Etale Cohomology

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This is a personal notes by Haodong Yao to study etale cohomology. It is a combination of reading Milne's *Etale Cohomology* book and watching Daniel Litt's lecture videos on : https://www.daniellitt.com/tale-cohomology. Whenever there is a tag, e.g. 0A0C, it refers to the Stacks Project. If you have any questions on this please feel free to contact me at haodong@math.columbia.edu. Any remarks and corrections are in particular welcome.

Introduction

Staring from Gauss's work on the reciprocity laws, the information of the number of solutions of polynomial equations in finite fields might be packed up into certain functions. In 1923, E. Artin observed the analogue of Riemann Zeta function $\zeta_K(s)$ in the function field case provides such an example. He also noticed that $\zeta_K(s) = Z(q^{-s})$ where $Z(u)$ is a rational function, satisfies a functional equation and all its zeros have absolute value $q^{\frac{1}{2}}$. From the Euler product it follows that the number of solutions are closely related to coefficients of $\log Z(u)$.

Similarly $Z_V(u)$ could be defined using an Euler product for any smooth hypersurface V over $\overline{k}$ where the base field $k = \mathbb{F}_q$ is finite. Expand $uZ'_V(u)/Z_V(u)$ gives the generating series $\sum_{v=1} N_v u^v$ where N_v is the number of points of V in $\mathbb{F}_{q^v}$. In 1931 F. K. Schmidt showed that for a projective smooth curve C of genus g defined over $\mathbb{F}_q$ the zeta function

$$Z_C(u) = \frac{P_{2g}(u)}{(1-u)(1-qu)}$$

where P_{2g} is a polynomial of degree $2g$ with integral coefficients and one has the function equation

$$Z_C(1/qu) = (qu^2)^{1-g} Z_C(u)$$

The *Riemann hypothesis* for C was therefore that the zeros of P_{2g} have absolute value $q^{\frac{1}{2}}$.

One special point about finite field is the Frobenius morphism Φ. In 1936, Hasse observed that for a curve C the number N_v is exactly the number of fixed points of Φ^v. By Lefschetz trace formula this number could be read from the action of Φ on cohomology groups

$$N_v = \sum_i (-1)^i \operatorname{Tr}((\Phi^v)^{(i)})$$

More generally if it could be possible to attach any hypersurface V over $\mathbb{F}_q$ a 'suitable' cohomology algebra $H^\bullet(V)$ then one could show that $Z_V(u)$ is rational, satisfies a functional equation and *Riemann hypothesis* on its zeros.

More precisely let X_0 be a smooth, absolutely irreducible, projective algebraic variety of dimension d over a finite field $k = \mathbb{F}_q$. Let N_v be the number of points of X_0 in $\mathbb{F}_{q^v}$. Let

$$\zeta_{X_0}(s) = Z(q^{-s}) = \prod_x (1 - (\mathrm{Norm}(x))^{-s})^{-1}$$

where x runs through all the closed points of X_0 and the norm of x is the number of elements in the residue field of x. Then one could show

$$Z'(u)/Z(u) = \sum_{v=1} N_v u^{v-1}$$

In 1949 Weil conjectured that:

1 $Z(u)$ is a rational function.

2 $Z(u)$ has a functional equation (hence so has $\zeta_{X_0}(s)$).

3
$$Z(u) = \frac{P_1(u)P_3(u)\dots P_{2d-1}(u)}{P_0(u)P_2(u)\dots P_{2d}(u)}$$
where $P_0(u) = 1-u$, $P_{2d}(u) = 1-q^d u$ and each $P_j(u)$ is a polynomial with integer coefficients with $P_j(0) = 1$. Furthermore, the roots of $P_j(u)$ all have the same absolute value $q^{j/2}$.

4 If $X = X_0 \otimes \overline{k}$ arises by 'good' reduction of constants from a complex algebraic variety $\widetilde{X}$ then the degree of $P_j(u)$ is precisely the j-th Betti number of $\widetilde{X}$.

Now consider the category of smooth irreducible projective algebraic varieties Y over a fixed algebraically closed field k. A Weil cohomology theory with coefficients in a fixed field L of char 0 is a series of contravariant functors $\{H^i(Y, L)\}_{i=0,1,2,\dots}$ with values in the category of vector spaces over L satisfying

1 The space $H^i(Y, L)$ are finite dimensional over L and vanish for $i > 2\dim(Y) = 2d$. $\dim H^0(Y, L) = \dim H^{2d}(Y, L) = 1$.

2 [Poincare duality] The product

$$H^i(Y, L) \times H^{2d-i}(Y, L) \longrightarrow H^{2d}(Y, L) \xrightarrow{\sim} L$$

is a perfect pairing, inducing identification of $H^{2d-i}(Y, L)$ with $H^i(Y, L)^\vee$.

3 There is a Kunneth formula.

4 There is a Lefschetz fixed point formula.

Other conditions might vary from people to people, including

a If Y arises by 'good' reduction of constants from a complex algebraic variety $\widetilde{Y}$ with Betti numbers $b_0, \dots, b_{2d}$ then

$$\dim H^i(Y, L) = b_i$$

b If Y' is a smooth subvariety of Y of dimension $d-1$ there are natural linear mappings $H^i(Y, L) \to H^i(Y', L)$ which are bijective for $i \le d-2$ and injective for $i = d-1$.

c If $h \in H^2(Y, L)$ corresponds by Poincare Duality to the homology class in $H^{2d-2}(Y, L)^\vee$ of a hyperplane section of Y then $L\colon a \mapsto h \cdot a$ gives isomorphisms $L^{d-i}\colon H^i(Y, L) \to H^{2d-i}(Y, L)$ for $i \le d$.

The requirements made are closely related to the Weil conjecture, for example, the Poincare duality will give the functional equation and the Lefschetz fixed point formula will give the rationality. The pursuit of a Weil cohomology theory gives birth to the etale cohomology. We shall define the etale cohomology theory, show it satisfies the conditions above, and study how it is applied in the proof of the Weil conjecture.

Notations

All rings are Noetherian and all schemes are locally Noetherian unless otherwise specified.

CHAPTER 1

Etale Morphisms

Slogan: A flat morphism is the algebraic analogue of a map whose fibres form a continuously varying family.

Slogan: Locally an etale morphism is an isomorphism.

Slogan: The strictly Henselian rings play the same role for the etale topology as local rings play for the Zariski topology.

Slogan: The fundamental group of a scheme classifies finite etale coverings.

1. Finite and Quasi-finite Morphisms

Finite morphism, condition only needed to be checked for an affine open covering.

Example 1.1. *Let X be an integral scheme with field of rational functions $R(X)$ and let L be a finite field extension of $R(X)$. The normalization of X in L is a pair (X', f) where X' is an integral scheme with $R(X') = L$ and $f\colon X' \to X$ is an affine morphism such that for all affine opens U of X, $\Gamma(f^{-1}U, \mathcal{O}_{X'})$ is the integral closure of $\Gamma(U, \mathcal{O}_X)$ in L. The existence is from relative normalization. If X is normal and L is finite separable over $R(X)$ then f is finite.*

Proposition 1.2. *A closed immersion is finite. Composite of finite morphisms is finite. Base change of finite morphism is finite. Finite morphism is proper.*

Proposition 1.3. *Let $f\colon X \to \operatorname{Spec} k$ be a morphism of finite type with k a field. TFAE.*

1 *X affine and is the spectrum of an Artin ring.*
2 *X finite and discrete.*
3 *X is finite.*
4 *X discrete.*
5 *f finite.*

Proof. See Atiyah-Macdonald Chapter 8 exercises 2,3 and 02NG. □

Definition 1.4. *An A-algebra B is quasi-finite if it is of finite type and $B \otimes_A \kappa(\mathfrak{p})$ is a finite $\kappa(\mathfrak{p})$-module for all $\mathfrak{p}$ prime. A morphism is quasi-finite if it is of finite type and has finite fibres. By the Proposition above we see $A \to B$ is quasi-finite iff $\operatorname{Spec} B \to \operatorname{Spec} A$ is quasi-finite and finite morphisms are quasi-finite.*

Being quasi-finite is a local property for ring maps. A quasi-compact morphism is quasi-finite iff locally it is quasi-finite.

Remark 1.5. *$A[T]/(P(T))$ is a quasi-finite A-algebra iff all coefficients of P generate the unit ideal. In case A is DVR, iff some coefficient of P is a unit. In case A is domain, it is finite iff the leading coefficient of P is a unit.*

Let A be a Noetherian domain and not a field and K be its fraction field. Then K is not finite A-algebra. In general a finite type morphism from a finite scheme is always quasi-finite. The fraction field of a Dedekind domain is finite type over it iff the Dedekind domain has only finitely many prime ideals.

Proposition 1.6. *An immersion is quasi-finite by Noetherian condition. Composite of quasi-finite morphisms is quasi-finite. Base change of quasi-finite morphism is quasi-finite.*

Theorem 1.7 (Zariski's Main Theorem). *If X quasi-compact, then any separated quasi-finite morphism $f\colon Y \to X$ factors through $Y \xrightarrow{f'} Y' \xrightarrow{g} X$ where f' is an open immersion and g finite. (Locally Noetherian scheme is quasi-separated, in general see 05K0)*

Remark 1.8. *There are many different versions of Zariski's Main Theorem. This one is reformulated by Grothendieck.*

Corollary 1.9. *Any proper quasi-finite morphism is finite.*

Remark 1.10. *If X is the affine line with origin doubled and $f\colon X \to \mathbb{A}^1$ is the natural map then f is universally closed and quasi-finite (even flat and etale) but not finite.*

Remark 1.11. *Let $f\colon Y \to X$ be separated and of finite type with X irreducible. If the fibre of the generic point $\eta \in X$ is finite then there is an open neighborhood $U \subset X$ of η such that $f^{-1}U \to U$ is finite.*

Proof. See 02ML. □

2. Flat Morphisms

Proposition 1.12. *A homomorphism $f\colon A \to B$ is flat iff the map*

$$I \otimes_A B \longrightarrow B \ , \ a \otimes b \longmapsto f(a)b$$

is injective for all ideals I of A.

Proposition 1.13. *If $f\colon A \to B$ is flat, then so is $S^{-1}A \to T^{-1}B$ for all multiplicative subsets $S \subset A$ and $T \subset B$ such that $f(S) \subset T$. Conversely if $A_{f^{-1}\mathfrak{m}} \to B_{\mathfrak{m}}$ is flat for all maximal ideals $\mathfrak{m}$ of B, then $A \to B$ is flat.*

Proof. $S^{-1}A \to S^{-1}B$ is flat and $S^{-1}B \to T^{-1}S^{-1}B$ is flat.
$(B \otimes_A M)_{\mathfrak{m}} \cong B_{\mathfrak{m}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. □

Remark 1.14. *If $a \in A$ is a nonzerodivisor and $f\colon A \to B$ flat then $f(a)$ nonzerodivisor in B. If A integral domain and B nonzero then f injective.*
Conversely any injective map from Dedekind domain to integral domain is flat. Localize at maximal ideals to get an injective map from DVR to domain. The generator is not mapped to zero.

Definition 1.15. *A morphism $f\colon Y \to X$ is flat if for every $y \in Y$ the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat. Equivalently f is flat if locally it is flat. Being flat is a local property for ring maps.*
If in addition every point in Y specializes to a closed point then it suffices to check f flat at all closed points of Y.

Proposition 1.16. *An open immersion is flat. The composite of two flat morphisms is flat. Base change of flat morphism is flat.*

Proposition 1.17. *Let B be a flat A-algebra. If the image of $b \in B$ in $B/(\mathfrak{m} \cap A)B$ is nonzerodivisor for any maximal ideal $\mathfrak{m}$ of B then $B/(b)$ is flat A-module.*

Proof. By localizing we may assume $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a local homomorphism. By assumption if $c \in B$ and $bc = 0$ then $c \in \mathfrak{m}B$. By the equation criterion for flatness of B over A we see actually $c \in \mathfrak{m}^r B$ for all r hence $c \in \bigcap \mathfrak{m}^r B \subset \bigcap \mathfrak{n}^r = 0$ hence b is a nonzerodivisor for B. Similarly for any ideal of A after base change to A/I we see b is a nonzerodivisor for B/IB. Then it is easy to show $I \otimes_A B/(b) \to B/(b)$ is injective. □

Remark 1.18.

a *For any ring A, $A[x_1,\dots,x_n]$ is a free A-module hence flat. Let $P\neq 0$ and $Z=A[x_1,\dots,x_n]/(P)$. Then we see Z is flat over A if for all maximal ideals $\mathfrak{m}$ of A, $Z\otimes_A \kappa(\mathfrak{m})\neq \kappa(\mathfrak{m})[x_1,\dots,x_n]$, iff the ideal generated by coefficients of P is A.*

Assume $\operatorname{Spec} A$ *is connected. Let $I\subset A$ be the ideal generated by coefficients of P. Claim if Z is flat over A then $I=A$. Suppose $I\neq A$ and consider the nonempty closed subset $V(I)\subset \operatorname{Spec} A$.*

Case 1: If there exists a prime ideal $\mathfrak{p}\subset A$ contained in a maximal ideal $\mathfrak{m}$ such that $\mathfrak{p}\notin V(I)$ and $I\subset\mathfrak{m}$ then after quotient $\mathfrak{p}$ we may assume A is domain. Then after localizing at $\mathfrak{m}$ we may assume A is local.

Case 2: If no such $\mathfrak{p}$ exists then for every $\mathfrak{p}\notin V(I)$ we must have $V(I)\cap V(\mathfrak{p})=\emptyset$. Since A Noetherian, A has only finitely many minimal prime ideals and thus $V(I)^c$ is a finite union of closed subsets, being both open and closed hence empty, i.e. $V(I)=\operatorname{Spec} A$. Then consider the proper ideal $Ann(I)\subset A$. There is a maximal ideal $\mathfrak{m}$ of A containing $Ann(I)$. Then after localizing at $\mathfrak{m}$ we may assume A is local.

