Platonic solids and their symmetries

GU4041

Columbia University

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Regular polyhedra

A regular polyhedron is a convex object in 3-dimensional space made up of a collection of regular n-gons (the *faces*) , all of the same size and all with the same n, that meet (when they do) at the same angle at the *edges*, and with the same number of faces meeting at each *vertex*.

There are regular n-gons for every $n \geq 3$, but it has been known since antiquity that there are only five distinct regular polyhedra.

These are the *tetrahedron* (four faces) ; the *hexahedron* (six faces), better known as the cube ; the *octahedron* (eight faces) ; the *dodecahedron* (twelve faces) ; and the *icosahedron* (twenty faces).

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A proof

There is a proof in Book XIII of Euclid's *Elements* that there are no other regular polyhedra.

Here is a modern proof:
Suppose we have r faces (each a regular n-gon) meeting at every vertex. The angle at the corner of the n-gon is $(n-2)\pi/n$ (basic Euclidean geometry!) Since the polyhedron is convex we must have

$$r \times (n-2)\pi/n < 2\pi \Rightarrow (r-2)(n-2) < 4.$$

The only positive integers satisfying this are $(n, r) = (3, 3)$ (the tetrahedron), $(4, 3)$ (the cube), $(3, 4)$ (the octahedron), $(5, 3)$ (the dodecahedron), $(3, 5)$ (the icosahedron).

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Plato's *Timaeus*

https://en.wikipedia.org/wiki/Platonic_solid

The name *Platonic solid* refers to their prominent mention in Plato's *Timaeus*, one of his most speculative dialogues, in which Plato posited that each of the four classical elements is made up of one of the regular polyhedra.

- Fire is composed of tetrahedra;
- Earth is composed of cubes;
- Air is made up of octahedra;
- Water is made up of icosahedra.

The dodecahedron, with the most complex faces, did not correspond to an element.

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Kepler

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Symmetries of the cube

We have seen that the rotations of the cube form the group S_4. If we include reflections, we obtain the group $S_4 \times \mathbb{Z}_2$.
The cube has six faces, twelve edges, and eight vertices. Place a new vertex at the center of each face, and connect each one to the four adjacent centers by a new edge. There are twelve new edges in all (15 pairs of vertices, not counting the pairs on opposite faces).
The result is an octahedron.

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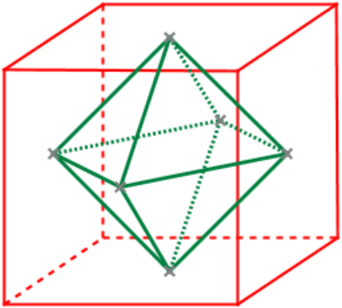
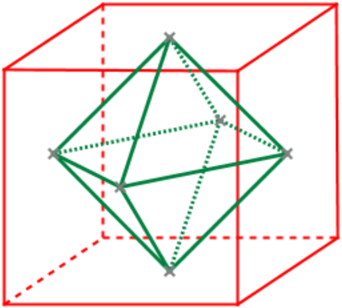
Octahedron



Any symmetry of the cube is also a symmetry of the octahedron, and vice versa.

So S_4 is the group of rotations of the octahedron.

Octahedron



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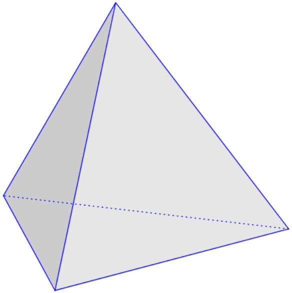
The tetrahedron



Any permutation of the four vertices of the tetrahedron is a symmetry. So S_4 is the symmetry group of the tetrahedron; A_4 its group of rotations.

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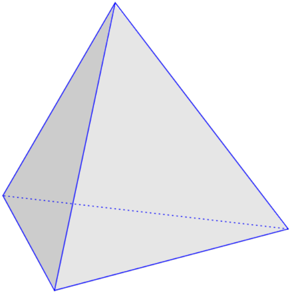

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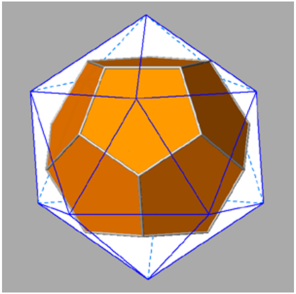
Icosahedral viruses

https://www.wired.com/story/glass-microbiology/

Icosahedron and dual dodecahedron

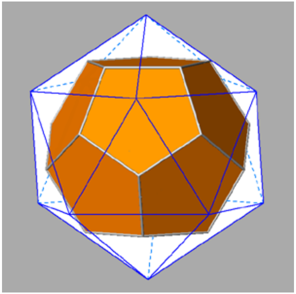


To determine the group G of rotations of the icosahedron, note that G acts transitively on the 20 faces, and there are 3 rotations that fix any face.

Similarly, G acts transitively on the 12 faces of the (dual) dodecahedron, and there are 5 rotations that fix any pentagon.

So the order of G is 60.

Icosahedron and dual dodecahedron



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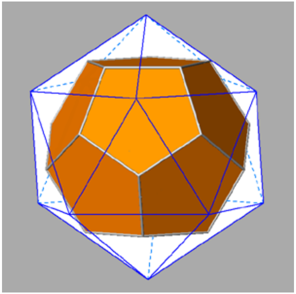

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Dodecahedral symmetry

`http://www-groups.mcs.st-andrews.ac.uk/~john/geometry/Lectures/L10.html`