A correction to *Conducteur des Représentations du groupe linéaire*

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Nadir Matringe has indicated to me that the paper *Conducteur des Représentations du groupe linéaire* ([JPSS81a], citeError2) contains an error. I correct the error in this note. The correct proof is actually simpler than the erroneous proof. Separately, Matringe has given a different, interesting proof of the result in question ([Mat11]).

We recall the result in question. Let F be a local field. We denote by α the absolute value, by q the cardinality of the residual field and finally by v the valuation of F. Thus $\alpha(x) = |x| = q^{-v(x)}$. Let ψ be an additive character of F whose conductor is the ring of integers $\mathcal{O}_F$. Let G_r be the group $GL(r)$ regarded as an algebraic group. We denote by dg the Haar measure of $G_r(F)$ for which the compact group $G_r(\mathcal{O}_F)$ has volume 1. Let N_r be the subgroup of upper triangular matrices with unit diagonal. We define a character

$$\theta_{r,\psi} : N_r(F) \to \mathbf{C}^\times$$

by the formula

$$\theta_{r,\psi}(u) = \psi\left(\sum_{1\leq i\leq r-1} u_{i,i+1}\right) .$$

We denote by du the Haar measure on $N_r(F)$ for which $N_r(\mathcal{O}_F)$ has measure 1. We have then a quotient invariant measure on $N_r(F)\backslash G_r(F)$.

Let S_r be the algebra of symmetric polynomials in

$$(X_1, X_1^{-1}, X_2, X_2^{-1}, \dots X_r, X_r^{-1}) .$$

Let H_r be the Hecke algebra. Let $\mathcal{S}_r : H_r \to S_r$ be the Satake isomorphism. Thus for any $r-$tuple of non-zero complex numbers $(x_1, x_2, \dots, x_r)$ we have an homomorphism $H_r \to \mathbf{C}$, defined by

$$\phi \mapsto \mathcal{S}(\phi)(x_1, x_2, \dots, x_r) .$$

There is a unique function $W : G_r(F) \to \mathbb{C}$ satisfying the following properties:

- $W(gk) = W(g)$ for $k \in G_r(\mathcal{O}_F)$,
- $W(ug) = \theta_\psi(u) W(g)$ for $u\ in N_r(F)$,
- for all $(x_1, x_2, \ldots, x_r)$ and all $\phi \in H_r$,

$$\int_{G_r(F)} W(gh)\phi(h)dh = \mathcal{S}(\phi)(x_1, x_2, \ldots, x_r)\, W(g)\,,$$

- $W(e) = 1$.

We will denote this function by $W(x_1, x_2, \ldots x_r; \psi)$ and its value at g by $W(g; x_1, x_2, \ldots x_r; \psi)$.

Let (π, V) be an irreducible admissible representation of $G_r(F)$. We assume that π is *generic*, that is, there is a non-zero linear form $\lambda : V \to \mathbf{C}$ such that

$$\lambda(\pi(u)v) = \theta_{r,\psi}(u)\, \lambda(v)$$

for all $u \in N_r(F)$ and all $v \in V$. Recall that such a form is unique, within a scalar factor. We denote by $\mathcal{W}(\pi; \psi)$ the space of functions of the form

$$g \mapsto \lambda(\pi(g)v)\,,$$

with $v \in V$. It is the *Whittaker model* of π. On the other hand, we have the $L-$factor $L(s, \pi)$ ([GJ72]). We denote by $P_\pi(X)$ the polynomial defined by $L(s, \pi) = P_\pi(q^{-s})^{-1}$. The main result of [JPSS81a] is the following Theorem.

Theorem 1 *There is an element $W \in \mathcal{W}(\pi; \psi)$ such that, for all $r-1$-tuple of non zero complex numbers $(x_1, x_2, \ldots, x_{r-1})$,*

$$\int_{N_{r-1}(F)\backslash G_{r-1}(F)} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; x_1, x_2, \ldots x_{r-1}; \overline{\psi}) |\det g|^{s-1/2} dg$$

$$= \prod_{1 \le i \le r-1} P_\pi(q^{-s} x_i)^{-1}\,.$$

In [JPS] it is shown that if we impose the extra condition

$$W\begin{pmatrix} gh & 0 \\ 0 & 1 \end{pmatrix} = W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

for all $h \in G_{r-1}(\mathcal{O}_F)$ and $g \in G_{r-1}(F)$ then W is unique. The vector W is called the *essential vector* of π and further properties of this vector are obtained in [JPSS81a].