Thus we may take A to be local. Consider the map

$$I\otimes_A A[x_1,\dots,x_n]/(P)\longrightarrow A[x_1,\dots,x_n]/(P)$$

Write $P=\sum_J a_J x_J$ then the element $\sum_J a_J\otimes x_J$ is mapped to 0 and we want to show this is a nonzero element. Suppose $I\subset\mathfrak{m}$ and consider the map

$$\begin{aligned}I\otimes_A A[x_1,\dots,x_n]/(P)&\longrightarrow I\otimes_A A[x_1,\dots,x_n]/\mathfrak{m}[x_1,\dots,x_n]\\&=I\otimes_A\kappa(\mathfrak{m})[x_1,\dots,x_n]\\&=(I\otimes_A\kappa(\mathfrak{m}))[x_1,\dots,x_n]\end{aligned}$$

thus if $\sum_J a_J\otimes x_J=0$ then its image $\sum_J(a_J\otimes 1)x_J$ is 0 in $(I\otimes_A\kappa(\mathfrak{m}))[x_1,\dots,x_n]$. But then each $a_J\otimes 1=0$ and by Nakayama Lemma we see $I=0$ contradiction!

Alternatively we have $I\otimes_A A[x_1,\dots,x_n]/(P)=I[x_1,\dots,x_n]/P\cdot I[x_1,\dots,x_n]$. Then it suffices to show $P\notin P\cdot I[x_1,\dots,x_n]$. Suppose $P=f\cdot P$ for some $f\in I[x_1,\dots,x_n]$ then $I=I^2$. In Case 1 where A is domain we have $I=\bigcap_{i>0}I^i=0$ and in Case 2 where I is nilpotent we have $I=0$, contradiction!

Similar statements hold for hypersurfaces in $\mathbf{P}^n_A$.

b *Part a generalizes to say if $f\colon Y\to X$ is flat then*

$$\dim(\mathcal{O}_{Y_x,y})=\dim(\mathcal{O}_{Y,y})-\dim(\mathcal{O}_{X,x})$$

where $x=f(y)$. For varieties this means $\dim(Y_x)=\dim Y-\dim X$ for any $x\in X$ closed with Y_x nonempty.

Conversely in case X and Y are regular and $f\colon Y\to X$ satisfies the equation for dimensions of fibres for all closed $y\in Y$ then f is flat.

c *Suppose we have $X\xrightarrow{f}Y\xrightarrow{h}S$ with X,Y,S locally Noetherian or f,h locally of finite presentation and let $g=hf$. If g flat and f_s flat for all $s\in S$ then f is flat and h is flat at $f(X)$.*

This essentially comes from the lemma (Ulrich, Torsten, Algebraic Geometry I, 14.25):

LEMMA 1.19. *Let $A\to B\to C$ be local maps of Noetherian local rings and let M be a finite C-module. Then M is flat A-module, $M\otimes_A\kappa(A)$ is flat $B\otimes_A\kappa(A)$-module iff B is flat A-module, M is flat B-module.*

d *If B is flat over A and $b_1,\dots,b_n\in B$ such that their images in $B/\mathfrak{m}B$ is a regular sequence for each $\mathfrak{n}\subset B$ maximal and $\mathfrak{m}=\mathfrak{n}\cap A$ then $B/(b_1,\dots,b_n)$ is flat over A.*

e *Let X be an integral scheme and Z a closed subscheme of $\mathbf{P}^n_X$. For each $x\in X$ let $p_x\in\mathbb{Q}[T]$ be the Hilbert polynomial of the fibre $Z_x\subset\mathbf{P}^n_{\kappa(x)}$. Then Z is flat over X iff p_x is independent of x. (Hartshorne III 9.9)*

Faithfully flat morphism of rings is injective.

PROPOSITION 1.20. *Let $f\colon A\to B$ be a flat morphism with $A\neq 0$. TFAE.*

a *f is f.f.*
b *a sequence of A-modules is exact whenever after tensoring with B it is exact.*
c $\operatorname{Spec} f$ *is surjective.*
d *for every maximal ideal $\mathfrak{m}$ of A, $f(\mathfrak{m})B \neq B$.*

COROLLARY 1.21. *Let $f\colon Y \to X$ be flat. Let $y \in Y$ and $x' \in X$ such that $x = f(y) \in \overline{\{x'\}}$. Then there is y' such that $y \in \overline{\{y'\}}$ and $f(y') = x'$. In particular flat morphisms map generic points to generic points.*

DEFINITION 1.22. *A morphism is faithfully flat if it is flat and surjective.*

PROPOSITION 1.23. *Let M be a finite A-module. TFAE.*

a *M is flat.*
b *$M_{\mathfrak{m}}$ is free $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of A.*
c *$\widetilde{M}$ is a locally free sheaf on* $\operatorname{Spec} A$.
d *M is a projective A-module.*
and if A domain
e $\dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p}))$ *is the same for all prime ideals $\mathfrak{p}$ of A.*

THEOREM 1.24. *Any flat morphism locally of finite type is open.*

PROOF. Chevalley's Theorem + Going Down condition. □

REMARK 1.25. *If $f\colon Y \to X$ is finite and flat, then it is open and closed. If X connected and Y nonempty, then f is surjective hence f.f.*

EXAMPLE 1.26. *Consider $A = k[t] \to B = k(t)$. Then B is flat over A. Thus* $\operatorname{Spec} B \coprod \operatorname{Spec} A \to \operatorname{Spec} A$ *is flat and surjective but the image of* $\operatorname{Spec} B$ *is not open.*

THEOREM 1.27. *Let $f\colon Y \to X$ be locally of finite-type. The set of points $y \in Y$ such that $\mathcal{O}_y$ is flat over $\mathcal{O}_{f(y)}$ is open in Y.*

PROOF. See 0399. □

Next we shall look into descent theory.

DEFINITION 1.28. *Let $\mathcal{C}$ be a category with fibre products. A morphism $Y \to X$ is called a strict epimorphism if the pull back diagram*

$$Y \times_X Y \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y \longrightarrow X$$

is a coequalizer, i.e. we have equalizer

$$\operatorname{Hom}(X, Z) \longrightarrow \operatorname{Hom}(Y, Z) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \operatorname{Hom}(Y \times_X Y, Z)$$

for all Z.

THEOREM 1.29. *Any f.f. morphism of finite type is a strict epimorphism.*

PROPOSITION 1.30. *If $f\colon A \to B$ is f.f. then the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{d^0} B^{\otimes 2} \longrightarrow \dots \longrightarrow B^{\otimes r} \xrightarrow{d^{r-1}} B^{\otimes r+1} \longrightarrow \dots$$

is exact where

$$d^{r-1} = \sum (-1)^i e_i \ , \ e_i(b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}$$

Proof. Clearly $d^r d^{r-1} = 0$. If f has a section then construct $k_r\colon B^{\otimes r+2} \to B^{\otimes r+1}$ by

$$k_r(b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0)b_1 \otimes \cdots \otimes b_{r+1}$$

it is easy to show k_r is a null homotopy thus the sequence is exact. In general tensor the sequence with B to get a section. □

Corollary 1.31. *Similarly if $f\colon A \to B$ is f.f. then for any A-module M the sequence*

$$0 \longrightarrow M \longrightarrow M \otimes_A B \xrightarrow{1\otimes d^0} M \otimes_A B^{\otimes 2} \longrightarrow \dots \longrightarrow M \otimes_A B^{\otimes r} \xrightarrow{1\otimes d^{r-1}} M \otimes_A B^{\otimes r+1} \longrightarrow \dots$$

is exact.

Proof of Theorem 1.29.

Proof. We have to show for any Z and any $h\colon Y \to Z$ with $hp_1 = hp_2$ there is a unique $g\colon X \to Z$ with $gf = h$.

Case (a): In case X, Y, Z are all affine apply the Proposition above.

Case (b): If X, Y are all affine and Z arbitrary. Firstly show the uniqueness by case (a). Then construct g locally. Let $x \in X$ and $y \in f^{-1}(x)$ and let U be an affine open neighborhood of $h(y)$ in Z. Then $f(h^{-1}U)$ is open in X thus we can find $x \in X_a \subset f(h^{-1}U)$. By $hp_1 = hp_2$ we see $f^{-1}X_a \subset h^{-1}U$. Since $f^{-1}X_a$ is also affine open reduce to case (a).

Case (c): In general again reduce to the case X affine by similar argument as in (b). Since f quasi-compact we can cover Y with a finite union of affine opens. Let Y' be the disjoint of there affines. Then Y' is affine and $Y' \to X$ is f.f. of finite type. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(X,Z) & \longrightarrow & \mathrm{Hom}(Y,Z) & \rightrightarrows & \mathrm{Hom}(Y \times_X Y, Z) \\
\| & & \downarrow & & \downarrow \\
\mathrm{Hom}(X,Z) & \longrightarrow & \mathrm{Hom}(Y',Z) & \rightrightarrows & \mathrm{Hom}(Y' \times_X Y', Z)
\end{array}$$

to show the first row is exact. □

Example 1.32. *Let k be a field. Consider the map $f\colon \operatorname{Spec} k[T] \to \operatorname{Spec} k[T^3, T^5]$. At ring level it is injective and integral hence on spectra it is surjective. Then it is easy to see f is an epimorphism. Consider the map $k[T] \to k[T]$ sending T to T^7. Clearly it fails to factor through $k[T^3, T^5]$. Let $u = T \otimes 1, v = 1 \otimes T \in k[T] \otimes_{k[T^3,T^5]} k[T]$. Note that $uv = vu$ and $u^3 = v^3, u^5 = v^5$. Then*

$$u^7 - v^7 = (u^2 + v^2)(u^5 - v^5) - u^2v^2(u^3 - v^3) = 0$$

and this shows f is not a strict epimorphism.

Remark 1.33. *Let $f\colon A \to B$ be f.f. and let M be an A-module. Write M' for the B-module $f_*M = B \otimes_A M$. The module $(e_0)_*M' = (B \otimes_A B) \otimes_B M'$ may be identified with $B \otimes_A M'$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(b \otimes m) = b_1 b \otimes b_2 m$. Similarly $(e_1)_*M'$ may be identified with $M' \otimes_A B$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(m \otimes b) = b_1 m \otimes b_2 b$. There is a canonical isomorphism $\phi\colon (e_1)_*M' \to (e_0)_*M'$ arising from $e_1 f = e_0 f$. Explicitly it is given by*

$$M' \otimes_A B \longrightarrow B \otimes_A M' \ , \ (b \otimes m) \otimes b' \longmapsto b \otimes (b' \otimes m)$$

and M is recovered from the pair (M', ϕ) by

$$M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}$$

Conversely given a pair (M', ϕ) where $\phi\colon M' \otimes_A B \to B \otimes_A M'$ define

$$\phi_1\colon B \otimes_A M' \otimes_A B \to B \otimes_A B \otimes_A M'$$
$$\phi_2\colon M' \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M'$$
$$\phi_3\colon M' \otimes_A B \otimes_A B \to B \otimes_A M' \otimes_A B$$

by tensoring ϕ with id_B in the first, second, third positions respectively. Then the pair (M', ϕ) arises from some A-module M as above iff $\phi_2 = \phi_1\phi_3$.

Assume the condition. Define $M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}$. This is an A-module. We want to show the canonical map $(b \otimes m \mapsto bm)\colon B \otimes_A M \to M'$ is an isomorphism. Consider the diagram

$$\begin{array}{ccc}
M' \otimes_A B & \overset{m\otimes b \longmapsto 1\otimes m\otimes b}{\underset{m\otimes b \longmapsto \phi(m\otimes 1)\otimes b}{\rightrightarrows}} & B \otimes_A M' \otimes_A B \\
\downarrow \phi & & \downarrow \phi_1 \\
B \otimes_A M' & \overset{b\otimes m \longmapsto 1\otimes b\otimes m}{\underset{b\otimes m \longmapsto b\otimes 1\otimes m}{\rightrightarrows}} & B \otimes_A B \otimes_A M'
\end{array}$$

by $\phi_2 = \phi_1\phi_3$ we see the diagram commutes with either the upper or the lower horizontal maps. Hence their equalizers are isomorphic, which is exactly given by the canonical map above.

Proposition 1.34. *Let $f\colon Y \to X$ be f.f. and quasi-compact. To give a quasi-coherent $\mathcal{O}_X$-module M is the same as to give a quasi-coherent $\mathcal{O}_Y$-module M' with an isomorphism $\phi\colon p_1^*M' \to p_2^*M'$ satisfying $p_{31}^*(\phi) = p_{32}^*(\phi)p_{21}^*(\phi)$ where p_{ij} are projections $Y \times_X Y \times_X Y \to Y \times_X Y$.*

By using the relation between schemes affine over a scheme and quasi-coherent algebras we also have

Proposition 1.35. *Let $f\colon Y \to X$ be f.f. and quasi-compact. To give a scheme Z affine over X is the same as to give a scheme Z' affine over Y with an isomorphism $\phi\colon p_1^*Z' \to p_2^*Z'$ satisfying $p_{31}^*(\phi) = p_{32}^*(\phi)p_{21}^*(\phi)$.*

Remark 1.36. *Suppose we have a fibre product*

$$\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow f' & & \downarrow f \\
X' & \longrightarrow & X
\end{array}$$

where $X' \to X$ is f.f. and quasi-compact or f.f. and locally of finite type, then if f' is quasi-compact/separated/of finite type/proper/open immersion/affine/finite/quasi-finite/flat/smooth/etale then so is f.

If $f\colon Y \to X$ is f.f. and Y integral/normal/regular then so is X.

Suppose Y is integral. Since f is surjective X is also irreducible. Then check reducedness on stalks.

