HW 10 Solutions

April 21, 2020

1. 14.4.2. Compute all X_g and G_x for the following permutation groups:
(a) $X = \{1, 2, 3\}, G = S_3$.
(b) $X = \{1, 2, 3, 4, 5, 6\}, G = \{(1), (12), (345), (354), (12)(345), (12)(354)\}$.
14.4.3. Compute the G-equivalence classes of X for each of the G-sets in Exercise 14.4.2. For each $x \in X$ verify that $|G| = |O_x| \cdot |G_x|$.

Solution. 14.4.2. (a) $X_1 = X, X_{(12)} = \{3\}, X_{(13)} = \{2\}, X_{(23)} = 1, X_{(123)} = \emptyset, X_{(132)} = \emptyset$. $G_1 = \langle (23) \rangle, G_2 = \langle (13) \rangle, G_3 = \langle (12) \rangle$.

(b) $X_1 = X, X_{(12)} = \{3, \ldots, 6\}, X_{(345)} = \{1, 2, 6\}, X_{(354)} = \{1, 2, 6\}, X_{(12)(344)} = \{6\}, X_{(12)(354)} = \{6\}$. $G_1 = \langle (345) \rangle, G_2 = \langle (345) \rangle, G_3 = \langle (12) \rangle = G_4 = G_5, G_6 = G$.

14.4.3. (a) For each x, $O_x = X$ and G_x has two elements as we saw above. Thus in all cases the equation reads $6 = 3 \cdot 2$.

(b) The orbits (or G-equivalence classes) are $\{1, 2\}, \{3, 4, 5\}, \{6\}$. For $x = 1, 2, |O_x| = 2$ and $|G_x| = 3$. For $x = 3, 4, 5$ $|O_x| = 3$ and $|G_x| = 2$, and for $x = 6, |O_x| = 6$ and $|G_x| = 1$. In all cases the product is $6 = |G|$.

2. List the conjugacy classes of the groups $Q_8, \mathbf{Z}_{12}, D_{14}$. Determine the number of elements in each conjugacy class and verify the class equation for each group.

Solution. We recall that if g is in the center of a group G, then its conjugacy class is $\{g\}$. Therefore for $G = \mathbf{Z}_{12}$, the conjugacy classes are the one-element subsets: $\{[0]\}, \{[1]\}, \ldots, \{[11]\}$. In this case the class equation says

$$12 = \#\mathbf{Z}_{12} = \sum_{i=0}^{11} \#\{[i]\} = \sum_{i=0}^{11} 1,$$

which is true.

For Q_8, the center is $\{\pm 1\}$, so two of the conjugacy classes are $\{1\}, \{-1\}$. Next, take any element y not in the center. Then for every $x \in Q_8$, either x commutes with y, in which case $xyx^{-1} = y$, or $xy = -yx$, which is equivalent to $xyx^{-1} = -y$. Thus the non-trivial conjugacy classes are $\{\pm i\}, \{\pm j\}, \{\pm k\}$. In this case, the class equation reads

$$8 = \#Q_8 = \#\{1\} + \#\{-1\} + \#\{\pm i\} + \#\{\pm j\} + \#\{\pm k\} = 1 + 1 + 2 + 2 + 2.$$

Finally for $D_{14} = \langle s, r | r^7 = s^2 = 1, sr = r^6 s \rangle$, the conjugacy class of 1 is $\{1\}$ of course. If we take a rotation $r^i, i = 1, \ldots, 6$, then r^i commutes with all other

r^j, and

$$(r^j s)r^i(r^j s)^{-1} = r^j(sr^i s^{-1})r^{-j} = r^j r^{-i} r^{-j} = r^{-i}.$$

Thus the conjugacy class of r^i is $\{r^i, r^{-i}\}$. Let's compute the conjugacy class of s. We have

$$r^i s r^{-i} = r^{2i} s.$$

Since 7 is odd, all rotations r^k can be written as r^{2i} for some i. Thus all $r^k s$ are conjugate. Note also that $(r^i s)s(r^i s)^{-1} = r^{2i}s$. Thus the last conjugacy class consists of all the reflections $r^i s$. We have

$$\begin{aligned}14 = \#D_{14} &= \#\{1\} + \#\{r, r^6\} + \#\{r^2, r^5\}\#\{r^3, r^4\} + \#\{s, rs, \ldots, r^6 s\}\\ &= 1 + 2 + 2 + 2 + 7.\end{aligned}$$

3. Let A be a set with n elements, and let $P(A)$ denote the set of subsets of A.

(i) How many elements does $P(A)$ have?

