INTRODUCTION TO MODERN ALGEBRA I, GU4041, FALL 2023

Midterm I, October 10, 2023

For any positive integer m, we denote by $\mathbb{Z}_m$ a cyclic group with m elements.

1. True or False? (Each question is worth 5 points.) If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) Let X be a set and let $R \subset X \times X$ be a relation. If R is an equivalence relation then its complement $X \times X \setminus R$ is *not* an equivalence relation.

(b) Let G be a set with two operations $\star : G \times G \to G$ and $\circ : G \times G \to G$. Suppose both of these operations make G into a group. Then the operation

$$\times : G \times G \to G : g \times h = (g \star h) \circ (g \star h)$$

makes G into a group.

(c) Let G be a group and let A be the set of bijective homomorphisms from G to itself. Then A is a group.

2. (15 points) (a) Carry out the following operations in modular arithmetic.

(i) In arithmetic modulo 29, solve the equation

$$[2] \cdot x + [5] = [12].$$

Exhibit the answer as the residue class of an integer between 0 and 28.

(ii) In arithmetic modulo 9 find the number a between 1 and 9 such that

$$1000000004^2 \equiv a \pmod{9}.$$

(b) List the set of generators of the group $\mathbb{Z}_8$.

3. (15 points) Which of the following is an equivalence relation? Justify your answer.

(a) On the set X of New York City streets, we say $a \sim b$ if a and b intersect at a corner.

(b) On the set $\mathbb{N}$ of natural numbers, we say $a \sim b$ if a/b is the square of a rational number.

(c) On the set $\mathbb{C}$ of complex numbers, we say $a \sim b$ if a is contained in the circle of radius 2 around b.

4. (20 points) Let G be a group with identity e and $g \in G$. Suppose there is no integer $n > 100$ such that $g^n = e$. Prove carefully that G is then an infinite group.

5. (15 points) (a) Let $M(2, \mathbb{R})$ be the set of 2×2 matrices. Show that it is not a group under multiplication. Which of the group axioms does $M(2, \mathbb{R})$ satisfy under multiplication?

(b) Let $I \in M(2, \mathbb{R})$ be the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. (You should have mentioned this matrix in part (a). Find two elements $x, y \in M(2, \mathbb{R})$ such that $x^2 = y^2 = I$ but x and y are not conjugate matrices.

(c) (Extra credit) Can you find three distinct matrices x, y, z with $x^2 = y^2 = z^2 = I$ such that no two are conjugate?

6. (20 points) List the sets of cyclic subgroups of the dihedral group D_6 and of the group $\mathbb{Z}_3 \times \mathbb{Z}_4$.