Suppose Y is normal. The map on stalks $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is faithfully flat hence injective. Then $\mathcal{O}_{Y,y}$ being normal implies $\mathcal{O}_{X,x}$ being normal by considering for any $a/b \in \mathcal{O}_{X,x}$ integral the map $\mathcal{O}_{X,x} \xrightarrow{a} \mathcal{O}_{X,x}/(b)$.

Suppose Y is regular. To show X is regular reduces to show that if $R \to S$ is a flat map between local Noetherian rings and S regular so is R. This is done by taking a finite projective resolution of residue field of R and tensoring with S.

Remark 1.37. *Similar results hold for fpqc descent. The first part in Remark 1.36 is 02KN and the second part is 02KJ and 033D. Numerous constructions in algebraic geometry are made using techniques of descent, such as constructing objects over a given space by first working over a somewhat larger space which projects down to the given space, or verifying a property of a space or a morphism by pulling back along a covering map.*

Proposition 1.38. *Let X be quasi-compact and $f\colon Y \to X$ be f.f. and locally of finite type. Then there is an affine scheme X' with a f.f. quasi-finite morphism $h\colon X' \to X$ and an X-morphism $g\colon X' \to Y$.*

3. Etale Morphisms

Definition 1.39. *Let k be a field and $\overline{k}$ be its algebraic closure. A k-algebra A is called separable if $\overline{A} = A \otimes_k \overline{k}$ has zero intersection of all maximal ideals.*

Proposition 1.40. *Let A be a finite algebra over field k. TFAE.*

a *A is separable over k.*
b *$\overline{A}$ is isomorphic to a finite product of $\overline{k}$.*
c *A is isomorphic to a finite product of separable field extensions of k.*
d *the discriminant of any basis of A over k is nonzero.*

Proof. Criteria for separable field extensions. □

Definition 1.41. *A locally finite type morphism $f\colon Y \to X$ is said to be unramified at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable field extension of $\kappa(x)$ where $x = f(y)$. A morphism is unramified if it is unramified everywhere. A morphism is unramified if locally it is unramified.*

Proposition 1.42. *Let $f\colon Y \to X$ locally of finite-type. TFAE.*

a *f unramified.*
b *for all $x \in X$, the fibre $Y_x \to \operatorname{Spec}\kappa(x)$ unramified.*
c *for all $\operatorname{Spec} k \to X$ with k separably closed the base change map is unramified.*
d *for all $x \in X$, Y_x has an open covering by spectrum of finite separable $\kappa(x)$-algebras.*
e *for all $x \in X$, Y_x is isomorphic to $\coprod \operatorname{Spec} k_i$ where k_i are finite separable field extensions of $\kappa(x)$, in particular if f is qc then f is quasi-finite.*

Proof. Use Proposition 1.40. □

Note that under the above definition even a closed immersion is unramified. This is weird since it is not the case in Riemann surfaces, i.e. not a local isomorphism. We need more restricted notion.

Definition 1.43. *A morphism of rings/schemes is called etale if it is flat and unramified (locally of finite type). A morphism is etale if locally it is etale.*

Proposition 1.44. *Open immersion is etale. The composite of two etale morphisms is etale. Base change of etale morphism is etale.*

Proof. Enough to check for unramified. Any immersion is unramified. Composite of unramified ring maps is unramified. For base change use Proposition 1.42 a and c. □

Example 1.45. *Let k be a field and P be a monic polynomial. Then $k[T]/P$ is separable/unramified/etale over k iff P is separable.*

This generalizes. A monic polynomial $P(T) \in A[T]$ is called separable if $(P, P') = A[T]$. Then P is separable iff its image is separable in $\kappa(\mathfrak{p})[T]$ for all prime ideals $\mathfrak{p}$ in A (Assume $(P, P') \subset \mathfrak{m}$ then consider $\mathfrak{p} = A \cap \mathfrak{m}$).

Let $B = A[T]/(P)$ where P is a monic polynomial. Then B is free A-module. Hence B is unramified (etale) over A iff P is separable. More generally for any $b \in B$, B_b is etale over A iff P' is a unit in B_b (Reduce to the field case then decompose P).

For example $B = A[T]/(T^r - a)$ is etale over A iff ra invertible in A.

For algebras generated by more than one element, there is the Jacobian criterion: Let $C = A[T_1, \ldots, T_n]$ and $P_1, \ldots, P_n \in C$ and $B = C/(P_1, \ldots, P_n)$. Then B is etale over A iff the image of $\det(\partial P_i/\partial T_j)$ is a unit in B.

In case Y and X are analytic manifolds this criterion indicates that it is an isomorphism on tangent spaces, hence locally an isomorphism.

Proposition 1.46. *Let $f\colon Y \to X$ be locally of finite type. TFAE.*

a *f is unramified.*
b *the sheaf $\Omega^1_{Y/X}$ is zero.*
c *the diagonal morphism $\Delta_{Y/X}\colon Y \to Y \times_X Y$ is an open immersion.*

Corollary 1.47. *Consider morphisms $Y \xrightarrow{g} X \xrightarrow{f} S$. If fg etale and f unramified then g is etale.*

Remark 1.48. *The notion of being unramified agrees with the notion in number theory for extension of rings of integral element for number fields. Moreover if $f\colon Y \to X$ is LFT then the annihilator of $\Omega^1_{Y/X}$ is called the different $\delta_{Y/X}$ of f. The closed subscheme of Y defined by $\delta_{Y/X}$ is called the branch locus of f. The open complement of the branch locus is precisely the set where $\Omega^1_{Y/X} = 0$. The theorem of purity of branch locus states that the branch locus if nonempty has pure codimension one in Y in each of the two cases: when f is f.f. and finite or when f is quasi-finite and dominant with Y regular and X normal.*

Proposition 1.49. *If $f\colon Y \to X$ is locally of finite type, then the set of points $y \in Y$ such that $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,f(y)}$ and $\Omega^1_{Y/X,y} = 0$ is open in Y. Thus there is a unique largest open set U in Y on which f is etale.*

Corollary 1.50. *A local homomorphism $A \to B$ is etale iff $\mathfrak{m}_A B = \mathfrak{m}_B$ and $\kappa(A) \to \kappa(B)$ is a finite separable extension.*

Remark 1.51. *Let $f\colon Y \to X$ be finite flat and X connected. Then $f_*\mathcal{O}_Y$ is locally free $\mathcal{O}_X$-module of constant rank r. There is a sheaf of ideals $\mathcal{D}_{Y/X}$ on X called the discriminant of f such that if U is an open affine in X with $B = \Gamma(f^{-1}U, \mathcal{O}_Y)$ free with basis $\{b_1, \dots, b_r\}$ over $A = \Gamma(U, \mathcal{O}_X)$ then $\Gamma(U, \mathcal{D}_{Y/X})$ is the principal ideal generated by $\det(\mathrm{tr}_{B/A}\, b_i b_j)$. Moreover f is unramified at all $y \in f^{-1}x$ iff $(\mathcal{D}_{Y/X})_x = \mathcal{O}_{X,x}$. Thus if f is unramified at all $y \in f^{-1}x$ for some $x \in X$ then there exists an open neighborhood V of x such that $f|_{f^{-1}V}$ is etale. If $B = A[T]/P$ with P monic then $\mathcal{D}_{B/A} = (D(P))$ where $D(P)$ is the discriminant of P, i.e. the resultant $\mathrm{res}(P, P')$, and $\delta_{B/A} = (P'(t))$ where $t = T \bmod P$.*

Proposition 1.52. *Any flat closed immersion is open immersion.*

Proof. Flat morphism locally of finite type is open. Hence we may assume f surjective. As f finite flat, $f_*\mathcal{O}_Y$ is locally free as an $\mathcal{O}_X$-module. Since f closed immersion, $\mathcal{O}_X = f_*\mathcal{O}_Y$. □

Remark 1.53. *Any etale, universally injective, separated morphism is an open immersion.*

Corollary 1.54. *If X is connected and $f\colon Y \to X$ is etale (etale and separated) then any section of f is an open immersion (an isomorphism onto an open connected component). Thus such sections are bijective with open (open and closed) subschemes Y_i of Y such that f induces an isomorphism $Y_i \to X$. In particular if f separated then a section is known when its value at a point is known.*

Proof. Assume f is etale and separated. Then a section s is a closed immersion and etale hence an open immersion. Thus s is an isomorphism onto its image, being an open and closed connected subset of Y hence an open connected component. □

Corollary 1.55. *Let $f, g\colon Y' \to Y$ be X-morphisms where Y' is connected and Y is etale and separated over X. If for some $y' \in Y'$ we have $f(y') = g(y') = y \in Y$ and the induced maps $\kappa(y) \to \kappa(y')$ are the same then $f = g$.*

Proof. The maps $f', g'\colon Y' \to Y' \times_X Y$ are sections of the projection and they agree at a point. □

Definition 1.56. *A standard etale morphism is an etale morphism of the form* $\operatorname{Spec} B_b \to \operatorname{Spec} A$ *where* $B = A[T]/(P)$ *for a monic polynomial* P *and* P' *unit in* B_b.

Theorem 1.57. *Assume* $f: Y \to X$ *is etale in some open neighborhood of* $y \in Y$. *Then there are affine open neighborhoods* V *and* U *of* y *and* $f(y)$ *such that* $f|_V: V \to U$ *is a standard etale morphism.*

Remark 1.58. *The same argument shows that unramified morphism locally is a composite of a closed immersion with a standard etale morphism.*

Corollary 1.59. *A morphism* $f: Y \to X$ *is etale iff for every* $y \in Y$ *there exists an affine open neighborhood* $V = \operatorname{Spec} C$ *of* y *and* $U = \operatorname{Spec} A$ *of* $x = f(y)$ *such that*

$$C = A[T_1, \dots, T_n]/(P_1, \dots, P_n)$$

and $\det(\partial P_i/\partial T_j)$ *is a unit in* C.

Proof. By Example 1.45 we see that locally f is etale hence it is etale.

Assume f is etale, we may assume it is standard etale. Then $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} C$ where $C = B_b$, $B = A[T]/(P)$. Then $C = A[T, S]/(P, bS - 1)$ and the determinant bP' is a unit in C. □

Next we see if $f: Y \to X$ is etale then Y inherits many good properties of X.

Proposition 1.60. *Let* $f: Y \to X$ *be etale. Then*

a $\dim(\mathcal{O}_{Y,y}) = \dim(\mathcal{O}_{X,f(y)})$ *for all* $y \in Y$.

b *If* X *is normal so is* Y. *If* X *is reduced so is* Y.

c *If* X *is regular so is* Y.

Proof. For (a) we may assume $X = \operatorname{Spec} A$ is the spectrum of a local ring and $Y = \operatorname{Spec} B$. Let $\mathfrak{q}$ be the prime ideal of B corresponding to y. Then $\operatorname{Spec} B_\mathfrak{q} \to \operatorname{Spec} A$ is surjective so $\dim(B_\mathfrak{q}) \geq \dim(A)$. Conversely by Zariski's Main Theorem we have $\operatorname{Spec} B \to Z \to \operatorname{Spec} A$ where $\operatorname{Spec} B$ is open in Z and $Z \to \operatorname{Spec} A$ is finite. Then $\dim(B_\mathfrak{q}) \leq \dim(B) \leq \dim(Z) \leq \dim(A)$.

For (b) see Milne's book.

For (c), let $y \in Y$. Then $\dim(\mathcal{O}_{Y,y}) = \dim(\mathcal{O}_{X,f(y)}) = d$ and $\mathfrak{m}_y = \mathfrak{m}_x \mathcal{O}_{Y,y}$ could be generated by d elements. □

Remark 1.61. *It follows that if* $f: Y \to X$ *is etale and surjective then*

$$\dim X = \sup_{x \in X} \dim \mathcal{O}_{X,x} = \sup_{y \in Y} \dim \mathcal{O}_{Y,y} = \dim Y$$

Proposition 1.62. *Let* $f: Y \to X$ *be etale and* X *normal. Then locally* f *is a standard etale morphism of the form* $\operatorname{Spec} C \to \operatorname{Spec} A$ *where* A *is an integral domain,* $C = B_b$, $B = A[T]/(P(T))$ *and* $P(T)$ *is irreducible over the field of fractions of* A.

Theorem 1.63. *Let* X *be a normal scheme and* $f: Y \to X$ *unramified. Then* f *is etale iff for any* $y \in Y$ *the map* $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ *is injective.*

Theorem 1.64. *Let* X *be a connected normal scheme and let* $K = R(X)$. *Let* L *be a finite separable extension of* K *and* X' *the normalization of* X *in* L. *Let* U *be any open subscheme of* X' *disjoint from the support of* $\Omega_{X'/X}$. *Then* $U \to X$ *is etale and conversely any separated etale morphism* $Y \to X$ *of finite type can be written as* $Y = \coprod U_i \to X$ *where each* $U_i \to X$ *is of this form.*

Definition 1.65. *Let* X *be a scheme and* F *a contravariant functor from schemes over* X *to sets. Then* F *is said to be formally smooth/unramified/etale if for any affine* X*-scheme* X' *and any closed subscheme* X'_0 *of* X' *defined by a square zero (nilpotent) ideal the map* $F(X') \to F(X'_0)$ *is surjective/injective/bijective.*

A scheme Y *over* X *is said to be formally smooth/unramified/etale over* X *if the functor* $h_Y = \operatorname{Hom}_X(-, Y)$ *is.*

PROPOSITION 1.66. *A morphism $f\colon Y \to X$ is etale iff it is formally etale and locally of finite presentation.*

THEOREM 1.67 (Topological invariance of etale morphisms). *Let X_0 be the closed subscheme of X defined by a nilpotent ideal. The functor $Y \mapsto Y_0 = Y \times_X X_0$ gives an equivalence between the category of etale X-schemes and etale X_0-schemes.*

For completeness we list some relations with smoothness.

