Solvable and nilpotent groups

GU4041

Columbia University

November 28, 2023

Outline

Commutators

Let G be a group and let $g, h \in G$. The *commutator* of g and h is the element

$$[g, h] = ghg^{-1}h^{-1}.$$

If g and h commute, then

$$[g, h] = ghg^{-1}h^{-1} = h(gg^{-1})h^{-1} = hh^{-1} = e.$$

So $[g, h]$ is trivial for all g and h if G is abelian.
If $f : G \to A$ is a homomorphism, then

$$f([g, h]) = [f(g), f(h)]$$

for all g, h. So if A is abelian, then $[g, h] \in \ker(f)$ for all g, h.

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Commutator subgroup

Definition

The **commutator subgroup** $[G,G] \subset G$ is the subgroup of G generated by all the elements $[g,h]$ for all $g,h \in G$.

We also call $[G,G]$ the **derived subgroup** and denote it G', or $D(G)$.

Proposition

Let $f : G \to A$ be a homomorphism with A abelian. Then $[G,G] \subseteq \ker(f)$.

Proof.

It suffices to show that if $g,h \in G$ then $[g,h] \in \ker(f)$; but we already saw that on the last slide. □

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Abelianization

Proposition

Let G be a group. Then $[G, G]$ is a normal subgroup. Moreover, $G/[G, G]$ is abelian.

Proof.

The grSuppose $g, h, j \in G$. It is easy to compute that the conjugate of a commutator is a commutator:

$$j[g, h]j^{-1} = [jgj^{-1}, jhj^{-1}].$$

Moreover, the conjugate of the product of two commutators is the product of two commutators: if $g, h, g', h', j \in G$, then

$$j[g, h][g', h']j^{-1} = [jgj^{-1}, jhj^{-1}][jg'j^{-1}, jh'j^{-1}].$$

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Abelianization, continued

Proof.

It follows that the conjugate of any product of commutators is again a product of commutators; thus the conjugate of any element of $[G, G]$ is again in $[G, G]$. Now consider the quotient map $f : G \to G/[G, G]$. Let $\bar{g}, \bar{h} \in G/[G, G]$, and suppose $\bar{g} = f(g)$, $\bar{h} = f(h)$. Then

$$[\bar{g}, \bar{h}] = f([g, h]);$$

but since $[g, h] \in \ker(f)$, we see that $\bar{g}$ and $\bar{h}$ commute. Thus $G/[G, G]$ is abelian. □

We call $G/[G, G]$ the *abelianization* of G.

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Definition

The group G is *solvable* if there is a finite sequence of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_r = \{e\}$$

such that

(1) Each G_{i+1} is a normal subgroup of G_i, and

(2) Each group G_i/G_{i+1} is abelian.

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For a solvable group, $G_{i+1} \supseteq D(G_i)$. So we have an equivalent definition: Let $D(G) = D^1(G) = [G, G]$, $D^2(G) = [D(G), D(G)]$, and define $D^{i+1}(G) = [D^i(G), D^i(G)]$ for $i \geq 1$.

Lemma

G is solvable if and only if, for some $r \geq 1$, $D^r(G) = \{e\}$.

Proof.

If $D^r(G) = \{e\}$ we can take $G_i = D^i(G)$ in the definition of solvable. Conversely, since we have $D(G_i) \subseteq G_{i+1}$ for each i, we see by induction that

$$D(G) \subseteq G_1, D^2(G) \subseteq D(G_1) \subseteq G_2$$

and in general $D^i(G) \subseteq G_i$. Thus $D^r(G) \subseteq G_r = \{e\}$. □

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Example

We will later use the Sylow theorems to prove:

Proposition

Let $p < q$ be prime numbers, and let G be a group of order pq. Then G contains a normal subgroup of order q.

Admitting this proposition, we have

Theorem

Let $p \neq q$ be prime numbers. Then any group G of order pq is solvable.

Proof.

We may assume $p < q$ and let $G_1 \subseteq G$ be the normal subgroup of order q. . Then G/G_1 is of order p, hence is abelian. And G_1 is of prime order, hence is also abelian. Thus we have $D(G) \subset G_1$ and $D^2(G) = D(G_1) = \{e\}$. □

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More about solvable groups

The importance of solvable groups becomes clearer in the study of Galois theory. It turns out that A_5 is the smallest group that is not solvable. This is used in Galois theory to show that the general polynomial of degree 5 cannot be solved by radicals.

One of the most difficult theorems in finite group theory is

Theorem (Feit-Thompson theorem)

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Every quotient group of a solvable group is solvable.

