Some group tables and group computations

The two groups of order 4 (up to isomorphism): (i) $\mathbb{Z}/4\mathbb{Z}$:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Aside from the trivial subgroup, $\mathbb{Z}/4\mathbb{Z}$ has one proper subgroup of order 2: $\langle 2 \rangle$.

(ii) The Klein 4-group V (isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$):

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

V has three subgroups of order 2: $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$.

The only group, up to isomorphism, of order 5, $\mathbb{Z}/5\mathbb{Z}$:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\mathbb{Z}/5\mathbb{Z}$ has no proper subgroups aside from the trivial subgroup.

The two groups of order 6: (i) $\mathbb{Z}/6\mathbb{Z}$:

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\mathbb{Z}/6\mathbb{Z}$ has one subgroup of order 2, namely $\langle 3 \rangle$, and one subgroup of order 3, namely $\langle 2 \rangle$.

(ii) The group table for $D_3 = S_3$: Assume that the vertices of an equilateral triangle are at the points $\mathbf{p}_1 = (1, 0) = (\cos 0, \sin 0)$, $\mathbf{p}_2 = (\cos 2\pi/3, \sin 2\pi/3)$, and $\mathbf{p}_3 = (\cos 4\pi/3, \sin 4\pi/3)$. Let $\rho = \rho_1$ be rotation about the angle $2\pi/3$, counterclockwise, and $\rho_2 = \rho^2 = \rho^{-1}$ be rotation about the angle $4\pi/3$, counterclockwise, or equivalently rotation by the angle $2\pi/3$, clockwise. Let $\tau = \tau_1$ be reflection about the point $\mathbf{p}_1$, i.e. τ_1 fixes $\mathbf{p}_1$ and interchanges $\mathbf{p}_2$ and $\mathbf{p}_3$, and similarly for τ_2, τ_3. Then one can check: $\rho_1\tau_1 = \tau_3$ and $\rho_2\tau_1 = \tau_2$. Clearly $\rho^3 = 1$ and $\tau^2 = \tau_i^2 = 1$ for all i. Hence every element of D_3 can be written as a product $\rho^a\tau^b$, where $a = 0, 1, 2$ and $b = 0, 1$, and in fact this representation is unique. Also, again by checking this directly, one can show that

$$\tau\rho\tau^{-1} = \tau\rho\tau = \rho^2,$$

which we can also write as

$$\tau\rho = \rho^2\tau.$$

This equation tells us how to multiply any two elements in D_3. For example,

$$\begin{aligned}\tau_1\tau_2 &= \tau\rho^2\tau = \tau\rho\rho\tau \\ &= \rho^2\tau\rho\tau = \rho^2\rho^2\tau\tau = \rho^4\tau^2 = \rho = \rho_1.\end{aligned}$$

$\cdot$	1	ρ_1	ρ_2	τ_1	τ_2	τ_3
1	1	ρ_1	ρ_2	τ_1	τ_2	τ_3
ρ_1	ρ_1	ρ_2	1	τ_3	τ_1	τ_2
ρ_2	ρ_2	1	ρ_1	τ_2	τ_3	τ_1
τ_1	τ_1	τ_2	τ_3	1	ρ_1	ρ_2
τ_2	τ_2	τ_3	τ_1	ρ_2	1	ρ_1
τ_3	τ_3	τ_1	τ_2	ρ_1	ρ_2	1

D_3 has one subgroup of order 3: $\langle \rho_1 \rangle = \langle \rho_2 \rangle$. It has three subgroups of order 2: $\langle \tau_1 \rangle$, $\langle \tau_2 \rangle$, and $\langle \tau_3 \rangle$.

The two nonabelian groups of order 8: (i) The dihedral group D_4: Here there are the four rotations 1, $\rho = \rho_1$, $\rho_2 = \rho^2$, $\rho_3 = \rho^3$, about the angles 0, $\pi/2 = 2\pi/4$, $\pi = 4\pi/4$, and $3\pi/2 = 6\pi/4$, and the reflections $\tau = \tau_1$ and τ_2 about the two diagonals of a square (τ_1 for the diagonal connecting vertices

1 and 3 and τ_2 connecting vertices 2 and 4) and μ_1, μ_2 for reflections about the perpendicular bisectors of a pair of sides (μ_1 for the reflection about the line bisecting the line segments $\overline{12}$ and $\overline{34}$, and μ_2 for the reflection about the line bisecting the line segments $\overline{14}$ and $\overline{23}$). One can check that $\rho\tau = \rho_1\tau_1 = \mu_1$, $\rho^2\tau = \rho_2\tau_1 = \mu_1$. The relations are $\rho^4 = 1$, $\tau^2 = 1$, and $\tau\rho\tau = \rho^{-1} = \rho^3$, or equivalently $\tau\rho = \rho^3\tau$.

$\cdot$	1	ρ_1	ρ_2	ρ_3	τ_1	τ_2	μ_1	μ_2
1	1	ρ_1	ρ_2	ρ_3	τ_1	τ_2	μ_1	μ_2
ρ_1	ρ_1	ρ_2	ρ_3	1	μ_1	μ_2	τ_2	τ_1
ρ_2	ρ_2	ρ_3	1	ρ_1	τ_2	τ_1	μ_2	μ_1
ρ_3	ρ_3	1	ρ_1	ρ_2	μ_2	μ_1	τ_1	τ_2
τ_1	τ_1	μ_2	τ_2	μ_1	1	ρ_2	ρ_3	ρ_1
τ_2	τ_2	μ_1	τ_1	μ_2	ρ_2	1	ρ_1	ρ_3
μ_1	μ_1	τ_1	μ_2	τ_2	ρ_1	ρ_3	1	ρ_2
μ_2	μ_2	τ_2	μ_1	τ_1	ρ_3	ρ_1	ρ_2	1

(ii) The quaternion group Q, given by the following table:

$\cdot$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Note that there are two elements of order 4 in D_4, ρ_1 and ρ_3, and five elements of order 2, ρ_2, τ_1, τ_2, μ_1, and μ_2. In Q, however, there are six elements of order 4, $\pm i$, $\pm j$, and $\pm k$, and one element of order 2, -1. In particular we see that D_4 and Q are not isomorphic. As for subgroups, Q had three subgroups of order 4 and they are all cyclic: $\langle i\rangle$, $\langle j\rangle$, and $\langle k\rangle$. (Note that for example $\langle i\rangle = \langle -i\rangle$.) There is one subgroup of order 2: $\langle -1\rangle$. As for D_4, there are five subgroups of order 2: $\langle\rho_2\rangle$, $\langle\tau_1\rangle$, $\langle\tau_2\rangle$, $\langle\mu_1\rangle$, and $\langle\mu_2\rangle$. There are three subgroups of order 4. One of them is cyclic, namely $\langle\rho_1\rangle = \langle\rho_3\rangle$. The other two are $\{1, \rho_2, \tau_2, \tau_2\}$ and $\{1, \rho_2, \mu_2, \mu_2\}$; both are isomorphic to the Klein 4-group V.