Fall 2023 Midterm II Answer Key

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1 Problem 1

(a). True. The groups of order 1,2,3,5,7 are all isomorphic to the unique cyclic groups of that order. The groups of order 4 are $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, both of which are abelian. There are 2 groups of order 6: $\mathbb{Z}_6$ and S_3, the latter of which is non-abelian. In conclusion, therefore, there is exactly one non-abelian group of order ≤ 7.
(b). Not true. There are three non-isomorphic abelian groups of order 24. They are:

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$
- $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3$
- $\mathbb{Z}_8 \times \mathbb{Z}_3$

(c). True. Let H^g be the image of H under conjugation by g. Note that for any $gh_1g^{-1}, gh_2g^{-1} \in H^g$ (where $h_1, h_2 \in H$), we have

$$gh_1g^{-1} \cdot gh_2g^{-1} = g(h_1h_2)g^{-1}.$$

So H^g is closed under multiplication. Moreover, H^g is closed under inverses, since if $p = ghg^{-1} \in H^g$, then $p^{-1} = gh^{-1}g^{-1} \in H^g$. Hence, H^g is indeed a subgroup of G.

2 Problem 2

(a). (i). (1)(2 6 4 7)(3 5) (ii). (1 5 3 2 7)(4 6)
(b). For (i), the order is 4. For (ii), the order is 10.
(c) There are $4! = 24$ ways to assign $\{1, 2, 3, 4\}$ to form a 4-cycle $(a\ b\ c\ d)$, but $(a\ b\ c\ d)$, $(b\ c\ d\ a)$, $(c\ d\ a\ b)$, and $(d\ a\ b\ c)$ represent the same cycle. So, there are $\frac{4!}{4} = 6$ distinct 4 cycles in S_4. They are (1 2 3 4), (1 2 4 3), (1 3 2 4), (1 3 4 2), (1 4 2 3), and (1 4 3 2). These 4-cycles are not in A_4, since for any 4-cycle $(a\ b\ c\ d)$, we can decompose it as $(a\ b)(b\ c)(c\ d)$, which is an odd number of transpositions.
(e). S_7 does not have an element of order 8, as there are no ways of making a 8-cycle from the elements in S_7.

3 Problem 3

The First Isomorphism Theorem: Let G, H be groups, and $\phi : G \to H$ be a group homomorphism. Then we have

$$G/\ker\phi \cong \operatorname{Im}\phi.$$

As an example, we can take the group homomorphism $\phi : S_3 \to \mathbb{Z}_2$ given by $\phi(\sigma) = sgn(\sigma)$. Note that ϕ is surjective, and the kernel of ϕ is A_3, consisting of the even permutations of $\{1, 2, 3\}$. Then the First Isomorphism Theorem says that $S_3/A_3 \cong \mathbb{Z}_2$. In this example, S_3 is non-abelian.

4 Problem 4

- Abelian groups of order 14
  - $\mathbb{Z}_2 \times \mathbb{Z}_7$
- Abelian groups of order 16
  - $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
  - $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$
  - $\mathbb{Z}_2 \times \mathbb{Z}_8$
  - $\mathbb{Z}_4 \times \mathbb{Z}_4$
  - $\mathbb{Z}_{16}$
- Abelian groups of order 20
  - $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$
  - $\mathbb{Z}_4 \times \mathbb{Z}_5$

5 Problem 5

(a). First, we show that $c_g : N \to N$ is a group homomorphism. For any $n_1, n_2 \in N$, we have

$$c_g(n_1)c_g(n_2) = (gn_1g^{-1})(gn_2g^{-1}) = g(n_1n_2)g^{-1} = c_g(n_1n_2).$$

Next, we show c_g is a bijection. If $c_g(n) = 1$ for some $n \in N$, then $gng^{-1} = 1$, or equivalently $gn = g$. Left multiplying both sides by g^{-1}, we get $n = 1$. Thus, $\ker(c_g) = 1$, so c_g is injective. To show surjectivity, suppose that $n \in N$. Let $m = g^{-1}ng = g^{-1}n(g^{-1})^{-1} \in N$, and observe that

$$c_g(m) = gmg^{-1} = gg^{-1}ngg^{-1} = n.$$

Therefore, c_g is surjective. We have shown that c_g is a bijective group homomorphism, and therefore a group isomorphism.

(b) Let $g, h \in G$. For any $n \in N$, we check that

$$c_g \circ c_h(n) = c_g(hnh^{-1}) = ghnh^{-1}g^{-1} = (gh)n(gh)^{-1} = c_{gh}(n).$$

Thus, $c_g \circ c_h = c_{g\cdot h}$. As stated in the problem, this gives a group homomorphism

$$f : G \to Aut(N)$$

given by $f(g) = c_g$.

(c) ($\Longleftarrow$) Suppose that G is abelian. For any $g \in G$ and $n \in N$, we see that

$$f(g)(n) = c_g(n) = gng^{-1} = gg^{-1}n = n,$$

so $f(g)$ is just the identity map, i.e., the identity element in $Aut(N)$. This shows that $\ker(f) = G$.
($\Longrightarrow$) Suppose that $\ker(f) = G$. Then for any $g \in G$ and $n \in N$, we have

$$gng^{-1} = f(g)(n) = n. \tag{1}$$

We claim that every element of G can be written in the form $x^a n$, for some $a \in \mathbb{Z}$ and $n \in N$.

To see why, note that G/N is of prime order p, so by Lagrange Theorem we know that G/N is cyclic. Let xN be the generator of G/N, where x is a representative. For any $g \in G$, we have that

$$gN = (xN)^a = x^a N,$$

for some $a \in \mathbb{Z}$. Then $gx^{-a} \in N$, which means that there exists $n \in N$ such that $gx^{-a} = n$. We then have $g = x^a n$, which proves the claim.

Now, let $g, h \in G$. By claim we can write $g = x^{a_1} n_1$, $h = x^{a_2} n_2$ for some $a_1, a_2 \in \mathbb{Z}$ and $n_1, n_2 \in N$. We compute that

$$\begin{aligned} gh &= (x^{a_1} n_1)(x^{a_2} n_2) \\ &= x^{a_1} x^{a_2} n_1 n_2 \\ &= x^{a_2} x^{a_1} n_2 n_1 \\ &= hg, \end{aligned}$$

where the second equality follows from equation (1), and the third equality follows from the assumption that N is abelian. This concludes the proof.