The proof of this theorem is incorrect in [JPS]. We give a simple proof here.

1 Review of some properties of the $L-$factor

Let $r \geq 2$ be an integer. Let $(t_1, t_2, \ldots, t_{r-1})$ be a $r-1$-tuple of complex numbers. We assume that

$$\mathrm{Re}(t_1) \geq \mathrm{Re} t_2 \geq \cdots \geq \mathrm{Re}(t_{r-1}) .$$

We denote by $\pi(t_1, t_2, \ldots t_{r-1})$ the corresponding principal series representation. It is the representation induced by the characters $\alpha^{t_1}, \alpha^{t_2}, \ldots \alpha^{t_{r-1}}$. Its space $I(t_1, t_2, \ldots t_{r-1})$ is the space of smooth functions $\phi : G_{r-1}(F) \to \mathbb{C}$ such that

$$\phi\left[\begin{pmatrix} a_1 & * & \ldots & \ldots & * \\ 0 & a_2 & \ldots & \ldots & * \\ 0 & 0 & \ldots & \ldots & a_r \end{pmatrix} g\right] =$$

$$|a_1|^{t_1+\frac{r-2}{2}} |a_2|^{t_2+\frac{r-2}{2}-1} \cdots |a_{r-1}|^{t_{r-1}-\frac{r-2}{2}} \phi(g) .$$

The space $I(t_1, t_2, \ldots t_{r-1})$ contains a unique vector ϕ_0 equal to 1 on $G_{r-1}(\mathcal{O}_F)$ and thus invariant under $G_{r-1}(\mathcal{O}_F)$. We recall a standard result.

Lemma 1 *The vector ϕ_0 is a cyclic vector for the representation $\pi(t_1, t_2, \ldots t_{r-1})$.*

PROOF: Indeed, if $\mathrm{Re}(t_1) = \mathrm{Re} t_2 = \cdots = \mathrm{Re}(t_{r-1})$, the representation is irreducible and our assertion is trivial. If not, we use Langlands' construction ([Sil78]). There is a certain intertwining operator N defined on the space of the representation and the kernel of N is a maximal invariant subspace. By direct computation $N\phi_0 \neq 0$ and our assertion follows. □

The representation $I(t_1, t_2, \ldots t_{r-1})$ admits a non-zero linear form λ such that, for $u \in N_{r-1}(F)$,

$$\lambda(\pi(u)\phi) = \theta_{r-1,\overline{\psi}}(u)\lambda(g) .$$

We denote by $\mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$ the space spanned by the functions of the form

$$g \mapsto W_\phi(g) \, , \, W_\phi(g) = \lambda(\pi(t_1, t_2, \ldots t_{r-1})(g)\phi) \, ,$$

with $\phi \in I(t_1, t_2, \ldots t_{r-1})$ We recall the following result ([JS83])

Lemma 2 *The map $\phi \mapsto W_\phi$ is injective.*

It follows that the image W_0 of ϕ_0 is a cyclic vector in $\mathcal{W}(t_1, t_2, \dots, t_{r-1}; \overline{\psi})$. Up to a multiplicative constant, W_0 is equal to the function $W(x_1, x_2, \dots, x_{r-1}; \overline{\psi})$.

