Isomorphism theorems

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GU4041

Columbia University

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Outline

Product of two subgroups

There are *three* isomorphism theorems, known by their numbers.

First we need to define the notion of a *product of subgroups*.

Lemma

Let $J, N \subseteq G$ be two subgroups, with N normal in G (we write $N \trianglelefteq G$). Then the set

$$J \cdot N = \{j \cdot n,\ j \in J, n \in N\}$$

is a subgroup of G.

Proof.

It suffices to show that $J \cdot N$ is closed under multiplication and inverses . If $j \cdot n \in JN$, then

$$(jn)^{-1} = n^{-1}j^{-1} = j^{-1} \cdot (jnj^{-1}) \in J \cdot N$$

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Proof.

Next, if $j_1, j_2 \in J$, $n_1, n_2 \in N$, then

$$(j_1 \cdot n_1)(j_2 \cdot n_2) = j_1 j_2 \cdot (j_2^{-1} n_1 j_2) n_2 \in J \cdot N,$$

again because N is normal. This completes the proof. □

First isomorphism theorem

Theorem

Let $f : G \to H$ be a homomorphism with kernel K. .
Then there is an isomorphism

$$G/K = G/Ker(f) \xrightarrow{\sim} Image(f).$$

If G and H are vector spaces and f is a linear transformation, this can be compared to the formula

$$\dim G - \dim \ker(f) = \dim \; Image(f).$$

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Second Isomorphism theorem

Theorem

Let G be a group, $H \subseteq G$ a subgroup, $N \trianglelefteq G$ a normal subgroup. Then the inclusion of H in $H \cdot N$ determines an isomorphism

$$H/H \cap N \xrightarrow{\sim} H \cdot N/N$$

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Third isomorphism theorem

First recall that if $N \trianglelefteq G$ is a normal subgroup, then there is a bijection between the set S of subgroups of the quotient G/N and the set T of subgroups of G containing N.

If $\pi : G \to G/N$ is the quotient map, this correspondence is defined as follows: to each subgroup $J \subset G/N$, we associate the preimage $\pi^{-1}(J) \subset G$.

This defines a function from S to T. The inverse function takes a subgroup $H \subset G$ containing N to its image $\pi(H) \subset G/N$.

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Proof of First Isomorphism Theorem

$$f : G \to H; \quad G/K = G/Ker(f) \xrightarrow{\sim} Image(f).$$

Proof.

Let $J = Image(f) \subset H$. Define $\alpha : G/K \to J$ by setting $\alpha(gK) = f(g)$.

First, α is *well-defined*; in other words, if $gK = g'K$ then $\alpha(gK) = \alpha(g'K)$. Now if $gK = g'K$ then $\exists k \in K$ such that $g' = gk$. Then

$$\alpha(gK) = f(g) = f(g) \cdot f(k) = f(gk) = f(g') = \alpha(g'K),$$

where the second equality follows because $f(k) = e$ for any $k \in \ker(f)$.

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Proof of First Isomorphism Theorem

Proof.

Next, the image of α (which a priori is in H) is in fact contained in J.
This is obvious by the definition of "image."
Third, α is surjective. Suppose $j \in J = Image(f)$. Thus there exists $g \in G$ such that $f(g) = j$. It follows that $\alpha(gK) = j$.
Finally α is injective. Suppose $\alpha(gK) = e$. Then $f(g) = e$, in other words $g \in \ker(f) = K$. So $gK = K$ which is the identity element of G/K. Thus α is injective.

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Proof of Second Isomorphism Theorem

Proof.

Consider the composition

$$H \hookrightarrow H \cdot N \to H \cdot N/N; \;\; h \mapsto h \cdot e_N \mapsto (h \cdot e_N)N \in H \cdot N/N.$$

Call the composition ϕ.

First, ϕ is *surjective*. Indeed, the map $\pi . H \cdot N \to H \cdot N/N$ is the surjective quotient map. Let $j \in H \cdot N/N$ and suppose $j = \pi(h \cdot n)$. Since $n \in N = \ker \pi$,

$$j = \pi(h \cdot n) = \pi(h) \cdot \pi(n) = \pi(h) = \pi(h \cdot e_N) = \phi(h).$$

Thus ϕ is surjective. □

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Next,

$$\ker(\phi) = \{h \,|h \cdot e_N \in \ker(\pi)\} = \{h \,|h \cdot e_N \in N\}.$$

But $h \cdot e_N \in N$ if and only if $h \in N$. Since $h \in H$, it follows that $\ker(\phi) = H \cap N$.

