Problem 1

1. True. Using classification theorem for finite abelian groups, there are precisely three isomorphism classes for an abelian group of order p^3: $\mathbb{Z}_{p^3}, \mathbb{Z}_p \times \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

2. False. From classification of finite abelian groups there is only one of them which is $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$.

3. False. WLOG say $q < p$. By Sylow's theorem it has a p-Sylow subgroup of index q. Therefore because $n_p | q$ and $n_p \equiv 1 \mod p$ we have $n_p = 1$, so the p-Sylow subgroup is normal. Therefore any group of order pq cannot be simple.

4. True. The 5-Sylow subgroup has order 125 and index 8. Now $n_5 | 8$ and $\equiv 1 \mod 5$, so it has to be 1.

Problem 2

(a)

Conjugacy classes in S_6 corresponds to partitions of $\{1, 2, 3, 4, 5, 6\}$, or equivalently by different types of disjoint cycles. Elements from each conjugacy class and their centralizer are given by

1. $e : S_6$ of size 6! and index 1.

2. (1 2): {everything that does not contain (1 2)}×⟨(1 2)⟩ of size 48 and index 15.

3. (1 2 3): {everything that does not contain (1 2 3)}×⟨(1 2 3)⟩ of size 18 and index 40.

4. (1 2)(3 4): ⟨(5 6), (1 3)(2 4), (1 2), (3 4)⟩ of size 16 and index 45.

5. (1 2 3 4): ⟨(5 6), (1 2 3 4)⟩ of size 8 and index 90.

6. (1 2 3)(4 5): ⟨(1 2 3), (4 5)⟩ of size 6 and index 120.

7. (1 2 3 4)(5 6): ⟨(1 2 3 4), (5 6)⟩ of size 8 and index 90.

8. (1 2 3 4 5): $\langle(1\ 2\ 3\ 4\ 5)\rangle$ of size 5 and index 144.
9. (1 2 3 4 5 6): $\langle(1\ 2\ 3\ 4\ 5\ 6)\rangle$ of size 6 and index 120.
10. (1 2 3)(4 5 6): $\langle(1\ 2\ 3), (4\ 5\ 6), (1\ 4)(2\ 5)(3\ 6)\rangle$ of size 18 and index 40.
11. (1 2)(3 4)(5 6): $\langle(1\ 3)(2\ 4), (1\ 5)(2\ 6), (3\ 5)(4\ 6), (1\ 3\ 5)(2\ 4\ 6)\rangle$ of size 48 and index 15.

$Z(S_6) = \{e\}$ has size one. Combined we have

$$1 + 15 + 40 + 45 + 90 + 120 + 90 + 144 + 120 + 40 + 15 = 720 = 6!.$$

(b)

$\{e\}$, (1 2 3), (1 2)(3 4), (1 2 3 4)(5 6), (1 2 3)(4 5 6) are conjugacy classes in A_6. (1 2 3 4 5) is not because (1 2 3 4 5) and (1 3 5 2 4) belong to two different conjugacy classes in A_6.

Problem 3

(a)

Indeed, from high school algebra knowledge we know $|zw| = |z||w|$ so this is a homomorphism. The image is $\mathbb{R}^{>0}$, the set of positive real numbers and the kernel is $U(1)$ or the unit circle or $\{z \in \mathbb{C}^\times \mid |z| = 1\}$.

(b)

Take any $G \subset \mathbb{C}^\times$ finite and $g \in G$. Then g has finite order, say $g^n = 1$. Then $|g^n| = |g|^n = 1$ and $|g| \in \mathbb{R}^{>0}$, so $|g| = 1$.

(c)

Since $G \subset \mathbb{C}^\times$ is finite and from (b) $G \subset U(1)$. Say $G = \{g_1, ..., g_m\}$. Now every element $g_i \in G$ has finite order, say n_i, and $g_i = e^{ic}$ for some $c \in \mathbb{R}$. Since $g_i^{n_i} = 1$ we deduce that $c = p\frac{2\pi}{n_i}$ for some integer p, and $g_i \in \langle e^{\frac{2i\pi}{n_i}}\rangle$. Therefore $G \subseteq \bigcup_{i=1}^m \langle e^{\frac{2i\pi}{n_i}}\rangle = \langle e^{\frac{2i\pi}{n}}\rangle$ where $n = \operatorname{lcm}_i n_i$. So we deduce that G is a subgroup of a cyclic group, so it is cyclic.