PROPOSITION 1.68. *Let $f\colon Y \to X$ be locally of finite type. TFAE.*

a *f is smooth.*
b *f is formally smooth and locally of finite presentation.*
c *for any $y \in Y$ there exist open affine neighborhood V of y and U of $f(y)$ such that $f|_V$ factors through $V \to V' \to U \hookrightarrow X$ where $V \to V'$ is etale and V' is affine n-space over U.*
d *for any $y \in Y$ there exist open affine neighborhood $V = \operatorname{Spec} C$ of y and $U = \operatorname{Spec} A$ of $x = f(y)$ such that*

$$C = A[T_1, \dots, T_n]/(P_1, \dots, P_m)\ ,\ m \leq n$$

and the ideal generated by the $m \times m$ minors of $(\partial P_i/\partial T_j)$ is C.
e *f is flat and for any algebraically closed geometric point $\overline{x}$ of X the fibre $Y_{\overline{x}} \to \overline{x}$ is smooth.*
f *the same as (e) but $Y_{\overline{x}}$ is regular.*
g *f is flat and $\Omega^1_{Y/X}$ is locally free of rank equal to the relative dimension of Y/X.*

REMARK 1.69. *In the case that f is of finite type, condition (e) might be interpreted as Y is a flat family of nonsingular varieties over X. The decomposition in (c) implies a finite type morphism is etale iff smooth and quasi-finite.*

PROPOSITION 1.70. *Let $f\colon Y \to X$ be smooth and surjective and assume X is quasi-compact. Then there exists an affine scheme X' with a surjective etale morphism $h\colon X' \to X$ and an X-morphism $X' \to Y$.*

REMARK 1.71. *Let $f\colon Y \to X$ be a morphism of smooth varieties over a field k. It is etale iff it induces an isomorphism on tangent spaces.*

For future applications we also need a lemma (04HN). This lemma and similar results in 04HK illustrates the idea that etale covering locally looks like the covering in topology or Riemann surfaces.

LEMMA 1.72. *Let $f\colon Y \to X$ be finite etale. Then for any $x \in X$ there exists an etale map $U \to X$ whose image contains x and such that Y_U is disjoint union of copies of U.*

4. Henselian Rings

Throughout this section, $(A, \mathfrak{m}, k)$ will be a (Noetherian) local ring.

Two polynomials f and g with coefficients in B are strictly coprime if they generate the unit ideal in $B[T]$.

DEFINITION 1.73. *A local ring A is called Henselian if for every monic polynomial f with coefficient in A such that $\overline{f} = g_0 h_0$ with g_0 and h_0 monic and coprime then we can write $f = gh$ where g and h monic and $\overline{g} = g_0$ and $\overline{h} = h_0$.*

REMARK 1.74. *If f monic and g are such that $\overline{f}$ and $\overline{g}$ are coprime then f and g are strictly coprime by Nakayama Lemma.*

The factorization is unique by our requirement on g being monic, equivalently $\deg g = \deg g_0$.

THEOREM 1.75. *Let x be the closed point of $X = \operatorname{Spec} A$. TFAE.*

a *A is Henselian.*

b *any finite* A*-algebra* B *is a direct product of local rings* $B = \coprod B_i$ *(the* B_i *are necessarily isomorphic to* $B_{\mathfrak{m}_i}$ *where* $\mathfrak{m}_i$ *are maximal ideals of* B*).*

c *if* $f\colon Y \to X$ *is quasi-finite and separated then* $Y = Y_0 \coprod Y_1 \coprod \cdots \coprod Y_n$ *where* $f(Y_0)$ *does not contain* x *and* Y_i *is finite over* X *and is the spectrum of a local ring for* $i \geq 1$*.*

d *if* $f\colon Y \to X$ *is etale and there is a point* $y \in Y$ *such that* $f(y) = x$ *and* $\kappa(x) = \kappa(y)$ *then* f *has a section.*

d' *let* $f_1, \ldots, f_n \in A[T_1, \ldots, T_n]$*. If there exists an* $a = (a_1, \ldots, a_n) \in k^n$ *such that* $\overline{f_i}(a) = 0$ *for all* i *and* $\det(\partial \overline{f_i}/\partial T_j)(a) \neq 0$ *then there exists some* $b \in A^n$ *such that* $\overline{b} = a$ *and* $f_i(b) = 0$ *for all* i*.*

e *let* $f \in A[T]$*. If* $\overline{f}$ *factors as* $\overline{f} = g_0 h_0$ *with* g_0 *monic and* g_0*,* h_0 *coprime then* f *factors as* $f = gh$ *with* g *monic and* $\overline{g} = g_0$*,* $\overline{h} = h_0$*.*

PROOF. $(a) \Longrightarrow (b)$: By going-up, any maximal ideal of B lies over $\mathfrak{m}$ thus B is local iff $B/\mathfrak{m}B$ is local. Note $B/\mathfrak{m}B$ is finite over k hence by Proposition 1.3 we see B has only finitely many maximal ideals, and they are all the prime ideals lying over $\mathfrak{m}$.

Assume B is of the form $B = A[T]/(f)$ with f monic. If $\overline{f}$ is a power of an irreducible polynomial then $B/\mathfrak{m}B$ is local and so is B. Otherwise by (a) we see $f = gh$ where g and h are monic, strictly coprime and of degree at least 1. Then $B = A[T]/(g) \times A[T]/(h)$ and reduces to previous case.

For the general case, if B is not local then $B/\mathfrak{m}B$ has at least two points hence disconnected, then we can find $b \in B$ such that $\overline{b}$ is a nontrivial idempotent in $B/\mathfrak{m}B$. Since B finite hence integral, we can find a monic polynomial f such that $f(b) = 0$. Let $C = A[T]/(f)$ and $\phi\colon C \to B$ sending T to b. Consider $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} C$. If $\operatorname{Spec} B$ is connected then so is its image. But by previous discussion we see C is a finite product of local rings, hence its spectrum has finitely many connected components, each being the spectrum of a local ring. Thus there is exactly one containing the image of $\operatorname{Spec} B$. It induces a map $C' = A[T]/(g) \to B$ where T maps to b and g is a power of monic h with $\overline{h}$ irreducible. If $T \notin (h)$ then T is invertible, hence so is b and $\overline{b}$ contradiction. Now $T \in (h)$ hence $T = h(T)$, then T is nilpotent and so is b and $\overline{b}$ contradiction. Thus B has a nontrivial idempotent and we can split B into a product. Since B has only finitely many maximal ideals, such process would eventually end.

$(b) \Longrightarrow (c)$: By Theorem 1.7 f factors through $Y \xrightarrow{f'} Y' \xrightarrow{g} X$ with f' open immersion and g finite. By (b) we see $Y' = \coprod \operatorname{Spec} \mathcal{O}_{Y',y}$ where y runs through the finitely many closed points of Y'. Then let $Y_* = \coprod \operatorname{Spec} \mathcal{O}_{Y',y}$ where y runs through the closed points of Y' which are in Y. Then Y_* is contained in Y and is both open and closed in Y. Then write $Y = Y_* \coprod Y_0$. Clearly $f(Y_0)$ does not contain x.

$(c) \Longrightarrow (d)$: Pick $y \in U = \operatorname{Spec} B \subset Y$ and consider $\operatorname{Spec} B \to \operatorname{Spec} A$. This map is of finite type and etale hence quasi-finite and separated. By (c) we may write $\operatorname{Spec} B$ as a finite disjoint union and assume $y \in \operatorname{Spec} B_i$ for some B_i local finite etale over A. Note B_i is flat hence finite free over A and by our condition $\kappa(\mathfrak{m}) = \kappa(\mathfrak{n})$, thus $B_i = A$.

$(d) \Longrightarrow (d')$: Let $B = A[T_1, \ldots, T_n]/(f_1, \ldots, f_n)$ and $J(T_1, \ldots, T_n) = \det(\partial f_i/\partial T_j)$. Then the condition implies there is a prime ideal $\mathfrak{q} = (T_i - \tilde{a}_i, \mathfrak{m})$ of B lying over $\mathfrak{m}$ such that $J \notin B_{\mathfrak{q}}$. Thus by Jacobian criterion B_J is etale over A. Clearly $y = \mathfrak{q}$ satisfies the condition in (d). Thus we have a section and let $a_i \in A$ be the image of $T_i - \tilde{a}_i$.

$(d') \Longrightarrow (e)$: Let $r = \deg g_0$, $s = n - r$ and consider the equation

$$f(T) = (T^r + b_{r-1}T^{r-1} + \cdots + \cdots + b_0)(c_s T^s + \cdots + c_0) = g(T)h(T)$$

the Jacobian of the system of equations is $\operatorname{res}(g, h)$. To apply (d') it suffices to check $\operatorname{res}(g_0, h_0) \neq 0$. This is implied by $r = \deg g_0$ and g_0, h_0 coprime.

$(e) \Longrightarrow (a)$: trivial. □

LEMMA 1.76. *A is Henselian if and only if for every monic $f \in A[T]$ and every simple root $a_0 \in k$ of $\overline{f}$ there exists $a \in A$ such that $f(a) = 0$ and $\overline{a} = a_0$.*

PROOF. Suppose A has the lifting of simple root property. Let $A \to B$ be an etale map with prime ideal $\mathfrak{q} \subset B$ lying over $\mathfrak{m}$ and $\kappa(\mathfrak{q}) = k$. It suffices to show we have a section $B \to A$. By standard etale structure we can find $b \in B, b \notin \mathfrak{q}$ such that $A \to B_b$ is standard etale. Thus we may assume $B = A[T]_g/(f)$ is standard etale. Since the prime $\mathfrak{q}$ lying over $\mathfrak{m}$ has residue field k it is of the form $(\mathfrak{m}, T - a')$ where $\overline{a'} = a_0$ is a root of $\overline{f}$ and not a root of $\overline{g}$. The condition that f' is invertible in B shows that $\overline{f'}(a_0) \neq 0$. Since f is monic we can find $a \in A$ with $\overline{a} = a_0$ and $f(a) = 0$. Then $g(a) \in A$ is a unit. The map $A[T]_g/(f) \to A$ sending T to a is a section. □

COROLLARY 1.77. *If A is Henselian, so is any finite local A-algebra and any quotient of A.*

PROPOSITION 1.78. *If A is Henselian, then the functor $B \mapsto B \otimes_A k$ gives an equivalence between finite etale A-algebras and finite etale k-algebras.*

PROOF. By the equivalent criteria it suffices to consider finite local etale algebras. The canonical map

$$\mathrm{Hom}_A(B, B') \longrightarrow \mathrm{Hom}_k(B \otimes k, B' \otimes k)$$

is injective by Corollary 1.55. To show surjectivity, a k-morphism $B \otimes k \to B' \otimes k$ induces an A-morphism $g \colon B \to B' \otimes k$ by composition with $B \to B \otimes k$ and hence an A-morphism

$$(b' \otimes b \longmapsto b'g(b)) \colon B' \otimes_A B \longrightarrow B' \otimes_A k$$

there is a unique point z in $\mathrm{Spec}\, B' \otimes k$ lying over $\mathfrak{m}_{B'}$ and its residue field is $\kappa(\mathfrak{m}_{B'})$. Now apply the equivalent criteria to the map $\mathrm{Spec}\, B' \otimes B \to \mathrm{Spec}\, B'$ where y is the image of z to get an A-morphism $B' \otimes_A B \to B'$ hence $B \to B'$. Thus the functor is fully faithful.

Note any local etale k-algebra k' has the form $k[T]/(f_0(T))$ where f_0 is an irreducible polynomial by standard etale theorem. Then $B = A[T]/(f(T))$ where $\overline{f} = f_0$ and f monic is finite etale over A and $B \otimes_A k = k'$. □

PROPOSITION 1.79. *Any complete local ring is Henselian.*

PROOF. Let B be an etale A-algebra and assume $s_0 \colon B \to k$. We want to show it could be lifted to a section $s \colon B \to A$. To do this we apply the formal etale property to $A/\mathfrak{m}^r \to A/\mathfrak{m}^{r-1}$ and lift s_0 step by step to get a compatible system of sections which induces a global section since A complete. □

REMARK 1.80.

a *The last two propositions show that the functor $B \mapsto B \otimes_A \widehat{A}$ is an equivalence between the category of finite etale A-algebras and its completion when A is Noetherian Henselian. When X is proper over a Noetherian Henselian ring A, this result extends to the category of schemes finite etale over X and over $\widehat{X} = X \otimes_A \widehat{A}$.*

b *Let X be proper over a Henselian ring A and let X_0 be its closed fibre. Then the functor $Y \mapsto Y \times_X X_0$ is an equivalence of categories between the category of finite etale schemes over X and over X_0.*

c *Let $f \colon Y \to X$ be separated and of finite type where $X = \mathrm{Spec}\, A$ with A Henselian. If y is an isolated point in the closed fibre Y_0 so that $Y_0 = \{y\} \coprod Y_0'$ then $Y = Y'' \coprod Y'$ with Y'' finite over X and Y'' and Y' having closed fibres $\{y\}$ and Y_0' respectively.*

d *If X is an analytic manifold over $\mathbb{C}$ then the local ring at any point is Henselian.*

REMARK 1.81. *Let $f \colon Y \to X$ be etale and suppose for some $y \in Y$ and $x = f(y) \in X$ we have $\kappa(y) = \kappa(x)$. Then we have induced map on completions $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{Y,y}}$. By 0394 we see this map is finite. To show it is etale it remains to show flatness. This is implied by Remark 1.18 (c). Thus it has a section and thus it is an isomorphism. Conversely suppose $f \colon Y \to X$ is a morphism of finite type between two locally Noetherian schemes. Let $y \in Y$ such that for $x = f(y) \in X$ we have $\kappa(x) = \kappa(y)$ and the induced map on completions $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{Y,y}}$ is an isomorphism. Then f is etale at y.*

This could be used to find an example of an injective unramified map of rings but not etale.

