INTRODUCTION TO MODERN ALGEBRA I, GU4041, FALL 2023

Midterm 2, November 16 2023

---

1. (15 points) True or False? (Each question is worth 5 points.) If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) There is only one non-abelian group of order ≤ 7.

(b) There are exactly two non-isomorphic abelian groups of order 24.

(c) Let H be a subgroup of the finite group G and let $g \in G$. Then the image of H under conjugation by g is a subgroup of G.

2. (25 points) (a) Write down the cycle decompositions of each of the following permutations in S_7.

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 6 & 5 & 7 & 3 & 4 & 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 2 & 6 & 3 & 4 & 1 \end{pmatrix}$

(b) Write down the order of each permutation in (a) as an element of S_7.

(c) Show that there are 6 distinct 4 cycles in S_4. Are they in A_4?

(d) Write each of the following permutations

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 2 & 5 & 6 & 7 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 6 & 3 & 4 & 7 \end{pmatrix}$

as a product of transpositions in S_7, and determine which belongs to the alternating group A_7.

(e) Does S_7 have an element of order 8? If so, give an example; if not, explain why not.

3. (20 points) State one of the three isomorphism theorems and illustrate it with an example in which one of the groups involved is non-abelian.

4. (15 points) List all the abelian groups of order 14, 16, and 20. In other words, for each number, give a list of abelian groups of that order such that any abelian group of that order is isomorphic to exactly one on your list.

5. Let G be a finite group and $N \subset G$ a normal subgroup. Suppose N is abelian. For any $g \in G$ consider the function $c_g : N \to N$, defined by

$$c_g(n) = gng^{-1}.$$

(a) (10 points) Show carefully that $c_g : N \to N$ is an automorphism. Prove first that it is a homomorphism, then that it is bijective.

(b) (10 points) Show carefully that if $g, h \in G$, then $c_g \circ c_h = c_{g \cdot h}$. Thus the map $g \mapsto c_g$ is a homomorphism

$$f : G \to Aut(N)$$

where $Aut(N)$ is the group of automorphisms of N.

(c) (5 points) Suppose G/N is of prime order p. Show that $\ker(f) = G$ if and only if G is abelian.