|a^{-1}r|\right)\leq |t^{-1}r^{-1}|$$

We first consider the contribution of the terms with $|a^{-1}r|\leq|\delta^{-1}|$:

$$\sum |a^{-1}r| \tag{91}$$

over

$$1\leq |r|\,,\ |a|\leq |r|$$
$$|r|\leq |a\delta^{-1}|\,,\ |r^2|\leq |at^{-1}|$$

If we change r to ra this becomes

$$\mu\left(\delta^{-1},\begin{Bmatrix}\sqrt[v]{t^{-1}a^{-1}}\\ \sqrt[v]{t^{-1}a^{-1}\varpi}\end{Bmatrix}:1,a^{-1}\right) \tag{92}$$

Next, we consider the contribution of the terms with $|\delta^{-1}|<|a^{-1}r|$:

$$\sum |\delta^{-1}|$$

over

$$1 \leq |r|\,,\ |\delta^{-1}a| < |r|$$
$$|r| \leq |\delta t^{-1}|$$

After a change of variables, this can be written as

$$|\delta^{-1}| \sum 1$$

over

$$1 \leq |r| \leq \inf\left(|\delta t^{-1}|, |\varpi\delta^2 t^{-1}a^{-1}|\right)$$

so that this is

$$|\delta^{-1}|\nu\left(\delta t^{-1}, \varpi\delta^2 t^{-1}a^{-1}\right)$$

In summary we have found that Ω_{Gl_2} is the sum of

$$\mu(t^{-1}, \delta^{-1}\varpi, a^{-1}\varpi) \tag{93}$$

$$\mu\left(\delta^{-1}, \left\{\begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array}\right\} : 1, a^{-1}\right) \tag{94}$$

$$|\delta^{-1}|\nu\left(\delta t^{-1}, \varpi\delta^2 t^{-1}a^{-1}\right) \tag{95}$$

If $|a| < |\delta|$ then the second term is zero and the first can be written as $\mu(t^{-1}, \delta^{-1}\varpi)$.

If $|a| < |t|$ then

$$|a^{-1}| > \left|\left\{\begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array}\right\}\right|$$

so that the second term is 0 and the first can be written again as $\mu(t^{-1}, \delta^{-1}\varpi)$.

Now assume $|a| \geq \sup(|\delta|, |t|)$. Then $\mu(t^{-1}, \delta^{-1}\varpi, a^{-1}\varpi) = \mu(a^{-1}\varpi)$. If $|a| \geq 1$ then $\mu(a^{-1}\varpi) = 0$ while the second term reduces to

$$\mu\left(\delta^{-1}, \left\{\begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array}\right\}\right)$$

and we obtain the Proposition. If $|a| < 1$ then the second term is in fact

$$\mu\left(\delta^{-1}, \left\{\begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array}\right\} : a^{-1}\right).$$

Adding $\mu(a^{-1}\varpi)$ to this and using (56) we obtain the Proposition. □

We now check the fundamental lemma for the case at hand. Of course $-b = a^2 - \delta^2\tau$ is a norm. Thus we may solve the conditions of matching (46) as follows:

$$a_1 = \delta\tau\,,\ c_1 = 0\,,\ b_1 = -\tau t a\,,\ t_1 = -\frac{\tau t}{2}\,.$$

Then $a_1^2 + b_1 c_1 = a_1^2 = \delta_1^2$ where $\delta_1 = \delta\tau$. Thus by section 6.3,

$$\Omega_{Sl_2}(X) =$$
$$\mu\left(t^{-1}, \delta^{-1}, \left\{\begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array}\right\}\right) + |\delta^{-1}|\nu(\delta t^{-1}\varpi, \delta^2 t^{-1}a^{-1}\varpi)\,.$$

If $|a| \geq \sup(|\delta|, |t|)$ then

$$|t^{-1}| \geq \left|\left\{\begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array}\right\}\right|,$$
$$|\delta^2 t^{-1}a^{-1}\varpi| \leq |\delta t^{-1}\varpi| < |\delta t^{-1}|\,.$$

Hence Ω_{SL_2} is equal to

$$\mu\left(\delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right) + |\delta^{-1}|\nu(\delta t^{-1}, \delta^2 t^{-1}a^{-1}\varpi)$$

which is Ω_{Gl_2} in this case.

Now assume $|a| < \sup(|\delta|, |t|)$. Suppose first $|t| \leq |a| < |\delta|$. Then $|\delta a^{-1}| > 1$, $|\delta t^{-1}| > 1$ and $|\delta^2| > |ta|$. Thus

$$|\delta^{-1}| \leq \left|\left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right| .$$

Recall $|\delta| \leq 1$. Hence

$$\begin{aligned} \Omega_{Sl_2} &= \mu(\delta^{-1}) + |\delta^{-1}|\nu(\delta t^{-1}\varpi) \\ &= \frac{|\delta^{-1}| - q^{-1}}{1 - q^{-1}} + |\delta^{-1}|(-v(\delta t^{-1})) \end{aligned}$$

while

$$\Omega_{Gl_2} = \mu(\delta^{-1}\varpi) + |\delta^{-1}|\nu(\delta t^{-1})$$

If $|\delta| < 1$ then we find

$$\Omega_{Gl_2} = \frac{|\delta^{-1}|q^{-1} - q^{-1}}{1 - q^{-1}} + |\delta^{-1}|(1 - v(\delta t^{-1}))$$

If $|\delta| = 1$ then we find

$$\Omega_{Gl_2} = 1 - v(\delta t^{-1})$$

In any case the two expressions are indeed equal.

Now assume $|\delta| \leq |a| < |t|$. Then

$$|t^{-1}| \leq \left|\left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right|$$

and both orbital integrals are equal to

$$\mu(t^{-1}) + |\delta^{-1}|\nu(\delta^2 t^{-1}a^{-1}\varpi) .$$

Finally assume $|a| < |\delta|$ and $|a| < |t|$. Then again

$$|t^{-1}| \leq \left|\left\{ \begin{array}{c} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{array} \right\}\right|$$

and Ω_{Sl_2} is equal to

$$\mu(t^{-1}, \delta^{-1}) + |\delta^{-1}|\nu(\delta t^{-1}\varpi)$$

while Ω_{Gl_2} is equal to

$$\mu(t^{-1}, \delta^{-1}\varpi) + |\delta^{-1}|\nu(\delta t^{-1}) .$$

If $1 > |\delta| > |t|$ then

$$\Omega_{Sl_2} = \mu(\delta^{-1}) + |\delta^{-1}|\nu(\delta t^{-1}\varpi) = \frac{|\delta^{-1}| - q^{-1}}{1 - q^{-1}} + |\delta^{-1}|(-v(\delta t^{-1})$$

while

$$\Omega_{Gl_2} = \mu(\delta^{-1}\varpi) + |\delta^{-1}|\nu(\delta t^{-1}) = \frac{|\delta^{-1}|q^{-1} - q^{-1}}{1 - q^{-1}} + |\delta^{-1}|(1 - v(\delta t^{-1}))$$

and those two expressions are indeed equal.

If $1 = |\delta| > |t|$ then

$$\Omega_{Sl_2} = \mu(\delta^{-1}) + |\delta^{-1}|\nu(\delta t^{-1}\varpi) = 1 - v(t^{-1})$$

while

$$\Omega_{Gl_2} = |\delta^{-1}|\nu(\delta t^{-1}) = 1 - v(t^{-1})$$

and the two expressions are indeed equal.

Now suppose $|\delta| = |t|$. Recall $|\delta| \leq 1$. Then

$$\Omega_{Sl_2} = \mu(\delta^{-1}) = \frac{|\delta|^{-1} - q^{-1}}{1 - q^{-1}}$$

while

$$\Omega_{Gl_2} = \mu(\delta^{-1}\varpi) + |\delta|^{-1}\nu(1) = \frac{|\delta|^{-1}q^{-1} - q^{-1}}{1 - q^{-1}} + |\delta|^{-1}$$

and the two expressions are indeed equal.

