Algebra 1 Midterm 2 Practice Solutions

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1) True or false:

a) No group of order 88 has a subgroup of order 16.

True: $16 \nmid 88$ so this follows from Lagrange's theorem.

b) False: $\mathbf{Z}_8, \mathbf{Z}_4 \times \mathbf{Z}_2$, and $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ are non-isomorphic abelian groups of order 8, and the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is a non-abelian one, so there are at least 4.

2) a) Write the cycle decomposition of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 1 & 4 & 7 & 8 & 5 & 2 \end{pmatrix}$$

$\sigma = (13)(268)(57)$.

b) Let G be a finite group with 8 elements, and consider the homomorphism $\alpha : G \to S_8$ from the proof of Cayley's theorem. Show there is no $g \in G$ such that $\alpha(g) = \sigma$.

By Lagrange's theorem, we would have $g^8 = e$, hence $\sigma^8 = \alpha(g)^8 = e$. However, σ has order $\operatorname{lcm}(2,3,2) = 6$ and $6 \nmid 8$, so we have a contradiction.

3)

a) List the normal subgroups of the dihedral group D_{34}. For which integers m is there a surjective homomorphism $\alpha : D_{34} \to \mathbb{Z}_m$? Suppose that we don't require α to be surjective?

We establish that we use the notation $D_{34} =< r, s : r^{17}, s^2, rsrs >$. Recall first that each element of D_{34} is of the form $r^k s^t$, for $k \in [0,16] \cap \mathbb{Z}, t \in \{0,1\}$. Additionally, to show that a subgroup is normal, it suffices to show that it is invariant under conjugation by both r and s, since the group is generated by those elements. Then we write down the subgroups of D_{34}, ordering by size. D_{34} is trivially a normal subgroup of itself, of order 34. $< r >$ is certainly a subgroup of order 17. It's normal, since it's index 2. There are no other subgroups of order 17, since if $H \leq G, r^k \in H$ for some k, then $< r > \subset H$. If $H \neq < r >$, then $|H| > |< r >|$, so by Lagrange's theorem, $|H| = 34$, so $H = D_{34}$. Any subgroup containing two distinct elements of the form $r^k s, r^{k'} s$ contains $r^{k-k'}$ by closure, which means it's D_{34}. The remainder of the nontrivial subgroups are of the form $\{e, r^k s\}$ for some k. To see this, we recall that adding an element of the form $r^{k'}$ forces us to include D_{34}, as does adding an element of the form $r^{k'} s$. Each of these are order 2, but they aren't normal; to see that, consider $s\{e, r^k s\}s^{-1} = \{ss^{-1}, sr^k\} = \{e, r^{17-k} s\}$. Then $k \neq 17 - k$, so they aren't normal. Finally, the trivial subgroup is normal, so to summarize, the normal subgroups are $D_{34}, < r >$, and $\{e\}$.

Suppose for some m, there is a surjective homomorphism $\alpha : D_{34} \to \mathbb{Z}_m$. Then its kernel needs to be a normal subgroup of D_{34}, which means $\ker(\alpha) \in D_{34}, < r >, \{e\}$. It is clear that $\ker(\alpha) \neq e$, since that would imply that there was an isomorphism from D_{34} to some cyclic group, which is false. So by the first isomorphism theorem, $|D_{34}| = |\ker(\alpha)||\mathrm{im}(\alpha)| \Rightarrow 34 = |\ker(\alpha)|m$. Since $|\ker(\alpha)| \in \{17, 34\}$, we have $m \in \{1, 2\}$. These are each possible: the trivial homomorphism sends $|D_{34}|$ to $\mathbb{Z}_1$, and the homomorphism $\Phi : D_{34} \to \mathbb{Z}_2 : \Phi(r) = 0, \Phi(s) = 1$ is surjective and well-defined. It's clear that it's surjective; the only possible challenge to well-definition comes from the relations; but $\Phi(r^{17}) = 0^{17} = 0, \Phi(s^2) = 1 + 1 = 0, \Phi(rsrs) = 0 + 1 + 0 + 1 = 0$. Then $m \in \{1, 2\}$.

Theres the trivial homomorphism $\Phi : D_{34} \to \mathbb{Z}_m, \Phi(g) = [0]$ for every m.

b) Quote a theorem that asserts that there is an isomorphism $\beta : \mathbb{Z}_2 \times \mathbb{Z}_9 \to \mathbb{Z}_{18}$

The Chinese remainder theorem asserts that if $n = \prod_{k=1}^{n} p_k^{i_k}$, then $\mathbb{Z}_n \cong \prod_{k=1}^{n} \mathbb{Z}_{p_k^{i_k}}$, so in particular, $18 = 2 \times 9$, so $\mathbb{Z}_{18} \cong \mathbb{Z}_2 \times \mathbb{Z}_9$. The classification of finitely generated abelian groups is a more powerful result that also implies this.

c) Let D_{36} be the dihedral group of symmetries of the regular 18-gon. We view $\mathbb{Z}_{18}$ as the subgroup of rotations in D_{36} and let $f \in D_{36}$ denote the reflection in the vertical axis. Let $K = \beta(\mathbb{Z}_9) \in \mathbb{Z}_{18}$ with β as in (b). Show that the subgroup $H \subseteq D_{36}$ generated by f and K is normal, and determine the group ${D_{36}} / {H}$.