DEFINITION 1.82. *For a Noetherian local ring A, $A \to \widehat{A}$ is local and flat hence injective. Thus any local ring is a subring of a Henselian ring. A local homomorphism $A \to A^h$ with A^h Henselian is called the Henselization of A if any other such morphisms factors uniquely through it.*

Next we will show three ways to construct the Henselization.

DEFINITION 1.83. *Let A be a local ring. An etale neighborhood of A is a pair $(B, \mathfrak{q})$ where B is an etale A-algebra and $\mathfrak{q}$ a prime ideal of B lying over $\mathfrak{m}$ such that the induced map $k \to \kappa(\mathfrak{q})$ is an isomorphism.*

LEMMA 1.84. *Let $(B, \mathfrak{q})$ and $(B', \mathfrak{q}')$ be etale neighborhoods of A.*

a *If $\operatorname{Spec} B'$ is connected, then there is at most one A-morphism $f: B \to B'$ such that $f^{-1}\mathfrak{q}' = \mathfrak{q}$.*
b *There is an etale neighborhood $(B'', \mathfrak{q}'')$ of A with $\operatorname{Spec} B''$ connected and A-morphisms $f: B \to B''$, $f': B' \to B''$ such that $f^{-1}\mathfrak{q}'' = \mathfrak{q}$ and $f'^{-1}\mathfrak{q}'' = \mathfrak{q}'$.*

PROOF. (a) is implied directly by Corollary 1.55.

Let $C = B \otimes_A B'$. We have a prime ideal $\mathfrak{q}''$ of C lying over $\mathfrak{q}$ and $\mathfrak{q}'$ with residue field k. Since C is Noetherian, $\operatorname{Spec} C$ is locally connected hence every connected component is open hence it has only finitely many connected components. Thus we can decompose C as products of rings until each subring has connected spectrum. Then we can find $c \in C, c \notin \mathfrak{q}''$ such that $\operatorname{Spec} C_c$ is connected and $B'' = C_c$ is the required etale neighborhood. □

By this Lemma we see in case A is Noetherian then the etale neighborhoods of A with connected spectra form a filtered direct system and its filtered direct limit is the same as the filtered colimit of all etale neighborhoods of A. Let A^h be its direct limit. Then A^h consists of triples $(B, \mathfrak{q}, b)$ with $b \in B$ and two such triples $(B, \mathfrak{q}, b)$ and $(B', \mathfrak{q}', b')$ define the same element iff there is a pair $(B'', \mathfrak{q}'')$ and morphisms $\varphi: (B, \mathfrak{q}) \to (B'', \mathfrak{q}'')$ and $\varphi': (B', \mathfrak{q}') \to (B'', \mathfrak{q}'')$ such that $\varphi(b) = \varphi'(b')$.

PROPOSITION 1.85. *A^h is a local ring with maximal ideal $\mathfrak{m}^h$ lying over $\mathfrak{m}$ and its residue field is the same as k. A^h is flat over A and is the Henselization of A.*

PROOF. Clearly A^h is a ring. Since filtered colimit is exact we see $\mathfrak{m}^h$ being the colimit of $\mathfrak{q}$ is a maximal ideal of A^h lying over $\mathfrak{m}$ with residue field k. Any element not in $\mathfrak{m}^h$ is invertible by passing to a localization. Thus A^h is a local ring. Clearly A^h is flat since it is the filtered colimit of flat modules. By looking at elements we see $\mathfrak{m}A^h = \mathfrak{m}^h$.

To show A^h is Henselian, let $P \in A^h[T]$ be monic and $a_0 \in k$ be a simple root of $\overline{P}$. Write $\overline{P} = (T - a_0)h_0(T)$. Then we can find an etale neighborhood $(B, \mathfrak{q})$ such that P is the image of a monic $Q \in B[T]$ and $T - a_0, h_0(T) \in B/\mathfrak{q}[T]$. Let a' be a lift of a_0 in B and consider $\mathfrak{q}' = (\mathfrak{q}, T - a') \subset B[T]/(Q)$. Then $\mathfrak{q}'$ is a prime ideal of $B' = B[T]/(Q)$ lying over $\mathfrak{q}$ and $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q}) = k$. Since a_0 is a simple root we have $Q' \notin \mathfrak{q}'$. Thus $B \to B'_{Q'}$ is standard etale and the pair $(B'_{Q'}, \mathfrak{q}')$ is again an etale neighborhood. Then the class of T in it serves as an element $a \in A^h$ killing P and reduces to a_0.

To show it is the Henselization of A, apply the next lemma. □

LEMMA 1.86. *Let $R \to S$ be a ring map with S Henselian. Given*

1 *an etale ring map $R \to A$;*
2 *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = \mathfrak{m}_S \cap R$;*
3 *a $\kappa(\mathfrak{p})$-algebra map $\tau: \kappa(\mathfrak{q}) \to \kappa(\mathfrak{m}_S)$.*

there exists a unique R-algebra map $f: A \to S$ such that $f^{-1}\mathfrak{m}_S = \mathfrak{q}$ and the reduction of f is τ.

PROOF. $A \otimes_R S$ is etale over S. The map τ gives rise to a prime ideal $\mathfrak{q}'$ lying over $\mathfrak{m}_S$ whose residue field is $\kappa(\mathfrak{m}_S)$. Thus we have a unique section $A \otimes_R S \to S$ and compose it to get $f: A \to S$. □

Thus we see for a general local ring A we have its Henselization A^h. The Henselization A^h would inherit many properties of A, for example if A is Noetherian, A^h will also be Noetherian. We will list these kind of permanence properties later after introducing the strict Henselization.

Notice that in the case A is Noetherian we have induced map $A^h \to \widehat{A}$ and by Lemma 1.19 it is flat hence injective. Now let $\widetilde{A}$ be the intersection of all Henselization subring H of $\widehat{A}$ containing A such that $\mathfrak{m}_H = \widehat{\mathfrak{m}} \cap H$.

PROPOSITION 1.87. *The inclusion $i\colon A \to \widetilde{A}$ is a Henselization of A.*

PROOF. It is easy to see $\widetilde{A}$ is Henselian. Thus we have induced map $A^h \to \widetilde{A}$. Compose it with the inclusion $\widetilde{A} \to \widehat{A}$ we see it is injective. On the other side, $\widetilde{A}$ is a subring of A^h by construction. Hence they equal. $\square$

REMARK 1.88. *The Henselization of A/J is A^h/JA^h.*

Every ring is a quotient of a normal ring (possibly too big to be Noetherian) and normal local rings come naturally into discussion. In case A is normal local, let K be its fraction field and K_s a fixed separable closure. Denote by G the Galois group of K_s over K. Then G acts on the integral closure B of A in K_s. Let $\mathfrak{n}$ be a maximal ideal of B lying over $\mathfrak{m}$ and let $D \subset G$ be the decomposition group of $\mathfrak{n}$, i.e. $D = \{\sigma \in G \mid \sigma(\mathfrak{n}) = \mathfrak{n}\}$. Consider the D-fixed field K_s^D and the integral closure B^D of A in K_s^D. Then $\mathfrak{n}^D$ is a maximal ideal of B^D lying over $\mathfrak{m}$. Let A' be the localization of B^D at $\mathfrak{n}^D$.

Suppose A' is not Henselian. There would exist a monic polynomial f irreducible over A' but whose reduction $\overline{f}$ factors into relatively prime factors. From such an f one can construct a finite Galois extension L of K_s^D such that the integral closure A'' of A' in L is not local. This is a contradiction since the Galois group of L over K_s^D permutes the prime ideals of A'' lying over $\mathfrak{n}^D$ hence can not be a quotient of D. To see A' is the Henselization of A, note that it is a union of etale neighborhoods of A. Thus A' is a Henselization of A.

EXAMPLE 1.89. *Let k be a field and A be the localization of $k[T_1, \ldots, T_n]$ at $(T_1, \ldots, T_n)$. The Henselization of A is the set of power series $P \in k[\![T_1, \ldots, T_n]\!]$ that are algebraic over A.*

DEFINITION 1.90. *If A is Henselian whose residue field is separably closed then A is called a strictly Henselian ring. The strict Henselization of a local ring A is a pair (A^{sh}, i) where A^{sh} is a strictly Henselian ring and $i\colon A \to A^{sh}$ is a local homomorphism such that any other local homomorphism $f\colon A \to H$ with H strictly Henselian extends to a local homomorphism $f'\colon A^{sh} \to H$ and f' is uniquely determined if the induced map $A^{sh}/\mathfrak{m}^{sh} \to H/\mathfrak{m}_H$ is given.*

LEMMA 1.91. *Fix a separable closure k_s of k. Then A^{sh} is the filtered colimit of B over all commutative diagrams*

$$\begin{array}{ccc} B & \longrightarrow & k_s \\ \uparrow & \nearrow & \\ A & & \end{array}$$

in which $A \to B$ is etale.

If $A = k$ is a field then A^{sh} is any separable closure of k. If A is normal, then A^{sh} can be constructed the same way as A^h except the decomposition group should be replaced by the inertia group.

DEFINITION 1.92. *Let X be a scheme and $\overline{x}$ be a geometric point. An etale neighborhood of $\overline{x}$ is a commutative diagram*

$$\begin{array}{ccc} \overline{x} & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X \end{array}$$

where $U \to X$ *is etale. Clearly* $\mathcal{O}^{sh}_{X,x} = \operatorname{colim}\Gamma(U, \mathcal{O}_U)$ *where the colimit is taken over all etale neighborhoods of* $\overline{x}$*. We will write* $\mathcal{O}_{X,\overline{x}}$ *for* $\mathcal{O}^{sh}_{X,x}$*.*

The next two lemmas come from Aaron Landesman, *The Smooth Base Change Theorem.*

LEMMA 1.93. *Let* $\{S_i\}$ *be an inverse system of qc schemes with affine transition maps. Then we can define the inverse limit* $S = \lim S_i$*. Let* $s \in S$ *with corresponding image* $s_i \in S_i$*. Then*

$$\operatorname{colim} \mathcal{O}^{sh}_{S_i,s_i} = \mathcal{O}^{sh}_{S,s}$$

LEMMA 1.94. *Suppose we have a fibre product*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

where $X \to S$ *is finite. For any point* $x' \in X'$ *lying over* s', s, x *of* S', S, X *respectively we have a natural map*

$$\mathcal{O}^{sh}_{X,x} \otimes_{\mathcal{O}^{sh}_{S,s}} \mathcal{O}^{sh}_{S',s'} \longrightarrow \mathcal{O}^{sh}_{X',x'}$$

and it is an isomorphism.

The next permanence properties come from 07QL.

PROPOSITION 1.95.

a $A \to A^h \to A^{sh}$ *are faithfully flat.*
b $\mathfrak{m}A^h = \mathfrak{m}^h$ *and* $\mathfrak{m}A^{sh} = \mathfrak{m}^{sh}$*.*
c *Let* P *denote a property for rings. Then if* P *=Noetherian, reduced, normal domain then* $P(A) \iff P(A^h) \iff P(A^{sh})$*.*
d $\dim(A) = \dim(A^h) = \dim(A^{sh})$*.*
e *If* A *is Noetherian then* A *is regular/DVR iff* A^h *regular/DVR iff* A^{sh} *regular/DVR.*
f *If* A *is Noetherian and* $\mathfrak{p} \subset A$ *be a prime, then*

$$A^{h/sh} \otimes_A \kappa(\mathfrak{p}) = \prod \kappa(\mathfrak{q}_i)$$

where $\mathfrak{q}_i$ *are the primes lying over* $\mathfrak{p}$*. Moreover the extensions* $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}_i)$ *are all separable algebraic.*

5. The Fundamental Group: Galois Coverings

The fundamental group of a general topological space X with base point x_0 may be defined either as the group of closed paths through x_0 modulo homotopy equivalence, or as the automorphism group of the universal covering space of X. The latter generalizes. Since etale being the most natural analogue of local homeomorphism, the fundamental group of a scheme should classify the etale coverings of X.

Let X be a connected scheme and $\overline{x}$ be a geometric point. Define a functor $F_{\overline{x}} \colon \mathrm{FEt}/X \to \mathrm{Sets}$ where FEt/X is the category of finite etale X-schemes by setting $F_{\overline{x}}(Y) = \operatorname{Hom}_X(\overline{x}, Y) = Y_{\overline{x}}$. Let $\pi_1(X, \overline{x}) = \operatorname{Aut}(F_{\overline{x}})$. It is given the topology such that

$$\pi_1(X, \overline{x}) \longrightarrow \operatorname{Aut}(F_{\overline{x}}(Y))$$

is continuous for all Y finite etale over X where the latter group is given the discrete topology. This makes the etale fundamental group into a profinite group.