If $|\delta| < |t|$ then both orbital integrals are equal to $\mu(t^{-1})$. So the fundamental lemma has been completely checked in this case. □

16. Proof of the fundamental Lemma: $a^2 + b$ is a square

Finally we consider the case where $a^2 + b = \delta^2$, $\delta \neq 0$. Recall we compute $\Omega_{Gl_2}(Y)$ as the sum

$$\sum_s \Omega_{Sl_2} \begin{pmatrix} a & bs^{-1} & 0 \\ s & -a & 1 \\ 0 & t & 0 \end{pmatrix} \eta(s)$$

and $a^2 + bs^{-1}s = a^2 + b = \delta^2$. Recall we have written the orbital integral Ω_{SL_2} as a sum of terms labeled $\Omega^{I}_{Sl_2}$, $\Omega^{II.1}_{Sl_2}$, $\Omega^{II.2}_{Sl_2}$, $\Omega^{III.1}_{Sl_2}$, $\Omega^{III.2}_{Sl_2}$ respectively. Correspondingly, we write $\Omega_{Gl_2}(Y)$ as the sum of terms labeled $\Omega^{I}_{Gl_2}$, $\Omega^{II.1}_{Gl_2}$ and so on. For instance,

$$\Omega^{I}_{Gl_2} = \sum_s \Omega^{I}_{Sl_2} \begin{pmatrix} a & bs^{-1} & 0 \\ s & -a & 1 \\ 0 & t & 0 \end{pmatrix} \eta(s)\,.$$

16.1. Computation of $\Omega^{I}_{Gl_2}$.

The term $\Omega^{I}_{Gl_2}$ can be computed as Ω_{Gl_2} in the previous case (where $a^2 + b$ is even and not a square). We write it as a sum

$$\Omega^{I}_{Gl_2} = \Omega^{I.1}_{Gl_2} + \Omega^{I.2}_{Gl_2} \tag{96}$$

where

$$\Omega^{I.1}_{Gl_2} = \begin{cases} \mu\left(\delta^{-1}, \begin{Bmatrix} \sqrt[v]{t^{-1}a^{-1}} \\ \sqrt[v]{t^{-1}a^{-1}\varpi} \end{Bmatrix}\right) & \text{if } |a| \geq \sup(|\delta|, |t|) \\ \mu(t^{-1}, \delta^{-1}\varpi) & \text{if } |a| < \sup(|\delta|, |t|) \end{cases} \tag{97}$$

and

$$\Omega^{I.2}_{Gl_2} = |\delta^{-1}|\nu(\delta t^{-1}, \delta^2 t^{-1} a^{-1}\varpi) \tag{98}$$

16.2. Computation of $\Omega_{Gl_2}^{II.1}$. After changing s into $s\delta^2$ we see that

$$\Omega_{Gl_2}^{II.1} = \sum_s \Omega_{Sl_2}^{II.1} \begin{pmatrix} a & bs^{-1}\delta^{-2} & 0 \\ s\delta^2 & -a & 1 \\ 0 & t & 0 \end{pmatrix} \eta(s)$$

and so, by Proposition 5, we get $\Omega_{Gl_2}^{II.1} = \Omega_{Gl_2}^{II.1.1} + \Omega_{Gl_2}^{II.1.2}$ where

$$\Omega_{Gl_2}^{II.1.1} = \sum_{|s|<1} \eta(s)\mu\left(t^{-1}, \delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{s\delta(a-\delta)^{-1}} \\ \sqrt[v]{s\delta(a-\delta)^{-1}\varpi} \end{array} \right\}\right) \tag{99}$$

and

$$\Omega_{Gl_2}^{II.1.2} = \sum_{|s|\geq 1} \eta(s)\mu\left(t^{-1}, \delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{s\delta(a-\delta)^{-1}} \\ \sqrt[v]{s\delta(a-\delta)^{-1}\varpi} \end{array} \right\} : \left\{ \begin{array}{c} \varpi^{-1}\sqrt[v]{s} \\ \sqrt[v]{s\varpi^{-1}} \end{array} \right\}\right) \tag{100}$$

Suppose first that $\delta(a-\delta)^{-1}$ is even. For $\Omega_{Gl_2}^{II.1.1}$ we write $s = r^2\varpi^2$ or $s = r^2\varpi$ with $|r| \leq 1$. We find, for $|r| \leq 1$, each term

$$\mu(t^{-1}, \delta^{-1}, \varpi r \sqrt[v]{\delta(a-\delta)^{-1}})$$

once with a + sign and once with a − sign. So we get zero. For $\Omega_{Gl_2}^{II.1.2}$ we write $s = r^2$ or $s = r^2\varpi^{-1}$ with $|r| \geq 1$. We find, for $|r| \geq 1$, each term

$$\mu(t^{-1}, \delta^{-1}, r\sqrt[v]{\delta(a-\delta)^{-1}}) : \varpi^{-1}r)$$

one with a + sign and once with a − sign. So we get 0. Thus $\Omega_{Gl_2}^{II.1} = 0$ if $\delta(a-\delta)^{-1}$ is even.

Now we assume $\delta(a-\delta)^{-1}$ is odd. For $\Omega_{Gl_2}^{II.1.1}$ we write $s = r^2$ or $s = r^2\varpi$ with $|r| \leq 1$. We have then added a term corresponding to $s = r^2$ with $|r| = 1$ that we must subtract. We find

$$-\mu\left(t^{-1}, \delta^{-1}, \sqrt[v]{\delta(a-\delta)^{-1}\varpi}\right) +$$

$$\sum_{|r|\leq 1} \mu\left(t^{-1}, \delta^{-1}, r\sqrt[v]{\delta(a-\delta)^{-1}\varpi}\right) - \sum_{|r|\leq 1} \mu\left(t^{-1}, \delta^{-1}, r\sqrt[v]{\delta(a-\delta)^{-1}\varpi}\right)$$

or

$$\Omega_{Gl_2}^{II.1.1} = -\mu\left(t^{-1}, \delta^{-1}, \sqrt[v]{\delta(a-\delta)^{-1}\varpi}\right) .$$

In particular, this is 0 unless $|\delta(a-\delta)^{-1}\varpi| \geq 1$. For $\Omega_{Gl_2}^{II.1.2}$ we write $s = r^2$ or $s = r^2\varpi^{-1}$ with $|r| \geq 1$. We find

$$\sum_{|r|\geq 1} \left(\mu\left(t^{-1}, \delta^{-1}, r\sqrt[v]{\delta(a-\delta)^{-1}\varpi} : \varpi^{-1}r\right) - \right.$$

$$\left.\mu\left(t^{-1}, \delta^{-1}, r\sqrt[v]{\delta(a-\delta)^{-1}\varpi^{-1}} : \varpi^{-1}r\right)\right)$$

$$= |\varpi^{-1}| \sum_{|r|\geq 1} |r| \left(\mu\left(t^{-1}r^{-1}\varpi, \delta^{-1}r^{-1}\varpi, \varpi\sqrt[v]{\delta(a-\delta)^{-1}\varpi}\right) - \right.$$

$$\left.\mu\left(t^{-1}r^{-1}\varpi, \delta^{-1}r^{-1}\varpi, \sqrt[v]{\delta(a-\delta)^{-1}\varpi}\right)\right) .$$

Once more we apply Lemma 9. We find this is zero unless $|\delta(a-\delta)^{-1}\varpi| \geq 1$. Then this is equal to

$$= -|\varpi^{-1}| \left|\sqrt[v]{\delta(a-\delta)^{-1}\varpi}\right| \sum_r |r|$$

where the sum is for

$$1 \leq |r|\,,\ |r| \leq \left| \frac{t^{-1}\varpi}{\sqrt[v]{\delta(a-\delta)^{-1}\varpi}} \right|\,,\ |r| \leq \left| \frac{\delta^{-1}\varpi}{\sqrt[v]{\delta(a-\delta)^{-1}\varpi}} \right|$$

Thus

$$\Omega_{Gl_2}^{II.1.2} =$$

$$-|\varpi^{-1}| \left| \sqrt[v]{\delta(a-\delta)^{-1}\varpi} \right| \mu \left(\frac{t^{-1}\varpi}{\sqrt[v]{\delta(a-\delta)^{-1}\varpi}}, \frac{\delta^{-1}\varpi}{\sqrt[v]{\delta(a-\delta)^{-1}\varpi}} \right).$$

Hence we find that $\Omega_{Gl_2}^{II.1}$ is zero unless $\delta(a-\delta)^{-1}$ is odd and $|\delta(a-\delta)^{-1}\varpi| \geq 1$. It is then given by

$$-|\varpi^{-1}| \left| \sqrt[v]{\delta(a-\delta)^{-1}\varpi} \right| \mu \left(\frac{t^{-1}\varpi}{\sqrt[v]{\delta(a-\delta)^{-1}\varpi}}, \frac{\delta^{-1}\varpi}{\sqrt[v]{\delta(a-\delta)^{-1}\varpi}} \right)$$

$$-\mu \left(t^{-1}, \delta^{-1}, \sqrt[v]{\delta(a-\delta)^{-1}\varpi} \right).$$