But the First Isomorphism Theorem implies that

$$H/\ker(\phi) \xrightarrow{\sim} Image(\phi).$$

We know $\ker(\phi) = H \cap N$ and $Image(\phi) = H \cdot N/N$ because ϕ is surjective. Thus

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Proof of Third Isomorphism Theorem

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Let $\pi : G \to G/N$ be the quotient map. We define a homomorphism

$$f : G/N \to G/H; gN \mapsto gH.$$

This is well-defined because $N \subseteq H$: if $g'N = gN$ then $g'H = gH$.
And it is a homomorphism because if $g_1, g_2 \in G$,

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because H is a normal subgroup. Moreover, f is surjective: if $j \in G/H$ then $j = gH$ for some $g \in G$, and then $j = f(gN)$. □

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Finally,

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which is just $\pi(H)$. But $\pi(H) = H/N$ under the bijection between subgroups of G/N and subgroups of G containing N.

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An example

Let $G = S_4$, $H = A_4 \supseteq N = K_4$. (We know N is normal in S_4 by a homework exercise.)
Then $H/N = A_4/K_4$ is a group of order 3, which must be the cyclic group $\mathbb{Z}_3$.

Question

$G/N = 6$. Is it isomorphic to $\mathbb{Z}_6$ or $S_3 = D_6$?

$\mathbb{Z}_6$ has an element of order 6. If $G/N = \mathbb{Z}_6$, then G must have an element of order at least 6. But S_4 has no such element. Thus $G/N = D_6$.
Of course $G/H = \mathbb{Z}_2$, H/N is the unique subgroup of order 3 in D_6, and $(G/N)/(H/N)$ is also $\mathbb{Z}_2$.
There are more interesting examples for finite abelian groups.

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The main theorem

Theorem

Let A be a finite abelian group. There is a sequence of prime numbers

$$p_1 \leq p_2 \leq \cdots \leq p_n$$

(not necessarily all distinct) and a sequence of positive integers

$$a_1, a_2, \ldots, a_n$$

(in no particular order) such that A is isomorphic to the direct product

$$A \xrightarrow{\sim} \mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}}.$$

In particular

$$|A| = \prod_{i=n}^{n} p_i^{a_i}.$$

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Prime factors

This can be broken down into two theorems.

Theorem (Theorem 1)

Let A be a finite abelian group. Let $q_1, \ldots, q_r$ be the distinct *primes dividing $|A|$, and say*

$$|A| = \prod_j q_j^{b_j}.$$

Then there are subgroups $A_j \subseteq A$, $j = 1, \ldots, r$, with $|A_j| = q_j^{b_j}$, and an isomorphism

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Abelian groups of prime power order

Theorem (Theorem 2)

Let p be a prime and let A be a finite abelian group of order p^N for some $N > 1$. Then there is a sequence of positive integers $c_1 \leq c_2 \cdots \leq c_s$ and an isomorphism

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Theorem 1 is essentially a series of applications of the Chinese Remainder Theorem, and is not very hard.

Theorem 2 is a more complicated induction argument.

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Additive notation

We will use *additive notation* for the abelian group A. So instead of writing $a \cdot b$ we write $a + b$, and instead of writing a^m we write ma, where m is any integer. We also write $-a$ instead of a^{-1} and 0 instead of e. Because A is abelian, we know $a + b = b + a$ for any $a, b \in A$.

Lemma

Let A be an abelian group. Then for any $m \in \mathbb{Z}$, the function $a \mapsto ma$ is a homomorphism.

Additive notation

We will use *additive notation* for the abelian group A. So instead of writing $a \cdot b$ we write $a + b$, and instead of writing a^m we write ma, where m is any integer. We also write $-a$ instead of a^{-1} and 0 instead of e. Because A is abelian, we know $a + b = b + a$ for any $a, b \in A$.

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Proof of the Lemma

Proof.