EXAMPLE 1.96.

a *Let* $X = \operatorname{Spec} k$ *and let* $\overline{x} = \operatorname{Spec} k_s$ *be a separable closure. Then finite etale maps to* X *are just finite separable extensions of* k *hence* $\pi_1(X, \overline{x}) = \operatorname{Gal}(k_s/k)$*.*

b *Let E be an elliptic curve over $k = \overline{k}$ with $\mathrm{char}(k) = p > 0$. Then one can show that*

$$\pi_1(X) = \lim E[n] = \begin{cases} \mathbb{Z}_p \times \prod_{\ell \neq p} \mathbb{Z}_\ell^2 & \text{if } E \text{ ordinary} \\ \prod_{\ell \neq p} \mathbb{Z}_\ell^2 & \text{if } E \text{ supersingular} \end{cases}$$

c *Let $X = \mathbb{A}_k^1$ where $\mathrm{char}(k) = p > 0$. Then*

$$\mathrm{Hom}(\pi_1(X)^{ab}, \mathbb{F}_p) = H^1(X_{et}, \mathbb{F}_p) = \mathrm{coker}(k[t] \xrightarrow{x \mapsto x^p - x} k[t])$$

hence $\pi_1(X)$ is far from being topologically finitely generated.

d *Let $X = \mathrm{Spec}\, A$ where A is Henselian. Let $\overline{x}$ be a geometric point over the closed point x of X. Then the equivalence of categories $FEt/X \to FEt/\kappa(x)$ induces an isomorphism $\pi_1(X, \overline{x}) = \mathrm{Gal}(\kappa(x))$. In particular if A is strictly Henselian then the etale fundamental group is just trivial. This agrees with the fact in topology since locally $\mathrm{Spec}\, \mathcal{O}_{X,\overline{x}}$ is just a small ball around a point hence contractible.*

e *Let $X = \mathrm{Spec}\, K$ where K is the fraction field of a strict Henselian DVR A. Then X is the algebraic analogue of a punctured disc in the plane. Serre showed that if the residue field $A/\mathfrak{m}$ has characteristic 0 then the Galois extensions of K are exactly the Kummer extensions K_n/K where $K_n = K[t^{1/n}]$ with t a uniformizer. The map*

$$\sigma \longmapsto \sigma(t^{1/n})/t^{1/n} \colon \mathrm{Gal}(K_n/K) \longrightarrow \mu_n(K)$$

is an isomorphism. Thus

$$\pi_1(X, \overline{x}) = \lim \mathrm{Gal}(K_n/K) = \lim \mu_n(K) = \hat{\mathbb{Z}}$$

If the residue field has characteristic p then this is no longer true because the existence of wild ramification. However Serre showed that any tamely ramified extension of K is still Kummer and the tame fundamental group

$$\pi_1^{tame}(X, \overline{x}) = \lim_{p \nmid n} \mu_n(K) = \lim_{p \nmid n} \mathbb{Z}/n\mathbb{Z}$$

In general if K is the fraction field of a DVR A then a finite separable extension L of K is called tamely ramified if for each valuation ring B of L lying over A the residue field extension $A/\mathfrak{m} \subset B/\mathfrak{n}$ is separable and the ramification index of B/A is not divisible by $p = \mathrm{char}(A/\mathfrak{m})$.

Let X be a connected normal scheme and D be a finite union of irreducible divisors on X and let $\{x_i\}$ be generic points of them. Then a map $f \colon Y \to X$ is called a tamely ramified covering if it is finite etale outside $X - D$, Y is connected and normal and $R(Y)/R(X)$ is tamely ramified with respect to the rings $\mathcal{O}_{X,x_i}$. The tame fundamental group π_1^{tame} is aimed to classify such coverings.

f *Let $X = \mathbf{P}_k^1$ where k is separably closed. If $k = \mathbb{C}$ then X is topologically a sphere and so it should have trivial fundamental group. In general let $f \colon Y \to X$ be a finite etale map with Y connected. Then by Riemann Hurwitz we see $-2n = 2g - 2 \geq -2$ where $n = \deg(f)$ hence $n = 1$ and f is an isomorphism. This shows that X has trivial fundamental group. The same argument shows that there is no nontrivial map $Y \to \mathbf{P}^1$ that is etale over $\mathbb{A}^1$ and tamely ramified at infinity.*

g *See more examples in Milne's book.*

Once the fundamental group has been constructed, the most important result is that it does classify finite etale maps to X.

THEOREM 1.97 (SGA 1). *Let $\overline{x}$ be a geometric point of a connected scheme X. Then the functor $F_{\overline{x}}$ defines an equivalence between the categories FEt/X and finite $\pi_1(X, \overline{x})$-sets with continuous left action.*

For any finite group G and any scheme X let G_X be the scheme $\coprod_{\sigma \in G} X_\sigma$ where $X_\sigma = X$ for each σ. Then G has a natural right action on G_X by requiring σ acts on X_τ to be the identity map $X_\tau \to X_{\tau\sigma}$.

Definition 1.98. *Let X and Y be connected. Let G be a finite group acting on Y over X. Then $Y \to X$ is called a Galois covering with Galois group G if it is f.f. and LFT and the map*

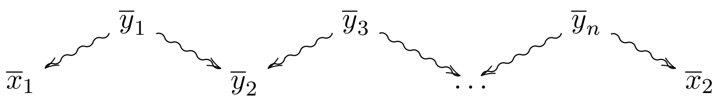
$$\psi\colon G_Y \longrightarrow Y \times_X Y \ , \ \psi|_{Y_\sigma} = (id, \sigma)$$

is an isomorphism.

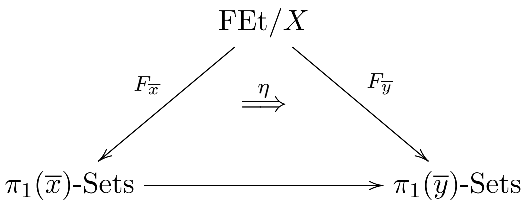
Remark 1.99. *$Y \to X$ is Galois with Galois group G iff there is a f.f. morphism $U \to X$ locally of finite type such that Y_U is isomorphic to G_U with G-action, iff for any $X' \to X$ the map $Y_{X'} \to X'$ is also Galois with Galois group G. Also any finite etale morphism $Y \to X$ of connected schemes can be embedded into a Galois extension, i.e. there exists a Galois covering $Y' \to X$ which factors through $Y \to X$.*

Let X be connected with geometric point $\overline{x}$. By Theorem 1.97, to give an $\overline{x}$-pointed Galois morphism $Y \to X$ with Galois group G is the same as to give a continuous morphism $\pi_1(X, \overline{x}) \to G$. Later we shall relate Galois coverings with torsors and classify them using the first etale cohomology group.

By Theorem 1.97 we could also see that if $\overline{x}_1$ and $\overline{x}_2$ are two different geometric points then $\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)$. We may choose a sequence of specializations and generalizations



and if $\overline{x}$ specializes to $\overline{y}$ then the isomorphism is given by



where $\eta(Y/X)(z) = \overline{z} \cap Y_y$.

In case we are over $\mathbb{C}$, the Riemann Existence Theorem applies and we get

Proposition 1.100 (Comparison). *Let X be a normal scheme over $\mathbb{C}$. Then there is a natural map*

$$\pi_1(X^{an}, \overline{x}^{an}) \longrightarrow \pi_1^{et}(X, \overline{x})$$

for any $x \in X(\mathbb{C})$. This map induces an isomorphism after completion.

Specialization Maps.

CHAPTER 2

Sheaf Theory

1. Presheaves and Sheaves

We shall be concerned with classes E of morphisms of schemes such that

- a all isomorphisms are in E
- b E is closed under composite
- c E is closed under base change

The full subcategory of X-schemes whose structure morphism is an E-morphism will be denoted as E/X.

Example 2.1. *The class E of (Zar) of all open immersions, (et) of all etale morphisms of finite type, (fl) if all flat morphisms locally of finite type. In these cases the E-morphisms are open and any open immersion is an E-morphism.*

Fix a base scheme, a class E and a full subcategory C/X and such that for any $Y \to X$ in C/X and any E-morphism $U \to Y$ the composite $U \to X$ is in C/X. An E-covering of Y of C/X is a family $(U_i \xrightarrow{g_i} Y)_{i\in I}$ of E-morphisms such that $Y = \bigcup g_i(U_i)$. The category C/X together with all such coverings is called the E-site over X, denoted by $(C/X)_E$. The small E-site is $(E/X)_E$ and if all E-morphisms are locally of finite type, the big E-site is $(LFT/X)_E$ where LFT/X is the full subcategory whose structure morphism is locally of finite type. A small site is the usual analogue of a topological space and a big site is the analogue of all topological spaces and continuous maps over a given space.

X_{Zar} denote the small (Zar)-site, X_{et} the small (et)-site, X_{fl} the big (fl)-site $(LFT/X)_{fl}$.

Remark 2.2. *The category C/X and the family of E-coverings satisfy*

1. *an isomorphism $U \to U$ in C/X is a covering*
2. *if $(U_i \to U)_i$ is a covering and for each i, $(V_{ij} \to U_i)_j$ is a covering then $(V_{ij} \to U)_{ij}$ is a covering*
3. *if $(U_i \to U)_i$ is a covering then for any morphism $V \to U$ in C/X, $(U_i \times_U V \to V)_i$ is a covering*

Definition 2.3. *A presheaf P on a site $(C/X)_E$ is a contravariant functor $(C/X) \to Ab$. Objectwisely we can define direct sum, kernel, cokernel, product, inverse limit, direct limit on the category of all presheaves $PSh((C/X)_E)$ and this is an abelian category. Arbitrary direct sum exists and filtered colimit is exact. Arbitrary product exist and product preserves exactness.*

Example 2.4.

- a *The constant presheaf for any abelian group.*
- b $\mathbb{G}_a$ *where* $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$.
- c $\mathbb{G}_m$ *where* $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^*$.
- d *The pullback presheaf $W(F)$ of a sheaf F of $\mathcal{O}_X$-modules.*

Definition 2.5. *A presheaf P is a sheaf if for any covering $(U_i \to U)_i$ we have*

(s_1) *if $s \in P(U)$ and $s|_{U_i} = 0$ for all i then $s = 0$*

(s_2) *if the family $(s_i)_i$, $s_i \in P(U_i)$ such that $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all i, j then they come from some $s \in P(U)$*

i.e. the sequence (S)

$$0 \longrightarrow P(U) \longrightarrow \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j)$$

is exact.

A presheaf P *that satisfies* (s_1) *is called a separated presheaf.*

PROPOSITION 2.6. *Let* Y *be Galois over* X *with Galois group* G. *Let* P *be a presheaf for the etale topology on* X *that takes disjoint sums to direct products. Then* G *acts on* $P(Y)$ *on the left and the sequence* (S) *for the covering* $(Y \to X)$ *is canonically identified with the sequence*

$$0 \longrightarrow P(X) \longrightarrow P(Y) \underset{(\sigma_1,\ldots,\sigma_n)}{\overset{(1,\ldots,1)}{\rightrightarrows}} P(Y)^n$$

where $G = \{\sigma_1, \ldots, \sigma_n\}$, *i.e.* $P(X) = P(Y)^G$.

PROOF. This follows from the definition that $Y \times_X Y \cong G_Y$. □

PROPOSITION 2.7. *Let* P *be a presheaf for the etale or flat site on* X. *Then* P *is a sheaf iff it satisfies*

a *for any* U *in* C/X *the restriction of* P *to the usual Zariski topology on* U *is a sheaf*

b *for any covering* $(U' \to U)$ *with* U *and* U' *both affine, the sequence* (S) *is exact*

PROOF. Condition (a) implies if a scheme $V = \coprod V_i$ then $P(V) = \prod P(V_i)$. Thus if we have a finite covering $(U_i \to U)_i$ where U_i and U are all affines then by consider $\coprod U_i \to U$ we see the sequence (S) holds for this covering.

Consider a general covering $(U'_j \xrightarrow{f_i} U)_j$. Write $U = \bigcup U_i$ as a union of affine opens. Then since each f_i is open and U_i quasi-compact, we may find finite U'_{il} affine opens each lying entirely in some U'_k and such that all U'_{il} cover every U'_j. Consider the diagram

$$\begin{array}{ccccc}
P(U) & \longrightarrow & \prod_j P(U'_j) & \rightrightarrows & \prod_{j,k} P(U'_j \times_U U'_k) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_i P(U_i) & \longrightarrow & \prod_i \prod_l P(U'_{il}) & \rightrightarrows & \prod_i \prod_{l,s} P(U'_{il} \times_{U_i} U'_{is}) \\
\downarrow\downarrow & & \downarrow\downarrow & & \\
\prod_{ik} P(U_i \cap U_k) & \longrightarrow & \prod_{ik} \prod_{ls} P(U'_{il} \times_U U'_{js}) & &
\end{array}$$

where the first column is exact and the second column factors through an exact sequence by condition (a). The second row is exact by the above discussions. Then $P(U) \to \prod_j P(U'_j)$ is injective and P is separated. Thus the bottom arrow is injective. An easy diagram chase shows the first row is also exact. □

COROLLARY 2.8. *For any quasi-coherent* $\mathcal{F}$, $W(\mathcal{F})$ *is a sheaf on* X_{fl} *hence* X_{et}.

PROOF. Check the condition for Proposition 2.7 using Corollary 1.31. □

REMARK 2.9. *Let* $\mathbb{G}_{a,X} = X \times \operatorname{Spec} \mathbb{Z}[T]$ *and* $\mathbb{G}_{m,X} = X \times \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$. *Then* $\mathbb{G}_a(U) = \operatorname{Hom}_X(U, \mathbb{G}_{a,X})$ *and* $\mathbb{G}_m(U) = \operatorname{Hom}_X(U, \mathbb{G}_{m,X})$. *Hence* $\mathbb{G}_a$ *and* $\mathbb{G}_m$ *are sheaves for the flat or etale sites by Theorem 1.29.*

DEFINITION 2.10. *An object* G *in the category of* X*-schemes is a (commutative) group scheme if there are morphisms*

$$m\colon G \times G \to G\ ,\ \epsilon\colon X \to G\ ,\ \operatorname{inv}\colon G \to G$$

plays the role of multiplication, unit and inverse respectively. Equivalently if $G(Y) = \mathrm{Hom}_X(Y, G)$ *is an abelian group for every* Y *and for every morphism* $Y' \to Y$ *the map* $G(Y) \to G(Y')$ *is a homomorphism. In particular any commutative group scheme defines a presheaf for any site on* X.

EXAMPLE 2.11. *Suppose* G *is a commutative group. Then naturally one can make* $G_X = \coprod_{\sigma \in G} X_\sigma$ *into a commutative group scheme where each* X_σ *is a copy of* X.