We claim this is $-\mu(t^{-1}, \delta^{-1})$. Indeed, this is clear if

$$\left| \sqrt[v]{\delta(a-\delta)^{-1}\varpi} \right| \geq \inf(|t^{-1}|, |\delta^{-1}|)$$

because the first term is then 0 and the second term equal to $-\mu(t^{-1}, \delta^{-1})$. Now assume that $\left| \sqrt[v]{\delta(a-\delta)^{-1}\varpi} \right| < \inf(|t^{-1}|, |\delta^{-1}|)$. Recall $|\delta| \leq 1$ and $|t| \leq 1$. To be definite assume $|t^{-1}| \leq |\delta^{-1}|$. Then our sum is

$$-|\varpi^{-1}| \left| \sqrt[v]{\delta(a-\delta)^{-1}\varpi} \right| \mu \left(\frac{t^{-1}\varpi}{\sqrt[v]{\delta(a-\delta)^{-1}\varpi}} \right)$$

$$-\mu \left(\sqrt[v]{\delta(a-\delta)^{-1}\varpi} \right)$$

$$= \frac{q^{-1} - |t^{-1}|}{1 - q^{-1}} = -\mu(t^{-1})$$

as was claimed. We have proved:

PROPOSITION 10. *$\Omega_{Gl_2}^{II.1}(Y) = 0$ unless $\delta(a-\delta)^{-1}$ is odd and $|(a-\delta)| \leq |\delta\varpi|$. Then*

$$\Omega_{Gl_2}^{II.1}(Y) = -\mu(t^{-1}, \delta^{-1}).$$

16.3. Computation of $\Omega_{Gl_2}^{II.2}$. As before

$$\Omega_{Gl_2}^{II.2}(Y) = \sum_s \Omega_{Sl_2}^{II.2} \begin{pmatrix} a & bs^{-1} & 0 \\ s & -a & 1 \\ 0 & t & 0 \end{pmatrix} \eta(s)$$

and we denote by $\Omega_{Gl_2}^{II.2.1}$ and $\Omega_{Gl_2}^{II.2.2}$ the respective contributions of the terms $|s| \leq 1$ and $|s| > 1$. Then

$$\Omega_{Gl_2}^{II.2}(Y) = \Omega_{Gl_2}^{II.2.1} + \Omega_{Gl_2}^{II.2.2}.$$

We now appeal to Proposition 5. To compute $\Omega_{Gl_2}^{II.2.1}$ we write $s = r^2$ or $s = r^2\varpi$ with $|r| \leq 1$. We find:

$$\Omega_{Gl_2}^{II.2.1} = |\delta^{-1}| \sum_{|r| \leq 1} \left[\nu\left(r^2\varpi\delta(a-\delta)^{-1}, \delta t^{-1}\varpi\right) - \nu\left(r^2\varpi^2\delta(a-\delta)^{-1}, \delta t^{-1}\varpi\right) \right]$$

By Lemma 10 this is

$$|\delta^{-1}|\sum 1$$

over

$$|r|\le 1\,,\ 1\le |r^2\varpi\delta(a-\delta)^{-1}|\le|\delta t^{-1}\varpi|\,.$$

This is 0 unless $|a-\delta|\le|\varpi\delta|$ and $|t\varpi^{-1}|\le|\delta|$. It can then be written as $|\delta^{-1}|$ times

$$\nu\left(1,\left\{\begin{array}{c}\sqrt[v]{(a-\delta)t^{-1}}\\ \sqrt[v]{(a-\delta)t^{-1}\varpi}\end{array}\right\}:\left\{\begin{array}{c}\varpi^{-1}\sqrt[v]{(a-\delta)\delta^{-1}}\\ \sqrt[v]{\varpi^{-1}(a-\delta)\delta^{-1}}\end{array}\right\}\right)$$

or

$$\nu\left(\left\{\begin{array}{c}\varpi\sqrt[v]{(a-\delta)^{-1}\delta}\\ \sqrt[v]{\varpi(a-\delta)^{-1}\delta}\end{array}\right\},\ \frac{\left\{\begin{array}{c}\sqrt[v]{(a-\delta)t^{-1}}\\ \sqrt[v]{(a-\delta)t^{-1}\varpi}\end{array}\right\}}{\left\{\begin{array}{c}\varpi^{-1}\sqrt[v]{(a-\delta)\delta^{-1}}\\ \sqrt[v]{\varpi^{-1}(a-\delta)\delta^{-1}}\end{array}\right\}}\right).$$

This can be further simplified

(101)
$$\Omega^{II.2.1}_{Gl_2}=|\delta^{-1}|\times$$

$$\begin{array}{ll}\nu\left(\varpi\sqrt[v]{\delta(a-\delta)^{-1}},\left\{\begin{array}{c}\varpi\sqrt[v]{\delta t^{-1}}\\ \varpi\sqrt[v]{\delta t^{-1}\varpi}\end{array}\right\}\right) & \text{if } \delta(a-\delta) \text{ is even}\\ \nu\left(\sqrt[v]{\varpi\delta(a-\delta)^{-1}},\left\{\begin{array}{c}\varpi\sqrt[v]{\delta t^{-1}}\\ \sqrt[v]{\delta t^{-1}\varpi}\end{array}\right\}\right) & \text{if } \delta(a-\delta) \text{ is odd}\end{array}.$$

To compute $\Omega^{II.2.2}_{Gl_2}$ we write $s=r^2$ or $s=r^2\varpi$ with $|r|>1$. We find:

$$\Omega^{II.2.2}_{Gl_2}=|\delta^{-1}|\sum_{|r|>1}\left[\nu\left(\varpi r\delta(a-\delta)^{-1},\delta r^{-1}t^{-1}\varpi\right)-\nu\left(\varpi r\delta(a-\delta)^{-1},\delta r^{-1}t^{-1}\right)\right].$$

By Lemma 10 this is

$$-|\delta^{-1}|\sum 1$$

over

$$|\varpi^{-1}|\le|r|\,,\ |\varpi^{-1}(a-\delta)t^{-1}|\le|r^2|\,,\ |r|\le|\delta t^{-1}|\,.$$

This is 0 unless

$$|a-\delta|\le|\delta^2t^{-1}\varpi|\,,\ |t\varpi^{-1}|\le|\delta|$$

and can be written then as:

$$-|\delta^{-1}|\nu\left(\delta t^{-1}:\varpi^{-1},\left\{\begin{array}{c}\varpi^{-1}\sqrt[v]{(a-\delta)t^{-1}}\\ \sqrt[v]{\varpi^{-1}(a-\delta)t^{-1}}\end{array}\right\}\right)$$

or

(102)
$$\Omega^{II.2.2}_{Gl_2}=-|\delta^{-1}|\nu\left(\varpi\delta t^{-1},\left\{\begin{array}{c}\varpi\delta t^{-1}\sqrt[v]{t(a-\delta)^{-1}}\\ \delta t^{-1}\sqrt[v]{\varpi t(a-\delta)^{-1}}\end{array}\right\}\right)$$

We can simplify our result:

PROPOSITION 11. *Suppose*

$$|a-\delta|\le|\varpi\delta|\,,\ |t\varpi^{-1}|\le|\delta|\,.$$

Then

$$\Omega^{II.2}_{Gl_2}(Y)=2^{-1}|\delta^{-1}|\left\{v(\delta t^{-1})+\begin{array}{|c|c|c|}\hline \delta t\text{ even} & \delta t\text{ odd} & \\ \hline 0 & -1 & \delta(a-\delta)\text{ even}\\ \hline 0 & 1 & \delta(a-\delta)\text{ odd}\\ \hline\end{array}\right\}$$

Suppose

$$|\delta| \leq |a-\delta| \leq |\varpi\delta^2 t^{-1}|\,,\ |t\varpi^{-1}| \leq |\delta|\,.$$

Then

$$\Omega^{II.2}_{Gl_2}(Y) =$$

$$2^{-1}|\delta^{-1}|\left\{ v\left(\delta t^{-1}\right) - v\left((a-\delta)\delta^{-1}\right) + \begin{array}{|c|c|c|}\hline \delta t\,\text{even} & \delta t\,\text{odd} & \\ \hline 0 & -1 & \delta(a-\delta)\,\text{even} \\ \hline -1 & 0 & \delta(a-\delta)\,\text{odd} \\ \hline \end{array} \right\}$$

In all other cases $\Omega^{II.2}_{Gl_2}(Y) = 0$.