(ii) Number the elements of A from 1 to n and let the symmetric group Σ_n act on A by permuting the elements. Thus if $n = 5$, the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

takes the first element to the second element, the second element to the third element, and so on. How many orbits does this action have?

(iii) Show that the action defined in (ii) defines an action of Σ_n on $P(A)$. How many orbits does this action have? How many elements are in each orbit? Justify your answer.

(iv) Write $P(A)$ as the union of the orbits described in (iii):

$$P(A) = \coprod_{i \in I} O_i.$$

Write $|P(A)| = \sum_{i \in I} |O_i|$. Use the binomial theorem to give another proof of this equality.

Solution. (i) It is well known that $|P(A)| = 2^n$, in fact, there is a nice bijection between $P(A)$ and the set of functions $A \to \{0, 1\}$. (ii) The action is transitive, i.e., there is only one orbit. In fact, $(1j)$ takes the first element to the j^{th} element, so 1 and j are in the same orbit for every j. (iii) For $\sigma \in \Sigma_n$ and a subset $B = \{a_1, \ldots, a_k\} \subset A$, we define

$$\sigma B = \{\sigma a_1, \ldots, \sigma a_k\}.$$

This is an action:

$$\begin{aligned}1 \cdot B &= \{1 \cdot a_1, \ldots, 1 \cdot a_k\} = \{a_1, \ldots, a_k\}\\ (\sigma\tau)B &= \{(\sigma\tau)a_1, \ldots, (\sigma\tau)a_k\} = \{\sigma(\tau a_1), \ldots, \sigma(\tau a_k)\}\\ &= \sigma(\tau B).\end{aligned}$$

Here we have used only the fact that Σ_n acts on A. For each $k = 0, 1, \dots, n$, there is an orbit O_k consisting of the set of subsets with k elements, and these are all the orbits. To prove this, since the sets O_k clearly partition $P(A)$, it will be enough to show that the O_k are in fact orbits. First, if $B \subset A$ then σB has the same number of elements as B. In fact, $b \mapsto \sigma b$ is a bijection $B \to \sigma B$ whose inverse is $c \mapsto \sigma^{-1} c$. Next we have to show the action of Σ_n on O_k is transitive. For this, it will be convenient to actually identify A with $\{1, \dots, n\}$. Then if $\{a_1, \dots, a_k\}$ is a subset of $\{1, \dots, n\}$ with k elements with complement $\{b_1, \dots, b_{n-k}\}$, define $\sigma \in \Sigma_n$ by

$$\sigma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ a_1 & \cdots & a_k & b_1 & \cdots & b_{n-k} \end{pmatrix}.$$

Then $\{a_1, \dots, a_k\} = \sigma\{1, \dots, k\}$. Thus we've shown $O_k = \Sigma_n \cdot \{1, \dots, k\}$, as needed. (iv) We know from combinatorics that $|O_k| = \binom{n}{k}$, so

$$2^n = |P(A)| = \sum_{k=0}^{n} |O_k| = \sum_{k=0}^{n} \binom{n}{k}.$$

We can also prove this using the binomial theorem:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

4. Describe all of the finite groups with exactly two conjugacy classes, and prove your claim.

Solution. Suppose G is such a group. Then $\{e\}$ is one conjugacy class, and therefore $G \setminus \{e\}$ is the other. Then by the orbit stabilizer formula, $\#G \setminus \{e\} = \#G - 1$ divides $\#G$. But an integer $n > 0$ is divisible by $n - 1$ iff $n = 2$. Thus G has two elements so $G \cong \mathbf{Z}/2$.