Problem 4 (a) $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication given by $i^2 = j^2 = k^2 = ijk = -1$.

Note that for $g, h \neq \pm 1$, $ghg^{-1}h^{-1} = gh(-g)(-h) = (gh)^2 = -1$.

If $g = \pm 1$, then for any $h \in Q_8$, $ghg^{-1}h^{-1} = hh^{-1} = 1$.

So, $D(Q_8)$ is the subgroup of Q_8 generated by ± 1.

$D(Q_8) = \{\pm 1\}$ = cyclic group of order 2.

Since $D(Q_8)$ is abelian, $D^2(Q_8) = \{1\}$ trivial.

(b) $D(S_4)$ contains all elements of the type $ghg^{-1}h^{-1}$ where $g, h \in S_4$. So, $D(S_4)$ contains $Cl(h) \cdot h^{-1}$ for all $h \in S_4$.

($Cl(h)$ ↑ conjugacy class of h)

We know that the conjugacy class of a permutation contains all permutations of its cycle type. Also, the inverse of a permutation has the same cycle type.

This means that $D(S_4)$ is generated by elements $\sigma_1 \sigma_2$ where σ_1 and σ_2 have the same cycle type. Such $\sigma_1\sigma_2$ is always even (sign of σ_1 = sign of σ_2), so $D(S_4) \subseteq A_4$.

On the other hand, every 3-cycle $(abc) = (ab)(bc)$ is in $D(S_4)$. (↑ ↗ same cycle shape) Since A_4 is generated by 3-cycles, we get $A_4 \subseteq D(S_4)$.

So, $D(S_4) = A_4$.

Now we want $D(A_4)$. For distinct a, b, c, d, the 3-cycle (abc) and product of disjoint transpositions $(ab)(cd)$ are in A_4.

$$(abc) \cdot (ab)(cd) \cdot ((abc))^{-1} ((ab)(cd))^{-1}$$
$$= (ac)(cd)(acb)(ab)(cd) = (ac)(cd)(cb)(cd)$$
$$= (ac)(bd).$$

So, $D(A_4)$ contains $(12)(34)$, $(13)(24)$ and $(14)(23)$. Note that each of these has order 2 and $(12)(34)(13)(24) = (14)(23)$.

Thus, $\{(1), (12)(34), (13)(24), (14)(23)\}$ is a subgroup of $D(A_4)$ and A_4 isomorphic to K_4. In fact, it is a normal subgroup since it contains the entire conjugacy class of $(12)(34)$. $|A_4| = \frac{4!}{2} = 12$. So, A_4/K_4 has order 3 and is therefore abelian (cyclic group of order 3). Then K_4 must contain $D(A_4)$. But $K_4 \subseteq D(A_4)$. Thus, $D^2(S_4) = D(A_4) = K_4$.

Problem 5 Let $g \in G$, $n \in N$. Since N is generated by S, $n = s_1 s_2 \cdots s_r$ for some $s_i \in S$. Then

$$gng^{-1} = g s_1 s_2 \cdots s_r g^{-1}$$
$$= (g s_1 g^{-1})(g s_2 g^{-1}) \cdots (g s_r g^{-1}).$$

Since S is a conjugacy class, every $g s_i g^{-1} \in S$. So, gng^{-1} above is generated by elements of S and lies in N.

For any $g \in G$, $gNg^{-1} \subseteq N \Rightarrow N$ is a normal subgroup.

Problem 6 (a) Non-abelian group of order 21:

Note that since 7 is a prime, $\mathrm{Aut}(\mathbb{Z}_7) = \mathbb{Z}_7^\times$ = cyclic group of order 6.