PROOF: In any case both $\Omega^{II.2.1}_{Gl_2}(Y)$ and $\Omega^{II.2.2}_{Gl_2}(Y)$ vanish unless $|t\varpi^{-1}| \leq |\delta|$. So we assume this is the case. Suppose $|a-\delta| \leq |\varpi\delta|$. Then $\Omega^{II.2.1}_{Gl_2}(Y)$ is non-zero. Since $|\delta t^{-1}\varpi| \geq 1$ we have also $|a-\delta| < |\delta^2 t^{-1}\varpi|$ so $\Omega^{II.2.2}_{Gl_2}(Y)$ is non-zero as well. We have then to consider 4 cases depending on the parity of $(a-\delta)\delta$ and $t\delta$. Suppose for instance that both are even. Then $\Omega^{II.2}_{Gl_2}(Y)$ is $|\delta^{-1}|$ times

$$\nu\left(\varpi\sqrt[v]{\delta(a-\delta)^{-1}}, \varpi\sqrt[v]{\delta t^{-1}}\right) - \nu\left(\varpi\delta t^{-1}, \varpi\delta t^{-1}\sqrt[v]{t(a-\delta)^{-1}}\right)$$

If $|a-\delta| \leq |t|$ then this

$$\begin{aligned} & \nu\left(\varpi\sqrt[v]{\delta t^{-1}}\right) - \nu\left(\varpi\delta t^{-1}\right) \\ = & \left(1 - v\left(\varpi\sqrt[v]{\delta t^{-1}}\right)\right) - \left(1 - v\left(\varpi\delta t^{-1}\right)\right) \\ = & \frac{1}{2}v(\delta t^{-1})\,. \end{aligned}$$

If, on the contrary, $|t| < |a-\delta|$ then this is

$$\begin{aligned} & \nu\left(\varpi\sqrt[v]{\delta(a-\delta)^{-1}}\right) - \nu\left(\varpi\delta t^{-1}\sqrt[v]{t(a-\delta)^{-1}}\right) \\ = & \left(1 - v\left(\varpi\sqrt[v]{\delta(a-\delta)^{-1}}\right)\right) - \left(1 - v\left(\varpi\delta t^{-1}\sqrt[v]{t(a-\delta)^{-1}}\right)\right) \\ = & \frac{1}{2}v(\delta t^{-1})\,. \end{aligned}$$

The other cases are treated in a similar way and we have proved the first assertion of the Proposition.

Now assume $|\delta| \leq |a-\delta|$. Then $\Omega^{II.2.1}_{Gl_2} = 0$ and $\Omega^{II.2.2}_{Gl_2} \neq 0$ if and only if $|a-\delta| \leq |\delta^2 t^{-1}\varpi|$. Note that these conditions imply $|(a-\delta)\varpi| \geq |t|$. Assume $t(a-\delta)$ even. Then $\Omega^{II.2.2}_{Gl_2}$ is equal to $|\delta^{-1}|$ times

$$-\nu\left(\varpi\delta t^{-1}, \varpi\delta t^{-1}\sqrt[v]{t(a-\delta)^{-1}}\right)\,.$$

Since $|(a-\delta)\varpi| \geq |t|$, this is in fact

$$-\nu\left(\varpi\delta t^{-1}\sqrt[v]{t(a-\delta)^{-1}}\right) = v(\delta) - \frac{1}{2}v(t) - \frac{1}{2}v(a-\delta)\,.$$

Assume now $t(a-\delta)$ odd. Then $\Omega^{II.2.2}_{Gl_2}$ is equal to $|\delta^{-1}|$ times

$$-\nu\left(\varpi\delta t^{-1}, \delta t^{-1}\sqrt[v]{\varpi t(a-\delta)^{-1}}\right)\,.$$

Since $|(a-\delta)\varpi| \geq |t|$ this is

$$-\nu\left(\delta t^{-1}\sqrt[v]{\varpi t(a-\delta)^{-1}}\right) = v(\delta) - \frac{1}{2}v(t) - \frac{1}{2}v(a-\delta) - \frac{1}{2}\,.$$

Thus we have completely proved the Proposition. □

16.4. Case where $-b$ is odd. We are now ready to compute Ω_{Gl_2} completely.

PROPOSITION 12. *If a^2+b is a square but $-b$ is not a norm then $\Omega_{Gl_2}(Y)=0$.*

PROOF: Assume that $-b$ is not a norm, that is, has odd valuation. Recall $-b=(a+\delta)(a-\delta)$. Thus $a+\delta$ and $a-\delta$ have different parities. In particular they have different absolute values. Thus, choosing the sign $\pm$ suitably, we must have $|a+\delta|=|a|=|\delta|$ and $|a-\delta|\le|\varpi\delta|$. In particular $(a-\delta)\delta$ is odd and $(a+\delta)\delta$ even. At this point we recall that the terms $\Omega^{III.1}$ and $\Omega^{III.2}$ are obtained from $\Omega^{II.1}$ and $\Omega^{II.2}$ by changing δ into $-\delta$. If $|a|=|\delta|\ge|t|$ then

$$\Omega^{I.1}_{Gl_2}=\mu\left(\delta^{-1},\left\{\begin{array}{c}\sqrt[v]{\delta^{-1}t^{-1}}\\ \sqrt[v]{\delta^{-1}t^{-1}\varpi}\end{array}\right\}\right)=\mu(\delta^{-1})\,.$$

If $|a|=|\delta|<|t|$ then

$$\Omega^{I.1}_{Gl_2}=\mu(t^{-1},\delta^{-1}\varpi)=\mu(t^{-1})\,.$$

Thus, in any case,

$$\Omega^{I.1}_{Gl_2}=\mu(t^{-1},\delta^{-1})\,.$$

On the other hand,

$$\Omega^{II.1}_{Gl_2}=-\mu(t^{-1},\delta^{-1})\,,\;\Omega^{III.1}_{Gl_2}=0\,.$$

Thus

$$\Omega^{I.1}_{Gl_2}+\Omega^{II.1}_{Gl_2}+\Omega^{III.1}_{Gl_2}=0\,.$$

We study the remaining terms. We have

$$\Omega^{I.2}_{Gl_2}=|\delta^{-1}|\nu(\delta t^{-1},\delta t^{-1}\varpi)=|\delta^{-1}|\nu(\delta t^{-1}\varpi)=\;.$$

This is 0 unless $|\delta|\ge|\varpi^{-1}t|$. Similarly, the terms $\Omega^{II.2}_{Gl_2}$ and $\Omega^{III.2}_{Gl_2}$ vanish unless $|\delta|\ge|\varpi^{-1}t|$. Thus we may assume $|\delta|\ge|\varpi^{-1}t|$. Then

$$\Omega^{I.2}_{Gl_2}=-|\delta^{-1}|v(\delta t^{-1})\,.$$

Since $|a-\delta|\le|\varpi\delta|$ and $(a-\delta)\delta$ is odd, we have

$$\Omega^{II.2}_{Gl_2}=2^{-1}|\delta^{-1}|\left\{v(\delta t^{-1})+\begin{array}{|c|c|}\hline \delta t\,\text{even} & \delta t\,\text{odd}\\ \hline 0 & 1\\ \hline\end{array}\right\}\,.$$

On the other hand since $|a+\delta|=|\delta|$ and $|\delta|\le|\delta^2t^{-1}\varpi|$ we get

$$\Omega^{III.2}_{Gl_2}=2^{-1}|\delta^{-1}|\left\{v(\delta t^{-1})+\begin{array}{|c|c|}\hline \delta t\,\text{even} & \delta t\,\text{odd}\\ \hline 0 & -1\\ \hline\end{array}\right\}\,.$$

Thus we do get

$$\Omega^{I.2}_{Gl_2}+\Omega^{II.2}_{Gl_2}+\Omega^{III.2}_{Gl_2}=0\,.$$

This concludes the proof. □

16.5. Case where b is even. We compute $\Omega_{Gl_2}(Y)$ when $a^2+b=\delta^2$, $\delta\neq 0$ and b is even. Then $a+\delta$ and $a-\delta$ have the same parity. The result is as follows:

PROPOSITION 13. *Suppose $a^2+b=\delta^2$, $\delta\neq 0$ and b is even. Then*

$$\Omega_{Gl_2}(Y)=\mu\left(t^{-1},\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right)\text{ if }|t|\geq|\delta| \tag{103}$$

$$\Omega_{Gl_2}(Y)=\mu\left(\delta^{-1},\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right)-\epsilon|\delta^{-1}|\text{ if }|\delta|>|t| \tag{104}$$

where

$$\epsilon=\left\{\begin{array}{ll}1 & \text{if }|a|\leq|\varpi\delta^2t^{-1}|\,,\ (a\pm\delta)t\text{ odd}\\ 0 & \text{otherwise}\end{array}\right. \tag{105}$$

PROOF: First we claim that $\Omega^{II.1}_{Gl_2}$ and $\Omega^{III.1}_{Gl_2}$ are both zero. Indeed, if $\Omega^{II.1}_{Gl_2}\neq 0$ then $|a-\delta|\leq|\varpi\delta|$ and $(a-\delta)\delta$ is odd. Then $(a+\delta)\delta$ is also odd. However $|a+\delta|=|\delta|$ and so we get a contradiction and $\Omega^{II.1}_{Gl_2}=0$. Likewise $\Omega^{III.1}_{Gl_2}=0$. We compute the other terms.