Now let π be a generic representation of $G_r(F)$. For $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(t_1, t_2, \dots, t_{r-1}; \overline{\psi})$ we consider the integral

$$\Psi(s, W, W') = \int_{N_{r-1}\backslash G_{r-1}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g) |\det g|^{s-1/2} dg$$

The integral converges absolutely if $\mathrm{Re}s >> 0$ and extends to a meromorphic function of s. In any case, it has a meaning as a formal Laurent series in the variable q^{-s} (see below). We recall a result from [JPSS83]

Lemma 3 *There are functions $W_j \in \mathcal{W}(\pi; \psi)$ and $W'_j \in \mathcal{W}(t_1, t_2, \dots, t_{r-1}; \overline{\psi})$, $1 \le j \le k$, such that*

$$\sum_{1 \le j \le k} \Psi(s, W_j, W'_j) = \prod_{1 \le i \le r-1} L(s + t_i, \pi)\,.$$

Since W_0 is a cyclic vector we see, after a change of notations, that there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers n_j, $1 \le j \le k$, such that

$$\sum_i q^{-n_i s} \Psi(s, W_i, W(x_1, x_2, \dots x_{r-1}; \overline{\psi})) = \prod_{1 \le i \le r-1} L(s + t_i, \pi)\,.$$

In our discussion $|x_1| \le |x_2| \le \cdots \le |x_{r-1}|$. However, the functions $W(x_1, x_2 \dots, x_{r-1}; \overline{\psi})$ are symmetric in the variables x_i. Thus we have the following result.

Lemma 4 *Given a $r-1$-tuple of non-zero complex numbers $(x_1, x_2, \dots x_{r-1})$ there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers n_j, $1 \le j \le k$, such that*

$$\sum_j q^{-n_j s} \Psi(s, W_j, W(x_1, x_2, \dots x_{r-1}; \overline{\psi})) = \prod_{1 \le i \le r-1} P_\pi(q^{-s} x_i)^{-1}\,.$$

2 The ideal I_π

First, we can define a function $W(X_1, X_2, \dots X_{r-1}; \overline{\psi})$ with values in S_{r-1} such that,n for every g and every $r-1$-tuple $(x_1, x_2, \dots, x_{r-1})$, the scalar $W(g; x_1, x_2, \dots x_{r-1})$ is the value of the polynomial $W(g; X_1, X_2, \dots X_{r-1}; \overline{\psi})$ at the point $(x_1, x_2, \dots, x_{r-1})$. For g in a set compact modulo $N_{r-1}(F)$ the polynomials $W(g; X_1, X_2, \ldots X_{r-1}; \overline{\psi})$ remain in a finite dimensional vector subspace of S_{r-1}. We have the relation

$$|\det g|^s W(g; x_1, x_2, \ldots, x_{r-1}; \overline{\psi}) = W(g; q^{-s}x_1, q^{-s}x_2, \ldots, q^{-s}x_{r-1}; \overline{\psi}) \,.$$

It follows that if $|\det g| = q^{-n}$ then the polynomial $W(g; X_1, X_2, \ldots X_{r-1}; \overline{\psi})$ is homogeneous of degree n. For each integer n define the integral

$$\Psi_n(W; X_1, X_2, \ldots X_{r-1}; \psi) =$$

$$\int_{|\det g| = q^{-n}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g, X_1, X_2, \ldots X_{r-1}; \overline{\psi}) |\det g|^{-1/2} dg$$

The support of the integrand is contained in a set compact modulo $N_{r-1}(F)$, which depends on W. In addition, there is an integer N_W such that the support of the integrand is empty if $n < N(W)$. The polynomial

$$\Psi_n(W; X_1, X_2, \ldots X_{r-1}; \psi$$

is homogeneous of degree n, that is,

$$X^n \Psi_n(W; X_1, X_2, \ldots X_{r-1}; \psi) = \Psi_n(W; XX_1, XX_2, \ldots XX_{r-1}; \psi) \,.$$

We consider the formal Laurent series

$$\Psi(X; W; X_1, X_2, \ldots X_{r-1}; \psi) = \sum_n X^n \Psi_n(W; X_1, X_2, \ldots X_{r-1}, \psi) \,,$$

or, more precisely,

$$\Psi(X; W; X_1, X_2, \ldots X_{r-1}; \psi) = \sum_{n \geq N_W} X^n \Psi_n(W; X_1, X_2, \ldots X_{r-1}; \psi)$$

