INTRODUCTION TO MODERN ALGEBRA I, GU4041, SPRING 2020

Midterm 2, April 9 2020

HONOR CODE AFFIRMATION.

I affirm that I will not plagiarize, use unauthorized materials, or give or receive illegitimate help on assignments, papers, or examinations. I will also uphold equity and honesty in the evaluation of my work and the work of others. I do so to sustain a community built around this Code of Honor.

Please sign and scan and submit this signed page:

Your name:

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This is an **open book test**. The exam will be available at 10:00 AM in each of three time zones:

- 10 AM Eastern Daylight Time (10 AM New York time) for students in North and South America
- 10 AM Central European Time (4 AM New York time) for students in Europe and Africa
- 10 AM Chinese Standard Time (10 PM April 8 New York time) for students in Asia and Oceania

You have 12 hours to complete and submit the exam on Courseworks. It is strongly recommended that you submit this signed page as soon as you receive the exam; this will let you know well in advance whether or not the system is working properly. Please contact CUIT immediately, at askcuit@columbia.edu, or call 212-854-1919, if you encounter any obstacle.

The professor will be available on Zoom at 851-930-118 to answer questions about the exam during the following periods: 10AM-12 noon Chinese Standard Time (10PM-midnight New York time April 8); 8AM-11:30 (more or less) New York time April 9 (for students in all three areas); and 8PM-10PM New York time April 9.

Questions by email will be impossible to manage; please *do not* send any.

The exam questions.

1. (10 points) True or False? (Each question is worth 5 points.) If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) Any finite group of order pq, where $p < q$ are two prime numbers, is abelian.

(b) Suppose H and J are two subgroups of a group G. Then $H \cdot J \subseteq G$ is a subgroup and

$$H/H \cap J \xrightarrow{\sim} H \cdot J/J.$$

2. (20 points) (a) Write down the cycle decompositions of the following products of permutations:

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}^2$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}^2$

(b) Write each of the following permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$$

as a product of transpositions in S_5, and determine which belongs to the alternating group A_5.

3. (15 points)

Find a non-abelian group G with two distinct normal subgroups H, K of index 2: $[G : H] = [G : K] = 2$. Show that $H \cap K$ is a normal subgroup of G and show that $G/(H \cap K)$ is an abelian group that is not cyclic.

4. (15 points) For any integer $n > 1$, let $A(n)$ be the number of non-isomorphic abelian groups of order n. Consider the numbers $A(28)$, $A(5)$, $A(33)$, $A(9)$, $A(32)$. Which is the largest? Which is the smallest? Are any two equal? Explain.

5. (20 points) Let G be a finite group and $H \triangleleft G$, $K \triangleleft G$. Suppose that $H \cap K = \{e\}$. Show that $|G| \leq [G : H][G : K]$ by constructing an injective homomorphism from G to a group of order $[G : H][G : K]$. When do we have $|G| = [G : H][G : K]$?

6. (20 points) Construct two non-isomorphic *non-abelian* groups of order 192, each of which contains a normal *abelian* subgroup of order 8. (Hint: at least one of them can be a direct product of smaller groups.)