We first consider the case $|\delta|<|t|$. Then the terms $\Omega^{I.2}_{Gl_2}$, $\Omega^{II.2}_{Gl_2}$, and $\Omega^{III.2}_{Gl_2}$ all vanish. Thus

$$\Omega_{Gl_2}(Y)=\Omega^{I.1}_{GL_2}\,.$$

We use the formula for $\Omega^{I.1}_{GL_2}$. If $|a|\geq|t|>|\delta|$ we find

$$\Omega_{Gl_2}(Y)=\mu\left(\delta^{-1},\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right)=\mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right).$$

If $|t|>|a|$ then

$$\Omega_{Gl_2}(Y)=\mu(t^{-1},\delta^{-1}\varpi)=\mu(t^{-1})$$

Now assume $|\delta|=|t|$. Then $\Omega^{II.2}_{Gl_2}=\Omega^{III.2}_{Gl_2}=0$. On the other hand,

$$\Omega^{I.2}_{Gl_2}=|\delta^{-1}|\nu(1,\delta a^{-1}\varpi)\,.$$

This is zero unless $|\delta|>|a|$ in which case this is $|\delta^{-1}|$. Thus, if $|a|\geq|\delta|=|t|$, we find

$$\Omega_{Gl_2}=\Omega^{I.1}_{Gl_2}=\mu\left(\delta^{-1},\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right)=\mu\left(t^{-1},\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right).$$

If $|a|<|\delta|=|t|$, then

$$\Omega_{Gl_2}=\Omega^{I.1}_{Gl_2}+\Omega^{I.2}_{Gl_2}=\mu(\delta^{-1}\varpi)+|\delta^{-1}|=\mu(\delta^{-1})$$

Thus if $|t|\geq|\delta|$ we find the first formula of the Proposition.

From now on, we assume $|\delta|>|t|$. Then we find

$$\Omega^{I.1}_{Gl_2}=\left\{\begin{array}{ll}\mu\left(\delta^{-1},\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right) & \text{if }|a|\geq|\delta|\\ \mu(\delta^{-1}\varpi) & \text{if }|a|<|\delta|\end{array}\right.$$

This can also be written

$$\Omega^{I.1}_{Gl_2}=\mu\left(\delta^{-1},\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right)+\left\{\begin{array}{ll}0 & \text{if }|a|\geq|\delta|\\ -|\delta^{-1}| & \text{if }|a|<|\delta|\end{array}\right.\,. \tag{106}$$

Similarly,

$$\Omega^{I.2}_{Gl_2} = \begin{cases} |\delta^{-1}|\nu(\delta^2 t^{-1} a^{-1}\varpi) & \text{if } |a| \geq |\delta| \\ |\delta^{-1}|\nu(\delta t^{-1}) & \text{if } |a| < |\delta| \end{cases}$$

Adding up these results we find:

$$\Omega^{I}_{Gl_2} = \mu\left(\delta^{-1}, \left\{ \begin{array}{c} \sqrt[v]{a^{-1}t^{-1}} \\ \sqrt[v]{a^{-1}t^{-1}\varpi} \end{array} \right\}\right) +$$

$$\begin{cases} 0 & \text{if } |a| \geq |\delta|\,,\ |a| \geq |\delta^2 t^{-1}| \\ -|\delta^{-1}|v(\delta^2 t^{-1} a^{-1}) & \text{if } |a| \geq |\delta|\,,\ |a| \leq |\delta^2 t^{-1}\varpi| \\ -|\delta^{-1}|v(\delta t^{-1}) & \text{if } |a| < |\delta| \end{cases} .$$

We compute the remaining terms.

Suppose $|a| \geq |\delta|$. Suppose first $|a+\delta| = |\delta - a| = |a|$ (or for short, $|\delta \pm a| = |a|$). Of course, this is always the case if $|a| > |\delta|$. Both $\Omega^{II.2}_{Gl_2}$ and $\Omega^{III.2}_{Gl_2}$ are 0 unless $|a| \leq |\varpi\delta^2 t^{-1}|$; then they are equal and

$$\Omega^{II.2}_{Gl_2} + \Omega^{III.2}_{Gl_2} = |\delta^{-1}| \left\{ v(\delta^2 t^{-1} a^{-1}) + \begin{array}{|c|c|} \hline (a\pm\delta)t\,\text{even} & (a\pm\delta)t\,\text{odd} \\ \hline 0 & -1 \\ \hline \end{array} \right\} .$$

Now suppose $|\delta| = |a|$ but $|\delta \pm a|$ is not equal to $|a| = |\delta|$ for both choices of $\pm$. Say $|\delta - a| \leq |\varpi\delta|$ and $|\delta + a| = |\delta|$. Both $\Omega^{II.2}_{Gl_2}$ and $\Omega^{III.2}_{Gl_2}$ are non-zero. In addition we remark that $\delta(\delta \pm a)$ have the same parity and are thus even. Thus we find again the same result. Note that here $|a| = |\delta| \leq |\varpi\delta^2 t^{-1}|$. We conclude that if $|a| \geq |\delta|$ then $\Omega^{II.2}_{Gl_2} + \Omega^{III.2}_{Gl_2} = 0$ unless $|a| \leq |\varpi\delta^2 t^{-1}|$. Then

$$\Omega^{II.2}_{Gl_2} + \Omega^{III.2}_{Gl_2} = |\delta^{-1}| \left\{ v(\delta^2 t^{-1} a^{-1}) + \begin{array}{|c|c|} \hline (a\pm\delta)t\,\text{even} & (a\pm\delta)t\,\text{odd} \\ \hline 0 & -1 \\ \hline \end{array} \right\} .$$

Finally, suppose $|a| < |\delta|$. Then $|a \pm \delta| = |\delta|$ so $(a \pm \delta)\delta$ is even and both $\Omega^{II.2}_{Gl_2}$ and $\Omega^{III.2}_{Gl_2}$ are non-zero with the same value. Then

$$\Omega^{II.2}_{Gl_2} + \Omega^{III.2}_{Gl_2} = |\delta^{-1}| \left\{ v(\delta t^{-1}) + \begin{array}{|c|c|} \hline (a\pm\delta)t\,\text{even} & (a\pm\delta)t\,\text{odd} \\ \hline 0 & -1 \\ \hline \end{array} \right\} .$$

Summing up, we find the second formula of the Proposition.

16.6. Verification of $\Omega_{Gl_2}(Y) = \Omega_{Sl_2}(X)$. We verify the identity of the fundamental lemma when $a^2 + b = \delta^2$, $\delta \neq 0$ and b is even. We solve the equations of matching (46) as before. We write

$$-\tau^2 b = y^2 - \tau a_1^2$$

and then we take

$$t_1 = -\frac{\tau t}{2}\,,\ c_1 = \frac{2}{t\tau}(y - \tau a)\,,\ b_1 = -\frac{t}{2}(y + \tau a)\,.$$

Then

$$a_1^2 + b_1 c_1 = \tau(a^2 + b) = \tau\delta^2\,.$$

Thus $a_1^2 + b_1 c_1$ is even but not a square. We need to compute $|c_1|$. We have

$$-\tau^2 b = y^2 - \tau a_1^2 = \tau^2 a^2 - \tau^2\delta^2\,.$$

Suppose $|a| \geq |\delta|$. If $|a| = |\delta|$ we choose δ in such a way that $|\delta - a| = |a|$. We have $|b| = |a^2 - \delta^2| \leq |a|^2$. From $-\tau^2 b = y^2 - \tau a_1^2$ we conclude that $|y| \leq |a|$ and $|a_1| \leq |a|$. From

$$y^2 - \tau^2 a^2 = \tau(a_1^2 - \tau\delta^2)$$

we conclude that

$$|(y-\tau a)(y+\tau a)| \leq |a|^2 .$$

Hence either $|y-\tau a| = |a|$ or $|y+\tau a| = |a|$. Thus we can choose y in such a way that $|y-\tau a| = |a|$. Then

$$|c_1| = |at^{-1}| = |(\delta - a)t^{-1}| .$$

Now suppose $|\delta| > |a|$. Then $|b| = |\delta|^2$. From $-\tau^2 b = y^2 - \tau a_1^2$ we conclude that $|y| \leq |\delta|$ and $|a_1| \leq |\delta|$. Suppose $|y| < |\delta|$. Then $|a_1| = |\delta|$. From $y^2 - \tau a_1^2 = \tau^2 a^2 - \tau^2 \delta^2$ we get

$$\tau = \left(1 - \frac{a^2}{\delta^2}\right) \frac{\tau^2\delta^2}{a_1^2} + \frac{y^2}{a_1^2} .$$

Thus τ is congruent to a square unit modulo $\varpi\mathcal{O}_F$ hence is a square, a contradiction. Thus $|y| = |\delta|$ and we find again

$$|c_1| = |\delta t^{-1}| = |(\delta - a)t^{-1}| .$$

Now we can write down the formula for $\Omega_{Sl_2}(X)$. It reads as follows.