If we multiply, this Laurent series by $\prod_{1 \leq i \leq r-1} P_\pi(XX_i)$ we obtain a new Laurent series

$$\Psi(X; W, X_1, X_2, \ldots X_{r-1}; \psi) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) =$$

$$\sum_{n \geq N_1(W)} X^n a_n(X_1, X_2, \ldots X_{r-1}; \psi) \,.$$

where $N_1(W)$ is another integer depending on W. We can replace π by its contragredient representation $\widetilde{\pi}$, W by $\widetilde{W}$, ψ by $\overline{\psi}$. Recall that $\widetilde{W}$ is defined by

$$\widetilde{W}(g) = W(w_r \, {}^t g^{-1})$$

where w_r is the permutation matrix whose non-zeero entries are on the anti-diagonal. The function $\widetilde{W}$ belongs to $\mathcal{W}(\widetilde{\pi}, \overline{\psi})$. We define similarly

$$\Psi(\widetilde{W}; X_1, X_2, \ldots, X_{r-1}; \psi) .$$

We have then the following functional equation

$$\Psi(q^{-1}X^{-1}; \widetilde{W}; X_1^{-1}, X_2^{-1}, \ldots X_{r-1}^{-1}; \psi) \prod_{i=1}^{r-1} P_{\widetilde{\pi}}(q^{-1}X^{-1}X_i^{-1})$$

$$= \prod_{i=1}^{r-1} \epsilon_\pi(XX_i, \psi)\, \Psi(X, W, X_1, X_2, \ldots X_{r-1}; \overline{\psi}) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) .$$

The ϵ factors are monomials. Thus there is another integer $N_2(W)$ such that in fact

$$\Psi(X, W, X_1, X_2, \ldots, X_{r-1}; \psi) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) =$$

$$\sum_{N_2(W) \geq n \geq N_1(W)} X^n a_n(X_1, X_2, \ldots X_{r-1}) .$$

From now on we drop the dependence on ψ from the notation. Hence the product $\Psi(X, W, X_1, X_2, \ldots X_{r-1}) \prod_{1 \leq i \leq r-1} P_\pi(XX_i)$ is in fact a polynomial in X with coefficients in S_{r-1}. Moreover, because the a_n are homogeneous of degree n, there is a polynomial $\Xi(W; X_1, X_2, \ldots X_{r-1}) \in S_{r-1}$ such that

$$\Psi(X; W; X_1, X_2, \ldots X_{r-1}) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) = \Xi(W; XX_1, XX_2, \ldots XX_{r-1}) .$$

In a precise way, let us write

$$\prod_{1 \leq i \leq r-1} P_\pi(X_i) = \sum_{m=0}^{R} P_m(X_1, X_2, \ldots, X_{r-1})$$

where P_m is homogeneous of degree m. Then

$$\Psi(X; W; X_1, X_2, \ldots X_{r-1}) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) = .$$

$$\sum_n X^n \sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \ldots, X_{r-1}) P_m(X_1, X_2, \ldots X_{r-1}) .$$

The polynomial $\Xi(W; X_1, XX_2, \dots X_{r-1})$ is then determined by the condition that its homogeneous component of degree n, $\Xi_n(W; X_1, X_2, \dots, X_{r-1})$ be given by

$$\Xi_n(W; X_1, X_2, \dots, X_{r-1}) =$$

$$\sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \dots, X_{r-1}) P_m(X_1, X_2, \dots X_{r-1})\,.$$

Let I_π be the vector space spanned by the polynomials $\Xi(W; X_1, X_2, \dots X_{r-1})$.

Lemma 5 *In fact I_π is an ideal.*

PROOF: Let Q be an element of S_{r-1}. Let ϕ be the corresponding element of H_{r-1}. Then

$$\int W(gh; X_1, X_2, \dots, X_{r-1})\phi(h)dh = W(g; X_1, X_2, \dots, X_{r-1})Q(X_1, X_2, \dots X_{r-1})\,.$$

Let W be an element of $\mathcal{W}(\pi, \psi)$. Define another element W_1 of $\mathcal{W}(\pi, \psi)$ by

$$W_1(g) = \int_{G_{r-1}} W\left[g \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right] \phi(h)|\det h|^{1/2} dh\,.$$

We claim that

$$\Xi(W_1; X_1X_2, \dots X_{r-1}) = \Xi(W; X_1X_2, \dots, X_{r-1})Q(X_1, X_2, \dots, X_{r-1})\,.$$

This will imply the Lemma.