COROLLARY 2.12. *Any presheaf defined by a commutative group scheme on* X *is a sheaf for the flat, etale and Zariski sites.*

EXAMPLE 2.13. *Fix a field* K *and consider the etale site on* $X = \mathrm{Spec}\, K$. *Then any scheme* U *etale and of finite type over* X *is finite disjoint union of spectrum of finite separable field extensions of* K. *Fix a separable closure* K_s *of* K *and let* $G = \mathrm{Gal}(K_s/K)$. *Denote* $\overline{x}$ *to be the geometric point of* K_s. *Then note* G *acts on* K_s *on the left hence on* $\overline{x}$ *on the right.*

Let P *be any presheaf on* X_{et}. *Define* $M_P = \mathrm{colim}\, P(K')$ *where* K' *runs through all finite sub-extensions* K'/K *of* K_s/K. *Then* G *acts on* M_P *and* $M_P = \bigcup M_P^H$ *where* H *runs through the open subgroups of* G. *Hence* M_P *is a discrete* G*-module.*

Conversely given any discrete G*-module* M *define*

$$F_M(U) = \mathrm{Hom}_G(F(U), M) \text{ where } F(U) = \mathrm{Hom}_X(\overline{x}, U)$$

then we have

a F_M *is a presheaf*
b $F_M(K') = M^H$ *where* $H = \mathrm{Gal}(K_s/K')$
c $F_M(\prod K_i) = \prod F_M(K_i)$ *for any finite product*

LEMMA 2.14. F_M *is a sheaf.*

PROOF. Part (c) of the properties shows it is a sheaf in the Zariski sense. To check on affine it suffices to take $\mathrm{Spec}\, L' \to \mathrm{Spec}\, L$ where $L' \supset L$ are both finite separable field extensions of K. Let L'' be a finite Galois extension of L containing L' and consider the diagram

$$\begin{array}{ccccc} F_M(L) & \longrightarrow & F_M(L') & \rightrightarrows & F_M(L' \otimes_L L') \\ \| & & \downarrow & & \downarrow \\ F_M(L) & \longrightarrow & F_M(L'') & \rightrightarrows & F_M(L'' \otimes_L L'') \end{array}$$

now the bottom row is exact by Proposition 2.6 and $F_M(L) = (F_M(L''))^{\mathrm{Gal}(L''/L)}$. Also $F_M(L) \to F_M(L')$ and $F_M(L') \to F_M(L'')$ are both injective, hence the top row is exact. □

THEOREM 2.15. *The constructions above give equivalence between categories of sheaves on* X_{et} *and discrete* G*-modules.*

PROOF. One check $\mathrm{Hom}_G(M, M') \to \mathrm{Hom}(F_M, F_{M'})$ is bijective and $F \to F_{M_F}$ is an isomorphism. □

REMARK 2.16. *For any profinite group* G *let* G*-sets be the category of finite sets with a continuous left* G*-action. A covering of a* G*-set* S *is just a surjective family* $(S_i \to S)$. *Then the category of sheaves on* G*-sets is equivalent to the category of discrete* G*-modules by sending a sheaf* F *to* $\mathrm{colim}\, F(G/H)$ *where* H *runs over all open subgroups and sending* M *to* $F(S) = \mathrm{Hom}_G(S, M)$.

In the situation where G *is the Galois group of* k, *the functor* $U \mapsto \mathrm{Hom}_X(\overline{x}, U)$ *defines an equivalence of categories from finite etale schemes over* X *to* G*-sets under which coverings correspond to coverings. Thus there are equivalences of categories*

$$\{\textit{sheaves on } G\textit{-sets}\} \sim \{\textit{sheaves on } X_{et}\} \sim \{\textit{discrete } G\textit{-modules}\}$$

Actually every profinite group arises as a Galois group of fields.

Remark 2.17. *Any presheaf on the etale or flat site representable by a group scheme is a sheaf. More generally any presheaf of sets on the etale or flat site representable by a scheme is a sheaf.*

One can show that on any category there is a finest topology relative to which all representable presheaves of sets are sheaves. This is called the canonical topology. Whenever the E-topology on C/X is coarser than the canonical topology, C/X can be embedded as a full subcategory of the category of sheaves of sets on $(C/X)_E$.

In some situations it may happen that all sheaves are representable or ind-representable. This is the case for etale sheaves over a field.

Remark 2.18. *Finite limits exist in the Zariski, etale and flat sites.*

Remark 2.19. *Sieve.*

2. The Category of Sheaves

Definition 2.20. *Let $(C'/X')_{E'}$ and $(C/X)_E$ be two sites. A morphism $\pi\colon X' \to X$ is a morphism of sites if*

a *for any Y in C/X the fiber product $Y \times_X X'$ is in C'/X'.*
b *for any E-morphism $U \to Y$ in C/X the base change $U \times_X X' \to Y \times_X X'$ is an E'-morphism.*

i.e. π gives a functor $C/X \to C'/X'$ taking covers to covers.

Example 2.21.
a *The identity map of X defines a morphism of sites if every E-map is also an E'-map.*
b *We have morphisms $X_{fl} \to X_{et} \to X_{Zar}$.*
c *Any morphism $\pi\colon X' \to X$ defines a morphism $\pi\colon X'_E \to X_E$ if it respects the underlying categories.*

Definition 2.22. *Let $\pi\colon X'_{E'} \to X_E$ be a morphism of sites. For any presheaf P' on $X'_{E'}$ we can associate the presheaf $\pi_p(P') = P' \circ \pi$ on X_E with $\Gamma(U, \pi_p(P')) = \Gamma(U \times_X X', P')$. This is called the direct image of P'. Clearly π_p defines a functor $PSh(X'_{E'}) \to PSh(X_E)$.*

Next we want to define the inverse image functor $\pi^p\colon PSh(X_E) \to PSh(X'_{E'})$ such that π^p is left adjoint to π_p.

Proposition 2.23. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small categories and let p be a functor $\mathcal{C} \to \mathcal{C}'$. Let $\mathcal{A}$ be a category with direct limits. Then the functor*

$$(f \longmapsto f \circ p)\colon \operatorname{Fun}(\mathcal{C}', \mathcal{A}) \longrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{A})$$

has a left adjoint.

One can check X_{et}, X_{Zar} are all small categories. To generalize refer to universes or basically bounded presheaves.

Leave the set-theoretical problem aside. Let $\pi^p P(U') = \operatorname{colim}_{(g,U)} P(U)$ where (g, U) makes a commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{g} & U \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\pi} & X \end{array}$$

with $U \to X$ in C/X. One can show $\pi^p P$ is a presheaf and π^p is left adjoint to π_p.

For any sites admitting finite limits (the Zariski, etale, flat sites) the colimit is filtered.

Example 2.24.
a *If P is constant presheaf then $\pi^p P$ is also constant presheaf defined by the same group.*
b *If $\pi\colon X' \to X$ is in C/X and $C'/X' = (C/X)/X'$ then $\pi^p P(U') = P(U')$.*
c *If $\pi\colon X \to X$ is the identity map then $\pi_p \pi^p = id$.*

Remark 2.25. *If π is the identity map $X_{fl} \to X_{et}$ or $X_{et} \to X_{Zar}$ then $\pi^p \mathbb{G}_a \neq \mathbb{G}_a$ in general.*

Proposition 2.26. *The functor π_p is exact and π^p is right exact. If finite inverse limits exist in C/X or π is in C/X and $C'/X' = (C/X)/X'$ then π^p is exact.*

Proposition 2.27. *If F is a sheaf so is $\pi_p F$.*

Slogan: stalks should be one abelian group defining a sheaf on one-point spaces.

In the etale site setting a point on X should be a geometric point. We shall discuss the stalks in this setting.

Definition 2.28. *Let x be a point of X. Let $\overline{x}$ be the spectrum of some separably closed field $\kappa(\overline{x})$ containing $\kappa(x)$ and $u_x \colon \overline{x} \to X$ denote the canonical map. For any presheaf P on X_{et} the stalk of P at $\overline{x}$ is the abelian group $P_{\overline{x}} = u_x^p P(\overline{x})$, i.e. $P_{\overline{x}} = \operatorname{colim} P(U)$ where U is finite type etale over X such that u_x factors through. Clearly $P_{\overline{x}}$ is independent of the choice of $\kappa(\overline{x})$.*

Remark 2.29.

- a *Taking stalk is exact.*
- b *The stalk $P_{\overline{x}}$ is acted on by $\operatorname{Gal}(\kappa(x)_{sep}/\kappa(x))$.*
- c *If U is finite type etale over X such that u_x factors through then there is associated canonical map $P(U) \to P_{\overline{x}}$ denoted by $s \mapsto s_{\overline{x}}$. Note that there might be several ways for u_x to factor through.*
- d *Let Y be a scheme locally of finite type over X. If (U_i) is a filtered inverse system of X-schemes with each U_i affine then the canonical map $\operatorname{colim} Y(U_i) \to Y(\lim U_i)$ is an isomorphism. Thus if P is a sheaf defined by a group scheme G that is LFT over X then*

$$P_{\overline{x}} = \operatorname{colim} G(U) = G(\lim U) = G(\mathcal{O}_{X,\overline{x}})$$

where $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{X,x}$. For example $(\mathbb{G}_a)_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$ and $(\mathbb{G}_m)_{\overline{x}} = \mathcal{O}^*_{X,\overline{x}}$.

Proposition 2.30. *Let F be a sheaf on X_{et}. If $s \in F(U)$ is nonzero then there is some $x \in X$ and an $\overline{x}$-point of U such that $s_{\overline{x}}$ is nonzero.*

Proof. Suppose not then we can find a covering of U to whom the restrictions of s are all zero. □

Theorem 2.31. *For any presheaf P on X_E there is an associated sheaf $P^\dagger$.*

We sketch two constructions.

• In the etale site setting, if X is a geometric point then sheafification is easy. For general X, choose $\overline{x}$ for every $x \in X$ and let $P^* = \prod_{x \in X} (u_x)_p (u_x^p P)^\dagger$ and $\phi \colon P \to P^*$ the induced map. Then let $P^\dagger$ to be the intersection of all subsheaves of P^* containing $\phi(P)$.

• For arbitrary site $(C/X)_E$ firstly for any U define $P_0(U)$ to be the set of all $s \in P(U)$ such that the restrictions of s are all zero for some covering of U. Then $P_1 = P/P_0$ is a separated presheaf. Define $P^\dagger(U) = \operatorname{colim} \check{H}^0(\mathcal{U}, P_1)$ where the filtered colimit runs over all coverings $\mathcal{U}$ of U and the zeroth Cech cohomology denotes compatible family of elements in the covering.

Remark 2.32.

- a *The above theorem says the natural inclusion functor of sheaves into presheaves has a left adjoint functor.*
- b *Let $\pi \colon X'_{E'} \to X_E$ such that π^p takes sheaves to sheaves. Then π^p commutes with sheafification.*
- c *u_x^p takes sheaves to sheaves hence sheafification has the same stalks as the original presheaf.*

Theorem 2.33.

- a *The inclusion functor $\operatorname{Sh}(X_E) \to \operatorname{PSh}(X_E)$ is left exact and preserves limits, the sheafification functor is exact and preserves colimits.*

b $0 \to F' \to F \to F''$ *is exact in sheaves iff exact in presheaves iff exact valuing on all* U*. For etale site, iff exact after taking stalks at all geometric points.*

c $\phi\colon F \to F'$ *is surjective morphism of sheaves iff for any* $s \in F'(U)$ *there is a covering* $(U_i \to U)$ *and elements* $s_i \in F(U_i)$ *such that* $\phi(s_i) = s|_{U_i}$*. For etale site, iff surjective on all stalks.*

d *Limits in sheaves are the same as their limits in presheaves. Colimits in sheaves are sheafification of their colimits in presheaves.*

e *The category of sheaves is abelian. Arbitrary products and direct sums exist and filtered colimit is exact.*

PROOF. A few facts which would be useful in the proof:

a A sheaf/presheaf in the etale site is zero iff its stalks are all zero.

b The presheaf kernel of a morphism of sheaves is its sheaf kernel. □

EXAMPLE 2.34. *It should be noted that arbitrary product might not be exact. Let* (M_i) *be a family of discrete* F*-modules where* G *is a profinite group. Let* $M_* = \prod M_i$ *and let* M *be the submodule* $\bigcup M_*^H$ *where the union runs through all open subgroups* H *of* G*. Then* M *is the product in the category of discrete* G*-modules. If* G *is infinite then this gives an example of products not being exact.*

REMARK 2.35.

a *Let* (F_i) *be a (pseudo)filtered system of sheaves on some site then their colimit* F *satisfy the sheaf condition for all finite coverings. In some cases this is enough to show* F *is a sheaf.*

b *If* X *is Jacobson, i.e. every closed subset of* X *is the closure of its set of closed points then only geometric points lying over closed points needed to be considered.*

c *The theorem above says again we can check properties of morphisms of sheaves on stalks.*

EXAMPLE 2.36.

a *The constant sheaf on* X *defined by an abelian group* M *is the sheafification of* P_M*. For the Zariski, etale and flat sites it is also the sheaf defined by the constant group scheme* M_X*. It suffices to show that the two takes the same values on quasi-compact/affine schemes. Let* π_0 *be the functor sending an affine scheme to its set of connected components. Then* π_0 *is left adjoint to the functor sending a set* T *to* $\coprod_{t\in T}(\operatorname{Spec}\mathbb{Z})_t$*. Thus* $\operatorname{Hom}_X(Y, M_X) = \operatorname{Hom}_{Sets}(\pi_0(Y), M)$*. Then it is easy to see we have a map from* P_M *to* M_X *satisfying the sheafification universal property on all quasi-compact schemes. In particular in etale site* $M_{\overline{x}} = M$*.*

b *Define a subsheaf* μ_n *of* $\mathbb{G}_m$ *by setting* $\mu_n(U) =$ *group of* n*-th roots of unity in* $\Gamma(U, \mathcal{O}_U)$*. This is the sheaf defined by the group scheme* $\operatorname{Spec}\mathbb{Z}[T]/(T^n - 1)$*. Consider the Kummer sequence*