If $|(\delta - a)t^{-1}| \leq 1$,

$$\Omega_{Sl_2}(X) =$$

$$\mu\left(t^{-1}, \delta^{-1}\left\{\begin{array}{c}\sqrt[v]{(\delta-a)t^{-1}}\\ \sqrt[v]{(\delta-a)t^{-1}\varpi}\end{array}\right\}, a^{-1}\left\{\begin{array}{c}\sqrt[v]{(\delta-a)t^{-1}}\\ \sqrt[v]{(\delta-a)t^{-1}\varpi}\end{array}\right\}\right) .$$

If $|(\delta - a)t^{-1}| > 1$,

$$\Omega_{Sl_2}(X) =$$

$$\mu\left(t^{-1}\left\{\begin{array}{c}\frac{1}{\sqrt[v]{(\delta-a)t^{-1}}}\\ \frac{1}{\sqrt[v]{(\delta-a)t^{-1}\varpi^{-1}}}\end{array}\right\}, \delta^{-1}\left\{\begin{array}{c}1\\ \varpi\end{array}\right\}, a^{-1}\left\{\begin{array}{c}\sqrt[v]{(\delta-a)t^{-1}}\\ \sqrt[v]{(\delta-a)t^{-1}\varpi}\end{array}\right\}\right)$$

Suppose first $|a| \geq |\delta|$. Recall that if $|a| = |\delta|$ then we choose δ in such a way that $|\delta - a| = |a|$. Thus $|\delta - a| = |a|$ in all cases. Then we find

$$\Omega_{Sl_2}(X) =$$

$$\begin{cases}\mu\left(t^{-1}, \delta^{-1}\left\{\begin{array}{c}\sqrt[v]{at^{-1}}\\ \sqrt[v]{at^{-1}\varpi}\end{array}\right\}, \left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right) & \text{if } |a| \leq |t| \\ \mu\left(\delta^{-1}\left\{\begin{array}{c}1\\ \varpi\end{array}\right\}, \left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right) & \text{if } |t| < |a|\end{cases}$$

Consider first the case $|a| \leq |t|$ so that $|\delta| \leq |a| \leq |t|$. This is

$$\Omega_{Sl_2}(X) = \mu\left(t^{-1}, \left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right) = \Omega_{Gl_2}(Y) .$$

Consider now the case $|t| < |a|$. If $|\delta| \leq |t|$ this is

$$\Omega_{Sl_2}(X) = \mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right) = \Omega_{Gl_2}(Y) .$$

If $|\delta| > |t|$ then we have to distinguish two cases. If $|a| > |\varpi\delta^2 t^{-1}|$ we find

$$\Omega_{Sl_2} = \mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right) = \mu\left(\delta^{-1}, \left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right)$$

which is again equal to Ω_{Gl_2} since $\epsilon = 0$ in this case. If $|a| \leq |\varpi\delta^2 t^{-1}|$ and at (or equivalently $(a-\delta)t$) is even we find

$$\Omega_{Sl_2}(X) = \mu(\delta^{-1}) .$$

Since $\epsilon = 0$ in this case, this is again Ω_{Gl_2}. If $|a| \leq |\varpi\delta^2 t^{-1}|$ and at (or equivalently $(a-\delta)t$) is odd we find

$$\Omega_{Sl_2}(X) = \mu(\delta^{-1}\varpi) = \mu(\delta^{-1}) - |\delta^{-1}| .$$

This is again equal to Ω_{Gl_2}, since $\epsilon = 1$ in this case.

We now discuss the case where $|a| < |\delta|$. Then $|a - \delta| = |\delta|$ and our expression for Ω_{Sl_2} simplifies:

$$\begin{cases} \mu\left(t^{-1}, \left\{ \begin{array}{c} \sqrt[v]{\delta^{-1}t^{-1}} \\ \sqrt[v]{\delta^{-1}t^{-1}\varpi} \end{array} \right\}\right) & \text{if } |\delta| \leq |t| \\ \mu\left(\left\{ \begin{array}{c} \sqrt[v]{\delta^{-1}t^{-1}} \\ \sqrt[v]{\delta^{-1}t^{-1}\varpi} \end{array} \right\}, \delta^{-1}\left\{ \begin{array}{c} 1 \\ \varpi \end{array} \right\}, a^{-1}\left\{ \begin{array}{c} \sqrt[v]{\delta t^{-1}} \\ \sqrt[v]{\delta t^{-1}\varpi} \end{array} \right\}\right) & \text{if } |t| < |\delta| \end{cases}$$

This simplifies further as follows:

$$\Omega_{Sl_2}(X) = \begin{cases} \mu\left(t^{-1}\right) & \text{if } |\delta| \leq |t| \\ \mu(\delta^{-1}) & \text{if } |t| < |\delta|\,,\ \delta t \text{ even} \\ \mu(\delta^{-1}\varpi) & \text{if } |t| < |\delta|\,,\ \delta t \text{ odd} \end{cases} .$$

Likewise, the expression for $\Omega_{Gl_2}(Y)$ simplifies as follows:

$$\Omega_{Gl_2}(Y) = \begin{cases} \mu\left(t^{-1}\right) & \text{if } |\delta| \leq |t| \\ \mu(\delta^{-1}) & \text{if } |t| < |\delta|\,,\ (a \pm \delta)t \text{ even} \\ \mu(\delta^{-1}) - |\delta^{-1}| & \text{if } |t| < |\delta|\,,\ (a \pm \delta t) \text{ odd} \end{cases} .$$

Again δt and $(\delta - a)t$ have the same parity and $\mu(\delta^{-1}\varpi) = \mu(\delta^{-1}) - |\delta^{-1}|$. Thus $\Omega_{Sl_2}(X) = \Omega_{GL_2}(Y)$ in all cases.

17. Proof of the fundamental Lemma: $a^2 + b = 0$

It remains to treat the case where $a^2 + b = 0$. THen $-b = a^2$ is a norm. We proceed as before. We write the integral for Ω_{Gl_2} as the sum of $\Omega^A_{GL_2}$ and $G^B_{Gl_2}$ corresponding respectively to the contributions of $|s| \leq 1$ and $|s| > 1$. We use Proposition 6. For $|s| \leq 1$ we write $s = r^2$ or $s = r^2\varpi$ with $|r| \leq 1$. We obtain

$$\begin{aligned} \Omega^A_{Gl_2} &= \sum_{|r| \leq 1} \left(\mu(t^{-1}, a^{-1}r) - \mu(t^{-1}, a^{-1}r\varpi)\right) \\ &= \mu(t^{-1}, a^{-1}) . \end{aligned}$$

For $|s| > 1$ we write $s = r^2$ or $s = r^2\varpi$ with $|r > |1$. We find

$$\Omega^B_{Gl_2} = \sum_{|r|>1} \left(\mu(t^{-1}r^{-1}, a^{-1}r) - \mu(t^{-1}r^{-1}, a^{-1}r\varpi)\right)$$

Applying Lemma 9 we find this is

$$\sum |a^{-1}r|$$

over

$$|\varpi^{-1}| \leq |r|\,,\ |a| \leq |r|\,,\ |r^2| \leq |at^{-1}| .$$

This is

$$\mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}: a^{-1}\varpi^{-1},1\right).$$

If $|a|\leq|t|$ then $\mu(t^{-1},a^{-1})=\mu(t^{-1})$ and $\mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}: a^{-1}\varpi^{-1},1\right)=0$. If $|a|\geq|t|$ then $\mu(t^{-1},a^{-1})=\mu(a^{-1})$. Moreover, if $|a|\leq 1$ then

$$\mu(a^{-1})+\mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}: a^{-1}\varpi^{-1},1\right)=\mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right).$$

If $|a|>1$ then $\mu(a^{-1})=0$ and

$$\mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}: a^{-1}\varpi^{-1},1\right)=\mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right)$$

Thus the above equality remains true. In summary,

$$\Omega_{Gl_2}(Y)=\left\{\begin{array}{ll}\mu(t^{-1}) & \text{if } |a|\leq|t|\\ \mu\left(\left\{\begin{array}{c}\sqrt[v]{a^{-1}t^{-1}}\\ \sqrt[v]{a^{-1}t^{-1}\varpi}\end{array}\right\}\right) & \text{if } |a|>|t|\end{array}\right.$$

On the other hand, the conditions of matching (46) can be solved with

$$a_1=0\,,\ b_1=0\,,\ c_1=\frac{-4a}{t}\,,\ t_1=-\frac{\tau t}{2}\,.$$

For the corresponding element X we find

$$\Omega_{Sl_2}(X)=\left\{\begin{array}{ll}\mu(t^{-1}) & \text{if } |a|\leq|t|\\ \mu\left(t^{-1}\left\{\begin{array}{c}\sqrt[v]{a^{-1}t}\\ \sqrt[v]{a^{-1}t\varpi}\end{array}\right\}\right) & \text{if } |a|>|t|\end{array}\right.$$

Clearly $\Omega_{Sl_2}(X)=\Omega_{Gl_2}(Y)$.