By linearity, it suffices to prove our claim when Q is homogeneous of degree t. Then ϕ is supported on the set of h such that $|\det h| = q^{-t}$. We have then, for every n,

$$\Psi_n(W_1; X_1, X_2, \dots, X_{r-1}) =$$

$$\int_{|\det g| = q^{-n}} \int W \begin{pmatrix} gh^{-1} & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, X_2, \dots, X_{r-1})\phi(h)|\det h|^{1/2} dh |\det g|^{-1/2} dg$$

$$= \int_{|\det g| = q^{-n+t}} \int W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(gh; X_1, X_2, \dots, X_{r-1})\phi(h) dh |\det g|^{-1/2} dg$$

$$= \int_{|\det g| = q^{-n+t}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, X_2, \dots, X_{r-1}) |\det g|^{-1/2} dg Q(X_1, X_2, \dots, X_{r-1})$$

$$= \Psi_{n-t}(W; X_1, X_2, \dots, X_{r-1})\ Q(X_1, X_2, \dots, X_{r-1})\,.$$

Hence

$$\Xi_n(W_1; X_1, X_2, \ldots X_{r-1}) =$$

$$\sum_{m=0}^{R} \Psi_{n-m}(W_1; X_1, X_2, \ldots, X_{r-1}) P_m(X_1, X_2, \ldots, X_{r-1})$$

$$= \sum_{m=0}^{R} \Psi_{n-m-t}(W; X_1, X_2, \ldots, X_{r-1}) P_m(X_1, X_2, \ldots, X_{r-1}) Q(X_1, X_2, \ldots, X_{r-1})$$

$$= \Xi_{n-t}(W; X_1, X_2, \ldots X_{r-1})\ Q(X_1, X_2, \ldots, X_{r-1})\,.$$

Since Q is homogeneous of degree t our assertion follows. □

3 Conclusion

Given a $r-1$-tuple of non-zero complex numbers $(x_1, x_2, \ldots, x_{r-1})$, Lemma 4 shows that we can find W_j and integers n_j such that

$$\sum_{1 \le j \le k} (q^{-s})^{n_j} \Xi(W_j, q^{-s}x_1, q^{-s}x_2, \ldots, q^{-s}x_{r-1}) = 1\,.$$

In particular,

$$\sum_{1 \le j \le k} \Xi(W_j, x_1, x_2, \ldots, x_{r-1}) = 1\,.$$

Thus the element

$$Q(X_1, X_2, \ldots X_{r-1}) = \sum_{1 \le j \le k} \Xi(W_j; X_1, X_2, \ldots X_{r-1})$$

of I_π does not vanish at $(x_1, x_2, \ldots, x_{r-1})$. By the Theorem of zeros of Hilbert $I_\pi = S_{r-1}$. In particular, there is W such that

$$\Xi(W; X_1, X_2, \ldots X_{r-1}) = 1\,.$$

This implies the Theorem.

Remark 1: The proof in [JPSS81a] is correct if $L(s, \pi)$ is identically 1. In general, the proof there only shows that the elements of I_π cannot all vanish on a coordinate hyperplane $X_i = x$.

Remark 2 : Consider an induced representation π of the form

$$\pi = I(\sigma_1 \otimes \alpha^{s_1}, \sigma_2 \otimes \alpha^{s_2}, \ldots, \sigma_k \otimes \alpha^{s_k})$$

where the representations $\sigma_1, \sigma_2, \ldots \sigma_k$ are tempered and $s_1, s_2, \ldots s+k$ are real numbers such that

$$s_1 > s_2 > \cdots > s_k\,.$$

The representation π may fail to be irreducible. But, in any case, it has a Whittaker model ([JS83]) and Theorem 1 is valid for the representation π.

Remark 3: The proof of Matringe uses the theory of derivatives of a representation. The present proof appears simple only because we use Lemma 3, the proof of which is quite elaborate (and can be obtained from the theory of derivatives).

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