We need to show that, for all $a, b \in A$,

$$m(a+b) = ma + mb.$$

We prove this for $m > 0$ by induction; the case of $m < 0$ is similar. For $m = 1$ there is nothing to prove. Suppose we know the equality for m. Then

$$(m+1)(a+b) = m(a+b) + (a+b) = (ma+mb) + (a+b)$$

by the induction hypothesis. But now by associativity

$$(ma+mb) + (a+b) = ma + (mb+a) + b = ma + (a+mb) + b$$

where the last equality is allowed because A is abelian. □

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$$ma + (a+mb) + b = (ma+a) + (mb+b) = (m+1)a + (m+1)b$$

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A Proposition

Proposition

Suppose A is an abelian group of order mn, where $(m, n) = 1$. *Then there are subgroups* $A_m, A_n \subseteq A$ *such that* $|A_m| = m$, $|A_n| = n$, *such that the inclusion defines an isomorphism*

$$A_n \times A_m \xrightarrow{\sim} A.$$

Proof.

Define

$$mA = \{ma, a \in A\}; nA = \{na, a \in A\}.$$

Claim $mA \cap nA = \{0\}$. Suppose $x \in mA \cap nA$. Then there are $a, b \in A$ such that

$$x = ma = nb.$$

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Proof.

Since $x = ma = nb$, we have

$$mx = m^2a = mnb = 0.$$

Similarly $nx = 0$.

But there are constants $\alpha, \beta \in \mathbb{Z}$ such that $\alpha m + \beta n = 1$. Thus

$$x = (\alpha m + \beta n)x = \alpha \cdot mx + \beta \cdot nx = 0.$$

So $mA \cap nA = \{0\}$ as claimed. □

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Proof.

Now define $A_n = mA$, $A_m = nA$ (careful!) Inclusion defines a homomorphism

$$f : A_n \times A_m \to A; \;\; f((u, v)) = u - v.$$

Suppose $(u, v) \in \ker f$. Then $u - v = 0$, so $u = v \in A_n \cap A_m = \{0\}$. Thus f is injective.
On the other hand, if $a \in A$, let $\alpha m + \beta n = 1$ as before. Write $u = \alpha \cdot ma \in A_n$, $v = -\beta \cdot na \in A_m$. Then

$$f((u, v)) = \alpha \cdot ma - (-\beta \cdot na) = (\alpha m + \beta n)a = a,$$

so f is surjective as well. Thus f is an isomorphism. □

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We see that

$$nm = |A| = |A_n| \cdot |A_m|.$$

But we still need to show that $|A_n| = n$ and $|A_m| = m$. It suffices to show that $|A_m|$ and n are relatively prime, because then n divides $nm = |A_n| \cdot |A_m|$ implies n divides $|A_n|$ by Gauss's Lemma; similarly m divides $|A_m|$, so we must have $n = |A_n|$ and $m = |A_m|$.
Thus suppose $p | gcd(|A_m|, n)$. Now we claim that $v \mapsto nv$ is an automorphism of A_m. Indeed, for $v = nb \in A_m$, $mv = mnb = 0$, so

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A key lemma

So p is an automorphism of $|A_m|$ but p divides the order of A_m. We pause for a key lemma:

Lemma

Let B be a finite abelian group of order divisible by p. Then B contains a non-zero element of order p.

This Lemma contradicts the earlier conclusion that $pv = 0 \Rightarrow v = 0$. So the Lemma completes the proof of the Proposition.

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Proof.

This is again an inductive proof. Say $|B| = pN$. If $N = 1$ then B is cyclic of order p and we know the result. Suppose we know the result for all $|B|$ of order pk with $k < N$. If B has no nontrivial proper subgroup, then B is cyclic of prime order; so B must have a proper subgroup $H \subsetneq B$, $|H| > 1$. If p divides $|H|$ then by induction H has a non-zero element of order p, and we are done. So assume p does not divide $r = |H|$. It follows that there is $g \in B/H$ of order p. □

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Let $\pi : B \to B/H$ be the quotient map, $\pi(b) = g \in B/H$. Thus $b \notin H$ but $\pi(pb) = pg = 0$, so $pb \in H$, so $rpb = 0$. Let $a = rb$, so $pa = 0$. We suppose $a = 0$ and derive a contradiction. Use Bezout's relation yet again. Since $(p, r) = 1$ there are integers γ, δ such that

$$b = (\gamma p + \delta r)b = \gamma pb + \delta a = \gamma pb + 0 \in H,$$

contradiction. □

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