$\mathbb{Z}_7^\times$ contains a cyclic group of order 3 as a subgroup.

$$C_3 = \{1, a, a^2\} \hookrightarrow \mathbb{Z}_7^\times = \mathrm{Aut}(\mathbb{Z}_7)$$
$$a \longmapsto [2]$$

So, there is a semidirect product $\mathbb{Z}_7 \rtimes C_3$ of order 21.

This is not abelian:

$$([1], a)([1], a^2) = ([1] + [2][1], a^3) = ([3], 1)$$
$$\neq ([1], a^2)([1], a) = ([1] + [4][1], a^3) = ([5], 1)$$

(b) Non-abelian group of order 55:

We have $\mathrm{Aut}(\mathbb{Z}_{11}) = \mathbb{Z}_{11}^\times$ = cyclic group of order 10. There is a homomorphism

$$C_5 = \{1, a, a^2, a^3, a^4\} \hookrightarrow \mathbb{Z}_{11}^\times = \mathrm{Aut}(\mathbb{Z}_{11})$$
$$a \longmapsto [4]$$

which leads to a semi-direct product $\mathbb{Z}_{11} \rtimes C_5$ of order 55. This is not abelian:

$$([1], a)([1], a^2) = ([1] + [4][1], a^3) = ([5], a^3)$$
$$\neq ([1], a^2)([1], a) = ([1] + [16][1], a^3) = ([6], a^3)$$

Problem 7 Let G be a group of order $56 = 8 \times 7 = 2^3 \times 7$. Then G contains a Sylow-7 subgroup of order 7. The number of such subgroups divides 56 and is congruent to 1 modulo 7. The only divisors of 56 which are 1 mod 7 are 1 and 8.

If there is only one Sylow-7 subgroup, then it must be normal and G cannot be simple.

Suppose that G has 8 distinct Sylow 7-subgroups. Each of these subgroups is isomorphic to $\mathbb{Z}_7$. So, each has 6 elements of order 7. Moreover, any two of these subgroups intersect trivially. Thus, G has at least $8 \times 6 = 48$ elements of order 7. Then the remaining 8 elements of G must form the unique Sylow-2 subgroup of G. (We know that G has at least one Sylow-2 subgroup of order 8 and such a subgroup cannot contain an element of order 7. Now we don't have enough elements to form more than one Sylow-2 subgroup.)

Then the Sylow-2 subgroup must be normal and G cannot be simple.

Problem 8

(a). Let G be a group of order p^2. By the class equation, every p-group has nontrivial center, so the center $Z(G)$ of G is nontrivial. Its order can be either p^2 or p. If its order is p^2, then $G = Z(G)$, and therefore G is abelian. If its order is p, then the quotient $G/Z(G)$ has order p, so $G/Z(G)$ is cyclic. This implies that G is abelian, so $G = Z(G)$, which is impossible since we assumed that $|Z(G)|$ has order p. Thus, G is abelian.
(b). Consider the Heisenburg group modulo p, consisting of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{Z}_p$. It is easy to check that this group satisfies the group axioms, since it is closed under multiplication, contains the identity matrix, and that

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x & xz - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

which belongs to the Heisenberg group modulo p. Moreover, this group clearly has order p^3, and it is nonabelian because

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

whereas

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 9

We prove a stronger statement: the size of any conjugacy class of G must divide n. For any $g \in G$, let $Cl(g)$ be its conjugacy class. Recall that $|Cl(g)| = [G : C_G(g)]$, where $C_G(g)$ is the centralizer of g. Because Z is a subgroup of $C_G(g)$, we have

$$n = [G : Z] = [G : C_G(g)][C_G(g) : Z].$$

In particular, we get that $[G : C_G(g)]$ divides n. Hence, $|Cl(g)|$ divides n.

Problem 10

The Klein group K_4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We can think of this as a vector space over $\mathbb{Z}_2$, and any autmorphism on $\mathbb{Z}_2 \times \mathbb{Z}_2$ is an invertible linear map, represented by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}_2$. There are exactly 6 such matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, $Aut(K_4)$ contains 6 elements. Alternatively, we can think of K_4 as

$$K = \{\text{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

Any automorphism of K_4 must fix the identity and permute the rest of the three elements, so $Aut(K_4) \cong S_3$, which contains 6 elements.