We have now completely proved the fundamental lemma for strongly regular elements.

18. Other regular elements

Recall the definition of a regular element. A matrix $X\in M(3\times 3,E)$ is **regular** if writing X in the form

$$\left(\begin{array}{cc}A & B\\ C & d\end{array}\right)$$

the column vectors B,AB are linearly independent and the row vectors C,CA are linearly independent. We have seen that if X is in $\mathfrak{g}(E)'$ then it is regular if and only if it is strongly regular. We consider now the elements X which are regular but not strongly regular. For such an element we have necessarily $A_2(X)=CB=0$.

Lemma 11. *Any element $X\in\mathfrak{g}(E)$ which is regular but not strongly regular is conjugate under $\iota Gl_2(E)$ to a unique matrix of the form*

$$\left(\begin{array}{ccc}0 & b & 0\\ c & 0 & 1\\ 1 & 0 & 0\end{array}\right)$$

with $b \neq 0$. *In addition*

$$\begin{array}{rcl} A_1(X) & = & -bc \\ B_1(X) & = & b \end{array}$$

PROOF: First B and C are not 0. After conjugation we may assume $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since $CB = 0$ we have

$$C = (t, 0)\,,\ t \neq 0\,.$$

Conjugating by a diagonal matrix in $Gl_2(E)$ we may assume $t = 1$. Thus we are reduced to the case of matrix of the form

$$\begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

If we conjugate by the matrix $\iota \begin{pmatrix} 1 & 0 \\ \frac{a}{b} & 1 \end{pmatrix}$ we arrive at a matrix of the prescribed form. The other assertions are obvious. □.

REMARK: Similarly, the element is conjugate to a unique matrix of the form

$$\begin{pmatrix} 0 & b & 0 \\ c & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Any element X of $\mathfrak{s}(F)$ which is regular but not strongly regular is conjugate under $Gl_2(F)$ to a unique element of the form

$$\xi = \begin{pmatrix} 0 & b & 0 \\ c & 0 & \sqrt{\tau} \\ \sqrt{\tau} & 0 & 0 \end{pmatrix}$$

with $b, c \in F\sqrt{\tau}$ and $b \neq 0$. Then

$$\begin{array}{rcl} A_1(X) & = & -bc \\ A_2(X) & = & b\tau \end{array}$$

Two such elements are conjugate under $Gl_2(F)$ if and only if they are conjugate under $Gl_2(E)$.

LEMMA 12. *Any element* X *of* $\mathfrak{u}(F)$ *which is regular but not strongly regular is conjugate under* $\iota U_{1,1}$ *to a unique element of the form*

$$\begin{pmatrix} 0 & b & 0 \\ c & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

with $b, c \in F\sqrt{\tau}$ *and* $b \neq 0$. *In addition*

$$\begin{array}{rcl} A_1(X) & = & -bc \\ B_1(X) & = & -b \end{array}$$

Two such elements are conjugate under $U_{1,1}$ *if and only if they are conjugate under* $Gl_2(E)$.

Proof: Write

$$X = \begin{pmatrix} a & b & z_1 \\ c & -a & z_2 \\ -\overline{z_2} & -\overline{z_1} & 0 \end{pmatrix} .$$

By assumption we have $\overline{z_2}z_1 + \overline{z_1}z_2 = 0$. Conjugating by a diagonal matrix in $U_{1,1}$ we may assume $z_2 = 1$. Then $z_1 + \overline{z_1} = 0$. Conjugating by the matrix $\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}$ we are reduced to the case where the matrix has the form

$$\begin{pmatrix} a & b & 0 \\ c & -a & 1 \\ -1 & 0 & 0 \end{pmatrix} .$$

We finish the proof as before. □

We see now that any element ξ' of $\mathfrak{s}(F)$ which is regular but not strongly regular matches an element ξ of $\mathfrak{u}(F)$. Explicitly

$$\xi = \begin{pmatrix} 0 & b & 0 \\ c & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

matches

$$\xi' = \begin{pmatrix} 0 & b' & 0 \\ c' & 0 & \sqrt{\tau} \\ \sqrt{\tau} & 0 & 0 \end{pmatrix}$$

if and only

$$bc = b'c' \, , \, -b = b'\tau \, .$$

As before we set

$$\begin{aligned} \Omega_U(\xi) &= \int_U f_0(\iota(u)\xi\iota(u)^{-1})du \\ \Omega_{Gl_2}(\xi') &= \int_{Gl_2(F)} \Phi_0(\iota(g)\xi'\iota(g)^{-1})\eta(\det g)dg \end{aligned}$$

The fundamental lemma asserts that if $\xi \to \xi'$ then

$$\Omega_U(\xi) = \tau(\xi')\Omega_{Gl_2}(\xi') \, .$$

To prove the lemma we proceed as before. We set

$$X = \Theta(\xi) \, , \, \xi' = \sqrt{\tau}Y \, .$$

Then

$$X = \begin{pmatrix} 0 & b_1 & 0 \\ c_1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

with

$$b_1 = b\sqrt{\tau} \, , \, c_1 = \frac{c}{\sqrt{\tau}} \, .$$

On the other hand

$$Y = \begin{pmatrix} 0 & b_2 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with

$$b_2 = \frac{b'}{\sqrt{\tau}} \, , \, c_2 = \frac{c'}{\sqrt{\tau}} \, .$$

Thus in terms of X and Y the matching conditions become

$$c_2 = -c_1\tau\,,\; b_2 = -\frac{b_1}{\tau^2}\,.$$

We have

$$|b_1| = |b_2|\,,\; |b_2| = |c_2|\,.$$

Moreover, if b_1c_1 (and thus b_2c_2) is even, then b_1c_1 is a square if and only if b_2c_2 is not a square.

THEOREM 2 (Remaining case of the fundamental Lemma). *If X and Y are as above and*

$$c_2 = -c_1\tau\,,\; b_2 = -\frac{b_1}{\tau^2}\,,$$

then

$$\Omega_{Sl_2}(X) = \eta(b_2)\Omega_{Gl_2}(Y)\,.$$

19. Orbital integrals for Sl_2

We compute the orbital integral under $SL_2(F)$ of

$$X = \begin{pmatrix} 0 & b & 0 \\ c & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}\,,$$

where $b \neq 0$, $c \neq 0$. We also write $\Omega_{Sl_2}(X) = \Omega_{Sl_2}(b, c)$.

We have

$$\Omega_{Sl_2}(X) = \int \Phi \begin{pmatrix} -bu & bm^2 & 0 \\ m^{-2}(c-u^2b) & ub & m^{-1} \\ -m^{-1} & 0 & 0 \end{pmatrix} du|m|^{-2}d^\times m\,.$$

If the integral is non zero then $|b| \leq 1$ and $|bc| \leq 1$. Explicitly the domain of integration is

$$1 \leq |m|\,,\; |bu| \leq 1\,,\; |bm^2| \leq 1\,,$$
$$|bc - u^2b^2| \leq |m^2b| \leq 1\,.$$

Under the assumption $|bc| \leq 1$ the condition $|ub| \leq 1$ is superfluous. After a change of variables, we can rewrite the integral as

$$|b|^{-1}\int du|m|^{-2}d^\times m$$

over

$$|bc - u^2| \leq |m^2b| \leq 1\,,\; 1 \leq |m|\,.$$

We divide the integral into the sum of the contribution $\Omega^1_{Sl_2}(X)$ of $|c| \leq |m^2|$ and the contribution $\Omega^2_{Sl_2}(X)$ of $|m^2| < |c|$.

We have

$$\Omega^1_{Sl_2}(X) = |b|^{-1}\int du|m|^{-2}d^\times m$$

over

$$|u^2| \leq |m^2b|\,,\; \sup(1, |c|) \leq |m^2| \leq |b|^{-1}\,.$$

This integral can be computed as follows

$$\Omega^1_{Sl_2}(X) =$$

$\lvert c\rvert \le 1$	b even		$\frac{\lvert b\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert \le 1$	b odd		$\frac{\lvert\varpi^{-1}b\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert > 1$		bc odd	$\frac{\lvert\varpi^{-1}bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert > 1$	b even	bc even	$\frac{\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert > 1$	b odd	bc even	$\frac{q^{-1}\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$

For $\Omega^2_{Sl_2}(X)$ we first compute the integral

$$\int_{|bc-u^2|\le|m^2b|} du\,.$$

It is 0 unless bc is a square then it is equal to $2|bc|^{-1/2}|bm^2|$. We have thus

$$\Omega^2_{Sl_2}(X) = |bc|^{-1/2}2\int_{1\le|m^2|<|c|} d^\times m\,.$$

This is 0 unless $|c|>1$. Then it is equal to

$$\Omega^2_{Sl_2}(X) = |bc|^{-1/2}\begin{cases} c \text{ even} & -v(c) \\ c \text{ odd} & 1-v(c)\end{cases}$$

Adding our two results we arrive at the following Proposition.

