Intro Modern Algebra I Midterm 1

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October 24, 2023

Problem 1

(a) True. $X \times X \backslash R$ is the relation $xR'y$ iff $\neg xRy$. Then given $(x, y) \in X \times X \backslash R$, $\neg xRy$ and $\neg yRx$(because R is symmetric) but xRx(reflexive), so R' is not transitive thus not an equivalence relation. For an example, let $X = \mathbb{R}$, R be $=$.

(b) False. Let $G = \mathbb{Z}_2$ and both operations be addition. Then $a \times b = (a+b)+(a+b) = 0$ for every a, b, and apparently there is no element a such that $a \times 1 = 1$. Therefore there is no identity element in this operation since there is no identity for 1.

(c) True. The group operation is composition of functions, and apparently composing two isomorphisms $G \to G$ gives another isomorphism $G \to G$. The identity function is the identity element, and the inverse element is the inverse function. Note: this group is usually denoted $\text{Aut}(G)$.

Problem 2

(a) (i) 2x+5=12 $\Rightarrow$ 2x=7 $\Rightarrow$ x=7*15 $\Rightarrow$ x=[18].

(ii) $(1000000004)^2 = (10^9+4)^2 = 10^{18} + 8 \times 10^9 + 16 \equiv 1^{18} + 8 \times 1^9 + 7 \pmod 9 = 1 + 8 + 7 = 7 \pmod 9$.

(b) The generators are those that are coprime with 8. So they are $\{1, 3, 5, 7\}$.

Problem 3

(a) No. W 116th St intersects with Broadway, and Broadway intersects with W 115th St, but the two streets do not intersect, so the relation is not transitive.

(b) Yes. $a/a = 1 = 1^2$. $a/b = (p/q)^2$ then $b/a = (q/p)^2$. If $a/b = (p/q)^2$ and $b/c = (r/s)^2$ then $a/c = (a/b)(b/c) = (pr/qs)^2$.

(c) No. Take 0, 1.5, 3. $0 \sim 1.5, 1.5 \sim 3$ but $0 \not\sim 3$ so the relation is not transitive.

Problem 4

Proof. It suffices to show g has infinite order, since then $\langle g \rangle \subset G$ would be infinite. Apparently g cannot have order > 100, but then if g has finite order $n \leq 100$ then $g^{100n} = e$ and $100n > 100$ since $n > 1$(otherwise $g = e$) which creates a contradiction. Therefore G has infinite cardinality. □

Problem 5

(a) It is apparent that $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity under multiplication, but then there is no matrix A such that $A \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = I$ by elementary linear algebra (since $\det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ so it is not invertible). Therefore this matrix does not have an inverse under multiplication and $M(2, \mathbb{R})$ does not form a group under multiplication. It does satisfy **closure** and **existence of identity**.

(b) $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) take $z = xy$.

This is not correct. Instead, take z = I. No two of x, y, and z are conjugate to each other. This is because z = I commutes with any matrix, so the only conjugate of z is z. Similarly, the only conjugate of y is y. This implies that x, y, and z=I gives a solution to the problem.

Another way to look at the question: if two matrices are conjugate then they have the same characteristic polynomial, hence they have the same eigenvalues. (The converse is not true.) You can compute the eigenvalues of x, y, and z = I to check that they are not conjugate.

On the other hand, z = xy is conjugate to x.

Problem 6

1. $D_6 = \langle a, b | a^2 = b^3 = 1, aba^{-1} = b^{-1} \rangle$. The cyclic subgroups are $\{1\}, \langle a \rangle, \langle b \rangle, \langle ab \rangle, \langle ab^2 \rangle$. This is similar to an earlier exercise. Main takeaway: subgroups of D_6 must have order 2 or 3, and by classification of finite groups they must be cyclic. It suffices to determine which elements have order 2 and which has order 3. Indeed, a, ab, ab^2 each has order 2 and they each generate a subgroup of order 2. b, b^2 have order 3 and they generate the same subgroup.

2. Since 3 and 4 are coprime, $\mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12}$. The cyclic subgroups are $\{(0,0)\}, \{0\} \times \{0,2\}, \mathbb{Z}_3 \times \{0\}, \{0\} \times \mathbb{Z}_4, \mathbb{Z}_3 \times \{0,2\}, \mathbb{Z}_3 \times \mathbb{Z}_4$.