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Subgroups of solvable groups

Let G be solvable and let

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_r = \{e\}$$

be a sequence of subgroups with $G_{i+1} \trianglelefteq G_i$ and G_i/G_{i+1} abelian. Let $H \subseteq G$ and define $H_i = H \cap G_i$. Then H_{i+1} is normal in H_i (check) and

$$H_{i+1} = H \cap G_{i+1} = H_i \cap G_{i+1}.$$

Thus

$$H_i/H_{i+1} = H_i/(H_i \cap G_{i+1}) \xrightarrow{\sim} G_{i+1} \cdot H_i/G_{i+1}$$

by the Second Isomorphism Theorem. But $G_{i+1} \cdot H_i \subseteq G_i$, so

$$G_{i+1} \cdot H_i/G_{i+1} \subset G_i/G_{i+1},$$

which is abelian. It follows that H_i/H_{i+1} is abelian, so H is solvable.

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Quotients of solvable groups

Let $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_r = \{e\}$ as above, and let $N \trianglelefteq G$, $H = G/N, f : G \to H$ the quotient map. Let $H_i = f(G_i)$ be the image. Obviously $H_{i+1} \trianglelefteq H_i$; we need to show H_i/H_{i+1} is abelian. We use $H_i = f(G_i) = G_i \cdot N/N$, so

$$H_i/H_{i+1} = (G_i \cdot N/N)/(G_{i+1} \cdot N/N) \xrightarrow{\sim} (G_i \cdot N)/(G_{i+1} \cdot N)$$

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$$(G_i \cdot N)/(G_{i+1} \cdot N) = G_i \cdot (G_{i+1} \cdot N)/(G_{i+1} \cdot N) \xrightarrow{\sim} G_i/G_i \cap (G_{i+1} \cdot N)$$

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$$G_i/G_i \cap (G_{i+1} \cdot N) \xrightarrow{\sim} (G_i/G_{i+1})/(G_i \cap G_{i+1} \cdot N)/G_{i+1}).$$

This is a quotient group of the abelian group G_i/G_{i+1}, hence is abelian.

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$$H_i/H_{i+1} = (G_i \cdot N/N)/(G_{i+1} \cdot N/N) \xrightarrow{\sim} (G_i \cdot N)/(G_{i+1} \cdot N)$$

by the Third Isomorphism Theorem. But $G_{i+1} \subseteq G_i$, so

$$(G_i \cdot N)/(G_{i+1} \cdot N) = G_i \cdot (G_{i+1} \cdot N)/(G_{i+1} \cdot N) \xrightarrow{\sim} G_i/G_i \cap (G_{i+1} \cdot N)$$

by the Second Isomorphism Theorem. By the Third Isomorphism Theorem again we have

$$G_i/G_i \cap (G_{i+1} \cdot N) \xrightarrow{\sim} (G_i/G_{i+1})/(G_i \cap G_{i+1} \cdot N)/G_{i+1}).$$

This is a quotient group of the abelian group G_i/G_{i+1}, hence is abelian.

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Nilpotent groups

Definition

The group G is *nilpotent* if there is a finite sequence of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_r = \{e\}$$

such that

(1) Each G_{i+1} is a normal subgroup of G_i, and

(2) Each group G_i/G_{i+1} is contained in the center of G/G_{i+1}.

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Every finite p-group is nilpotent

Theorem

Let G be a finite p-group. Then G is nilpotent (and in particular solvable).

Proof.

We induct from below. Let $Z = Z(G)$. Since G is a p-group, we know that Z is of order at least p. Now G/Z is again a finite p-group. So by induction it has a *central series*

$$G/Z = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_r = \{e\}$$

as in the definition. By the correspondence principle, each H_i corresponds to a subgroup $G_i \subseteq G$ that contains Z. Let $G_r = Z$, $G_{r+1} = \{e\}$. Then for $i < r$, $G_i/G_{i+1} = H_i/H_{i+1}$ is contained in the center of G/G_{i+1}, and for $i = r$ this is the definition of Z. □

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Comment

In the last step of the proof

$$G_i/G_{i+1} = H_i/H_{i+1}$$

we claimed that G_i/G_{i+1} is contained in the center of G/G_{i+1}.
Here we implicitly used the fact that

$$H_i/H_{i+1} = (G_i/Z)/(G_{i+1}/Z) \xrightarrow{\sim} G_i/G_{i+1}$$

by the Third Isomorphism Theorem and also that

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Since H_i/H_{i+1} is central in H/H_{i+1}, it follows that
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By copying the proofs of the hereditary properties of solvable groups, we obtain the corresponding properties for nilpotent groups:

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