Composition series

GU4041

Columbia University

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We have seen examples of chains of normal subgroups:

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \ldots G_r = \{e\} \qquad (1)$$

in which each group G_{i+1} is normal in the preceding group G_i (though not necessarily normal in G). Such a series is often called *subnormal*, and this is the terminology we use.

For example, there is the sequence of derived subgroups

$$G \supseteq D(G) = [G, G] \supseteq D^2(G) = [D(G), D(G)] \ldots$$

which ends with $D^r(G) = \{e\}$ if G is a solvable group, in which $D^i(G)/D^{i+1}(G)$ is abelian.

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Existence of composition series

A subnormal series as above is called a *composition series* if each of the quotient groups G_i/G_{i+1} is *simple*; in particular, $G_i \neq G_{i+1}$ for all i.

Lemma

Let G be a finite group. Then G has a composition series.

Proof.

We induct on the order of G. We know that a group of order 1 has a composition series. Suppose every group of order less than $|G|$ has a composition series. If G is simple, then we are done. If not, then G has a non-trivial proper normal subgroup N.

By induction, N and G/N both have composition series. □

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Existence of composition series, continued

Say

$$G/N = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{r-1} \supseteq H_r = \{e\}.$$

is a composition series. By the correspondence principle, each H_i corresponds to a subgroup G_i containing N, with $H_i = G_i/N$ for all i.

By the Third Isomorphism Theorem,

$$G_i/G_{i+1} \xrightarrow{\sim} (G_i/N)/(G_{i+1}/N) = H_i/H_{i+1}$$

which is simple.

On the other hand, $H_r = N$ has a composition series

$$N = G_r \supseteq G_{r+1} \cdots \supseteq G_N = \{e\}.$$

Then

$$G = G_0 \supseteq G_1 \cdots \supseteq N = G_r \supseteq G_{r+1} \cdots \supseteq G_N = \{e\}$$

is a composition series for G.

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Simple factors in a composition series

We write the collection of simple factors (J_α, m_α) where J_α is a simple group and m_α is the number of time it appears as a quotient G_i/G_{i+1}.
We call m_α the *multiplicity* of the simple factor J_α.
We call it a collection rather than a set, because the same element can appear more than once; sometimes this is called a *multiset*.

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Cyclic groups of prime power order

The cyclic group $\mathbb{Z}_{p^a}$ has a composition series:

$$\mathbb{Z}_{p^a} \supseteq (p) \supseteq (p^2) \supseteq \cdots \supseteq (p^{a-1}) \supseteq \{0\}$$

where (p^i) denotes the multiples of p^i modulo p^a, for any $i \leq a$. We can use the Third Isomorphism Theorem (see the notes online) to determine the simple factors.

Conclusion: the collection of simple factors of $\mathbb{Z}_{p^a}$ is $(\mathbb{Z}_p, a)$ (multiplicity a).

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Cyclic groups

Let $n \in \mathbb{Z}$. Write $n = \prod_i p_i^{a_i}$ as a product of prime factors. Then the cyclic group $\mathbb{Z}_n$ is isomorphic to a product of cyclic groups $\mathbb{Z}_{p_i^{a_i}}$ and the collection of simple factors of $\mathbb{Z}_n$ is the union of the simple factors of all the $\mathbb{Z}_{p_i^{a_i}}$:

$$(\mathbb{Z}_{p_i}, a_i).$$

Abelian groups

We know that any abelian group is isomorphic to a direct product of cyclic groups:

$$\prod_i \prod_j \mathbb{Z}_{p_i^{a_{ij}}}$$

where the p_i are distinct prime numbers and the $a_{ij}j$ are positive integers. The only simple abelian groups are the cyclic groups of prime order. So the collection of simple factors is

$$\{(\mathbb{Z}_{p_i}, m_i = \sum_j a_{ij})\}.$$

In other words, $\mathbb{Z}_{p_i}$ occurs as a simple factor a_{ij} times in the cyclic group $\mathbb{Z}_{p_i^{a_{ij}}}$, and the total multiplicity is the sum of the multiplicities in the simple factors.

The Jordan-Hölder Theorem

Theorem (Jordan-Hölder Theorem)

Let G be a finite group. Suppose G has two composition series:

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \cdots \supseteq G_{r+1} = \{e\}$$

$$G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{s+1} = \{e\}.$$

Then $r = s$ and the two collections of quotients

$$\{G_i/G_{i+1}\}, \ \{H_j/H_{j+1}\}$$

are equal (not taking order into account).

The proof is in the notes online.

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