We first identify K and f in terms of the symbols r and s, our previous notation, where $D_{36} =< r, s : r^{18}, s^2, rsrs >$. s is just f, whereas $K =< r^2 >$, the subgroup of $< r > \cong \mathbb{Z}_{18}$ isomorphic to $\mathbb{Z}_9$. Then $< r^2, s >$ is the subgroup generated by f and K; it has order 18. To see this, we note that it has at least 10 elements; $(r^2)^k$ for $k \in [0, 8] \cap \mathbb{Z}$, and s; then by Lagrange's theorem, it has order at least 18. It also doesnt contain r, so it has order at most 18, so it has order 18, so it's normal, and the quotient ${D_{34}} / {< r^2, s >}$ has order 2, so it's isomorphic to $\mathbb{Z}_2$.

4) List all the non-isomorphic abelian groups of order 75, 76, 77, and 72

$75 = 3 \times 5^2$, so the abelian groups are $\mathbb{Z}_3 \times \mathbb{Z}_{25}$ and $\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
$76 = 2^2 \times 19$, so the abelian groups are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \nVdash \nrightarrow$ and $\mathbb{Z}_4 \times \mathbb{Z}_{19}$.
$77 = 7 \times 11$, so the only abelian group is $\mathbb{Z}_7 \times \mathbb{Z}_{11}$.
$72 = 2^2 \times 3^2$, so the abelian groups are $\mathbb{Z}_4 \times \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9, \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

5) Let N, N' be subgroups of a group G, such that $N \lhd N'$. Let H be any subgroup of G. Let $K = N' \cap H$.
a) Show that $(N \cap H) \lhd K$.

To show that $(N \cap H) \lhd K$, we need to show that $kgk^{-1} \in (N \cap H)$ for any $g \in (N \cap H)$ and any $k \in K$. We know that $g \in N$ and $g \in H$, and, since $K = (N' \cap H)$, that $k \in N'$ and $k \in H$.
Since $N \lhd N'$, we see that $kgk^{-1} \in N$ because $k \in N'$ and $g \in N$.
Meanwhile, since H is a subgroup, we see that $kgk^{-1} \in H$ because $k \in H$ and $g \in H$.
Hence $kgk^{-1} \in (N \cap H)$, as required.

b) Show that KN is a subgroup of G.

We need to show that $KN := \{kn : k \in K, n \in N\} \subseteq G$ is closed under multiplication, contains the identity, and contains inverses. Observe that any product nk, where $n \in N$ and $k \in K$, can be written as $k \cdot k^{-1}nk$, where $k^{-1}nk \in N$ because $N \lhd N'$ and $K \subseteq N'$.

- For any k_1n_1, $k_2n_2 \in KN$, we can write $(k_1n_1)(k_2n_2) = (k_1k_2)(k_2^{-1}n_1k_2n_2) \in KN$.
- Since $1 \in K$ and $1 \in N$, we have $1 = 1 \cdot 1 \in KN$.
- For any $kn \in KN$, its inverse is $(kn)^{-1} = n^{-1}k^{-1} = k^{-1} \cdot kn^{-1}k^{-1} \in KN$.

Hence KN is a subgroup of G.

c) Show that $K/(N \cap H)$ is isomorphic to a subgroup of N'/N.

The first isomorphism theorem states the following:

If $f : G_1 \to G_2$ is a group homomorphism, then $(\ker f) \trianglelefteq G_1$ and $G_1/(\ker f) \cong (\mathrm{im} f)$.

Consider $f : K \xrightarrow{\imath} N' \xrightarrow{\rho} N'/N$, where $\imath$ is inclusion (as $K = (N' \cap H) \subseteq N'$) and ρ is reduction mod N. The kernel of f is by construction $(\ker f) = f^{-1}(\{1\}) = (\imath^{-1} \circ \rho^{-1})(\{1\}) = \imath^{-1}(N) = (N \cap K) = (N \cap H)$. The image of f is a subgroup of N'/N.

Hence $K/(N \cap H)$ is isomorphic to a subgroup of N'/N by the first isomorphism theorem.

6)

a) How many elements of each order are there in the alternating group A_5?

The group A_5 consists of

- the identity (order 1)
- products of two disjoint 2-cycles (order 2), of which there are $\frac{1}{2}\binom{5}{2}\binom{3}{2} = 15$
- 3-cycles (order 3), of which there are $\binom{5}{3}(3-1)! = 20$
- 5-cycles (order 5), of which there are $\binom{5}{5}(5-1)! = 24$

As a check, the total number of elements is $1 + 15 + 20 + 24 = 60 = \frac{1}{2}\,5!$, as expected.

b) Show using (a) that A_5 is not isomorphic to the direct product $D_{10} \times S_3$.

From the presentation $D_{10} = \langle r, s \mid r^5 = s^2 = 1, rs = sr^{-1} \rangle$, we see that r has order 5 in D_{10}.

Meanwhile, the 3-cycle (123) has order 3 in S_3.

Using the fact that the order of $g_1 \times g_2$ in $G_1 \times G_2$ is the l.c.m. of the orders of g_1 in G_1 and g_2 in G_2, we find that $r \times (123) \in D_{10} \times S_3$ has order $\mathrm{lcm}(5, 3) = 15$.

Since there is an element of order 15 in $D_{10} \times S_3$ but there is no such element in A_5, we deduce that these two groups are not isomorphic.