Proposition 14. *$\Omega_{Sl_2}(b,c)$ is given by the following formula.*

$\lvert c\rvert \le 1$	b even		$\frac{\lvert b\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert \le 1$	b odd		$\frac{\lvert\varpi^{-1}b\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert > 1$		bc odd	$\frac{\lvert\varpi^{-1}bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert > 1$	b even	bc even non square	$\frac{\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert > 1$	b odd	bc even non square	$\frac{q^{-1}\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert > 1$		bc square	$\frac{\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}} - v(c)\lvert bc\rvert^{-1/2}$

20. Orbital integrals for $Gl_2(F)$

We let

$$Y = \begin{pmatrix} 0 & b & 0 \\ c & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and we write $\Omega_{Gl_2}(Y) = \Omega_{Gl_2}(b,c)$. We have

$$\Omega_{Gl_2}(Y) = \int_{Gl_2(F)} \Phi(\iota(g)Y\iota(g)^{-1})\eta(\det g)dg$$

Explicitly this is

$$\int \Phi\begin{pmatrix} -b\alpha u & b\alpha m^2 & 0 \\ m^{-2}(c\alpha^{-1}-u^2b\alpha) & b\alpha u & m^{-1} \\ \alpha^{-1}m^{-1} & 0 & 0 \end{pmatrix}\eta(\alpha)d^\times\alpha du|m|^{-2}d^\times m\,.$$

or

$$\int \eta(\alpha)d^\times\alpha du|m|^{-2}d^\times m$$

over

$$|m^{-1}|\le 1\,,\ |\alpha^{-1}m^{-1}|\le 1$$

$$|b\alpha u| \le 1\,,\ |b\alpha m^2| \le 1$$
$$|cb - u^2 b^2\alpha^2| \le |m^2 b\alpha|\,.$$

As before, if the integral is non zero then $|b| \le 1$ and $|bc| \le 1$. Under these assumptions the condition $|b\alpha u| \le 1$ is superfluous. After a change of variables this becomes

$$|b|^{-1}\int\int \eta(\alpha)|\alpha|^{-1}d^\times\alpha du|m|^{-2}d^\times m$$

over

$$1 \le |m|, |\alpha|^{-1} \le |m|\,,$$
$$|cb - u^2| \le |m^2 b\alpha| \le 1\,.$$

After a new change of variables, we get

$$|b|^{-1}\int\int \eta(\alpha)|\alpha|^{-1}d^\times\alpha du d^\times m$$

over

$$1 \le |m| \le |\alpha| \le |b|^{-1}\,,$$
$$|bc - u^2| \le |\alpha b|\,.$$

Now, if $|\alpha| \ge 1$ then

$$\int_{1\le|m|\le|\alpha|} d^\times m = 1 - v(\alpha)\,.$$

Thus we get

$$|b|^{-1}\int \eta(\alpha)|\alpha|^{-1}(1 - v(\alpha))d^\times\alpha du$$

over

$$1 \le |\alpha| \le |b|^{-1}\,,\ |bc - u^2| \le |\alpha b|$$

or, after a new change of variables,

$$\eta(b)\int \eta(\alpha)|\alpha|^{-1}(1 - v(\alpha) + v(b))d^\times\alpha du$$

over

$$|b| \le |\alpha| \le 1\,,\ |bc - u^2| \le |\alpha|\,,$$

We divide the integral into the sum of the contribution $\Omega^1_{Gl}(Y)$ of $|bc| \le |\alpha|$ and the contribution $\Omega^2_{Gl}(Y)$ of $|bc| > |\alpha|$.

To compute $\Omega^1_{Gl}(Y)$ we may write $\alpha = \omega^{2s}$ or $\alpha = \omega^{2s+1}$ with $s \ge 0$ and sum over s. We set $A = b$ or $A = bc$ in such a way that

$$|A| = \sup(|b|, |bc|)\,.$$

We get

$$\Omega^1_{Gl}(\xi) =$$
$$\eta(b)\sum_{s\ge0,|A|\le|\varpi^{2s}|}(1 - 2s + v(b))q^s$$
$$-\eta(b)\sum_{s\ge0,|A|\le|\varpi^{2s+1}|}(v(b) - 2s)q^s\,.$$

If $|A| = |\varpi^{2r}|$ the first sum is for $0 \le s \le r$ and the second sum if for $0 \le s \le r - 1$. We find

$$\eta(b)\left(\sum_{0\le s\le r} q^s + (v(b) - 2r)q^r\right) =$$

$$\eta(b)\left(\frac{|A|^{-1/2}-q^{-1}}{1-q^{-1}}+(v(b)-2r)|A|^{-1/2}\right).$$

If $|c| \leq 1$, then $A=b$, b is even, and we are left with

$$\eta(b)\frac{|b^{-1}|^{1/2}-q^{-1}}{1-q^{-1}}.$$

If $|c|>1$ then $A=bc$, bc is even, and we are left with

$$\eta(b)\left(\frac{|bc|^{-1/2}-q^{-1}}{1-q^{-1}}-v(c)|bc|^{-1/2}\right).$$

If $|A|=|\varpi^{2r+1}|$ then both sums are for $0\leq s\leq r$. We are left with

$$\eta(b)(\sum_{0\leq s\leq r} q^s)=\eta(b)\frac{|\varpi|^{1/2}|A|^{-1/2}-q^{-1}}{1-q^{-1}}.$$

Now we compute $\Omega^2_{Gl}(Y)$. Now $|b|\leq|\alpha|<|bc|$. Thus in order to have a non-zero result we need $|c|>1$. The integral

$$\int_{|bc-u^2|\leq|\alpha|} du$$

is 0 unless bc is a square. Then it is equal to $2|\alpha||bc|^{-1/2}$. Thus we find

$$2\eta(b)|bc|^{-1/2}\int_{|b|\leq|\alpha|<|bc|}(1-v(\alpha)+v(b))\eta(\alpha)d^\times\alpha$$

or

$$2|bc|^{-1/2}\int_{1\leq|\alpha|<|c|}(1-v(\alpha))\eta(\alpha)d^\times\alpha$$

$$=2|bc|^{-1/2}\int_{1\leq|\alpha|<|c|}\eta(\alpha)d^\times\alpha+2|bc|^{-1/2}\int_{|c|^{-1}<|\alpha|\leq 1}v(\alpha)\eta(\alpha)d^\times\alpha.$$

Let us write $|c^{-1}|=|\varpi^r|$ and use the formula

$$\sum_{n=0}^{r-1} n(-1)^n=\frac{1}{4}(-1+(-1)^r-2(-1)^r r).$$

The first integral is 0 unless r is odd in which case it is 1. We find

$$\Omega_{Gl_2}(Y)=\begin{cases} c \text{ even} & |bc|^{-1/2}v(c)\\ c \text{ odd} & |bc|^{-1/2}(1-v(c))\end{cases}$$

Adding our two results we arrive at the following Proposition.

PROPOSITION 15. *$\Omega_{Gl_2}(b,c)$ is given by the following formula.*

$\lvert c\rvert\leq 1$	b even		$\eta(b)\frac{\lvert b\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert\leq 1$	b odd		$\eta(b)\frac{\lvert\varpi^{-1}b\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert>1$		bc odd	$\eta(b)\frac{\lvert\varpi^{-1}bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert>1$		bc even non square	$\eta(b)\left(\frac{\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}-v(c)\lvert bc\rvert^{-1/2}\right)$
$\lvert c\rvert>1$	b even	bc square	$\eta(b)\frac{\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$
$\lvert c\rvert>1$	b odd	bc square	$\eta(b)\frac{q^{-1}\lvert bc\rvert^{-1/2}-q^{-1}}{1-q^{-1}}$

21. Verifcation of $\Omega_{Sl_2}(X) = \eta(b_2)\Omega_{Gl_2}(Y)$

Under our condition of matching we have

$$|b_1| = |b_2|\,,\ |c_1| = |c_2|\,.$$

In addition if b_1c_1 and b_2c_2 are even then b_1c_1 is a square if and only b_2c_2 is not a square. By direct inspection we find

$$\Omega_{Sl_2}(b_1, c_1) = \eta(b_2)\Omega_{Gl_2}(b_2, c_2)\,.$$

This concludes the proof of the fundamental Lemma.

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