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 0$$

where the map takes n*-th power. Clearly it is left exact. Note that if* A *is a strictly Henselian ring such that* n *is a unit in* A *then every unit in* A *has an* n*-th root. Thus the Kummer sequence is also exact in* $\operatorname{Sh}(X_{et})$ *if the characteristic of* $\kappa(x)$ *does not divide* n *for any* $x \in X$*.*

The Kummer sequence is also exact in $\operatorname{Sh}(X_{fl})$*. For any* $U \to X$ *in* LFT/X *and* $u \in \Gamma(U, \mathcal{O}_U)^*$ *let* (U_i) *be a flat affine covering locally of finite type and let* $U_i' \to U_i$ *be the map* $A_i \to A_i[T]/(T^n - u_i)$ *where* $A_i = \Gamma(U_i, \mathcal{O}_{U_i})$ *and* u_i *is the image of* u *to* U_i*. Then consider the covering* (U_i') *of* U*.*

c *Let* $(\mathbb{Z}/p\mathbb{Z})_X$ *be the constant sheaf defined by the abelian group* $\mathbb{Z}/p\mathbb{Z}$ *where* X *is a scheme over* $\mathbb{F}_p$*. As in char* p *we have*

$$T^p - T = \prod_{i=0}^{p-1}(T - i)$$

we see

$$\mathbb{F}_p[T]/(T^p - T) \cong \prod_{i=0}^{p-1}\mathbb{F}_p$$

hence $(\mathbb{Z}/p\mathbb{Z})_X = (\operatorname{Spec}\mathbb{F}_p[T]/(T^p - T))_X$ *and thus*

$$(\mathbb{Z}/p\mathbb{Z})_X(U) = \{a \in \Gamma(U, \mathcal{O}_U) \mid a^p - a = 0\}$$

Consider the Artin-Schreier sequence of sheaves

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \longrightarrow 0$$

where F *is the* p*-th power map. If* A *is strictly Henselian ring then* $F - 1\colon A \to A$ *is surjective and so the sequence is exact for the etale and flat sites.*

d *Let* X *be a scheme of char* p *and let* α_p *be the sheaf of* $\mathbb{G}_a$ *defined by*

$$\alpha_p(U) = \{a \in \Gamma(U, \mathcal{O}_U) \mid a^p = 0\}$$

then α_p *is the sheaf defined by the group scheme* $\mathbb{F}_p[T]/(T^p)$*. The infinitesimal sequence*

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \longrightarrow 0$$

is exact in flat site but not Zariski or etale sites.

3. Direct and Inverse Images of Sheaves

Definition 2.37. *Suppose* $\pi\colon X' \to X$ *defines a morphism of sites* $(C'/X')_{E'} \to (C/X)_E$*. The direct image of a sheaf* F' *on* $X'_{E'}$ *is defined to be* $\pi_* F' = \pi_p F'$ *and the inverse image of a sheaf* F *on* X_E *is defined to be* $\pi^* F = (\pi^p F)^\dagger$*. Then* π^* *is left adjoint to* π_**. If* π^p *is exact then* π^* *is also exact.*

Remark 2.38.

a *If* $\pi\colon X' \to X$ *is in* C/X *then* π^* *is simply the restriction functor.*

b *If* $\pi\colon X_{fl} \to X_{et}$ *then* $\pi_*\pi^* = id$*. In general this is not true.*

c *Let* $\pi\colon X' \to X$ *be a morphism and* G *be a group scheme on* X*. Assume the* E*-topology is coarser than the canonical topology so that* G *defines sheaves* G_X *and* $G_{X'}$ *on* X_E *and* $X'_{E'}$*. The map* $\pi^p G_X \to G_{X'}$ *sending the element of* $\Gamma(U', G)$ *represented by* (s, g) *with* $g\colon U' \to U$ *and* $s \in \Gamma(U, G)$ *to* $sg \in G_X(U') = G_{X'}(U') = \Gamma(U', G_{X'})$ *factors uniquely through* $\pi^* G_X$ *hence we get a canonical map* $\phi_G\colon \pi^* G_X \to G_{X'}$*.*

This map may not be an isomorphism in general. There are two important cases where ϕ_G *is an isomorphism, where* π^* *is a restriction map and where* G *is in* C/X*.*

d $(\pi'\pi)_* = \pi'_*\pi_*$ *and* $(\pi'\pi)^* = \pi^*{\pi'}^*$

Theorem 2.39. *Let* $\pi\colon X' \to X$ *be a morphism.*

a *For any sheaf* F *on* X_{et} *and any* $x' \in X'$*,* $(\pi^* F)_{\overline{x'}} = F_{\overline{\pi(x')}}$*. In particular if* π *is the canonical morphism* $\operatorname{Spec}\mathcal{O}_{X,\overline{x}} \to X$ *then*

$$F_{\overline{x}} = \Gamma(\operatorname{Spec}\mathcal{O}_{X,\overline{x}}, \pi^* F)$$

b *Assume* π *is quasi-compact. Let* $x \in X$ *and* $\overline{x} = \operatorname{Spec}\kappa(x)_s$*. Let* $f\colon \widetilde{X} = \operatorname{Spec}\mathcal{O}_{X,\overline{x}} \to X$ *and* $\widetilde{X}' = X' \times_X \widetilde{X}$ *then* $(\pi_* F)_{\overline{x}} = \Gamma(\widetilde{X}', (f')^* F)$*.*

Proof. (a). Write $x = \pi(x')$ and we may take $\overline{x} = \overline{x}'$. The rest follows from definition.

(b). Write down the definitions and apply the next lemma. □

Lemma 2.40. *Let* X *be a scheme and let* (Y_i) *be a filtered inverse system of* X*-schemes such that each* Y_i *is qcqs and all transition maps are affine. Then* $Y = \lim Y_i$ *exists and for any* Z *locally of finite type over* X *we have* $\operatorname{Hom}_X(Y, Z) = \operatorname{colim}\operatorname{Hom}_X(Y_i, Z)$*.*

This is one phenomenon related to inverse limit of schemes. For future applications we also list some other properties here. The results come from 01YT.

Lemma 2.41. *Same setting as above. Let* $Z \to Y$ *be a morphism of finite presentation then there exists some* i *and* $Z_i \to Y_i$ *of finite presentation such that* $Z = Z_i \times_{Y_i} Y$*.*

LEMMA 2.42. *Same setting as above. Suppose for some i there is a map $f_i\colon S_i \to T_i$ of qcqs schemes over Y_i whose base change to Y is affine/separated then for some $i' \geq i$ the base change $f_{i'}$ is affine/separated. Moreover if f is LFT then the same holds for finite/unramified/closed immersion and if f is LFP then the same holds for flat/finite locally free/smooth/etale/open immersion/isomorphism/surjective and so on.*

COROLLARY 2.43.

a *Let $i\colon Z \to X$ be a closed immersion and F be a sheaf on Z_{et}. Then for any $x \in X$ we have*

$$(i_*F)_{\overline{x}} = 0 \quad \text{if } x \notin i(Z)$$
$$(i_*F)_{\overline{x}} = F_{\overline{x_0}} \quad \text{if } x = i(x_0), x_0 \in Z$$

b *Let $j\colon U \to X$ be an open immersion and F be a sheaf on U_{et}. If $x \in j(U), x = j(x_0)$ then $(j_*F)_{\overline{x}} = F_{\overline{x_0}}$.*

c *Let $\pi\colon X' \to X$ be a finite morphism and F a sheaf on X'_{et}. For any $x \in X$, $(\pi_*F)_{\overline{x}} = \prod F_{\overline{x'}}^{d(x')}$ where the product is taken over all x' lying over x and $d(x')$ denotes the separable degree of $\kappa(x')$ over $\kappa(x)$. In particular if π is etale of constant degree d then $(\pi_*F)_{\overline{x}} = F_{\overline{x'}}^{d}$ for any x' lying over x.*

COROLLARY 2.44. *If π is finite then π_* is exact.*

REMARK 2.45. *As we will see later, for a proper morphism $\pi\colon X' \to X$ we have $(\pi_*F)_{\overline{x}} = \Gamma(X'_{\overline{x}}, F)$.*

The next example is a key ingredient to compute the cohomology of smooth curves.

EXAMPLE 2.46. *Let X be integral and quasi-compact. Let $g\colon \eta \to X$ be the generic point. Then for any $U \to X$ etale we have*

$$\Gamma(U, g_*\mathbb{G}_{m,\eta}) = R(U)^*$$

where $R(U)$ is the ring of rational functions on U. This is true by direct descriptions if U is quasi-compact and in general by gluing both sides. There is a canonical injection $\phi\colon \mathbb{G}_{m,X} \to g_*\mathbb{G}_{m,\eta}$ that on any U is simply the inclusion $\Gamma(U, \mathcal{O}_U^*) \to R(U)^*$. This injection reduces to the affine case in which the intersection of all minimal primes ideals is just zero. Denote the cokernel of ϕ by the sheaf of Cartier divisors, Div_X on X_{et}.

In case X is regular, Cartier divisors may be interpreted as Weil divisors. Let X_1 be the set of points of X of codimension 1, i.e. $\dim(\mathcal{O}_{X,x}) = 1$ so is a DVR. The sheaf D_X of Weil divisors on X_{et} is defined to be $\bigoplus_{x \in X_1} (i_x)_*\mathbb{Z}$ where $\mathbb{Z}$ denotes the constant sheaf. Then for any U etale over X we have $\Gamma(U, D_X) = \bigoplus_{u \in U_1} \mathbb{Z}$. We may define a map $\psi\colon g_*\mathbb{G}_{m,\eta} \to D_X$ by requiring that $f \in R(U)^*$ sent to $(\mathrm{ord}_u(f))_u$ where ord_u is the discrete valuation defined by $\mathcal{O}_{U,u}$. The map is well-defined and the sequence

$$0 \longrightarrow \mathbb{G}_{m,X} \xrightarrow{\phi} g_*\mathbb{G}_{m,\eta} \xrightarrow{\psi} D_X \longrightarrow 0$$

is exact. It is called the divisor class sequence.

To see this, note that at each $\overline{x}$ the stalk sequence is

$$0 \longrightarrow A^* \longrightarrow L^* \longrightarrow \bigoplus \mathbb{Z} \longrightarrow 0$$

where $A = \mathcal{O}_{X,\overline{x}}$ and L is the fraction field of A and the direct sum is taken over the primes of height 1. Since X is regular, so is A. Thus at stalks the sequence is exact.

In the special case of open immersion and closed immersion, we have more results. Consider the situation: X is a scheme, U is an open subscheme and Z is a closed subscheme such that $X = U \coprod Z$ as sets. Denote the inclusion maps $i\colon Z \to X$ and $j\colon U \to X$.

If F is a sheaf on X_{et} we have a canonical map $F \to j_*j^*F$. Apply i^* to this we get $\phi_F\colon F_1 = i^*F \to i^*j_*f^*F = i^*j_*F_2$. Thus associated to F there is a triple (F_1, F_2, ϕ_F) where $F_1 \in Sh(Z_{et})$, $F_2 \in Sh(U_{et})$ and $\phi_F\colon F_1 \to i^*j_*F_2$.

Denote $T(X)$ to be the category of the triples above. A morphism of the triples is a pair on sheaves commuting with the maps ϕ and ϕ'.

THEOREM 2.47. *There is an equivalence between the category $Sh(X_{et})$ with $T(X)$ given by $F \mapsto (i^*F, j^*F, \phi_F)$.*

PROOF. The functor $t\colon Sh(X_{et}) \to T(X)$ is given by $F \mapsto (i^*F, j^*F, \phi_F)$.

The functor $s\colon T(X) \to Sh(X_{et})$ is given by $(F_1, F_2, \phi) \mapsto s(F_1, F_2, \phi)$ where $s(F_1, F_2, \phi)$ is the fibre product

$$\begin{array}{ccc} s(F_1,F_2,\phi) & \longrightarrow & j_*F_2 \\ \downarrow & & \downarrow \\ i_*F_1 & \xrightarrow{i_*(\phi)} & i_*i^*j_*F_2 \end{array}$$

One checks $F \to st(F)$ is an isomorphism and more on stalks. □

If Y is any subscheme of X and F is a sheaf on X_{et} then we say F has its support on Y if $F_{\overline{x}} = 0$ for any $x \notin Y$.

COROLLARY 2.48. *If $i\colon Z \to X$ is a closed immersion then the functor i_* induces an equivalence of category $Sh(Z_{et})$ with the full subcategory of $Sh(X_{et})$ supported in $i(Z)$.*

REMARK 2.49. *A sequence in $T(X)$ is exact iff the sequences of sheaves in Z_{et} and U_{et} are both exact. For example there is an exact sequence in $T(X)$*

$$0 \longrightarrow (0, j^*F, 0) \longrightarrow (i^*F, j^*F, \phi_F) \longrightarrow (i^*F, 0, 0) \longrightarrow 0$$

It is possible to define extra functors

$$j_!\colon Sh(U_{et}) \to Sh(X_{et}) \text{ and } i^!\colon Sh(X_{et}) \to Sh(Z_{et})$$

In terms of $T(X)$, they are described together with other pushforwards and pullbacks as follows:

$$\begin{aligned} i^*\colon (F_1,F_2,\phi) &\mapsto F_1 & j_!\colon F_2 &\mapsto (0,F_2,0) \\ i_*\colon F_1 &\mapsto (F_1,0,0) & j^*\colon (F_1,F_2,\phi) &\mapsto F_2 \\ i^!\colon (F_1,F_2,\phi) &\mapsto \ker\phi & j_*\colon F_2 &\mapsto (i^*j_*F_2, F_2, 1) \end{aligned}$$

$j_!$ is called extension by zero and $i^!$ is called subsheaf of sections with support on Z.

Alternatively $j_!F$ is the sheaf associated to the presheaf

$$V \longmapsto \begin{cases} F(U \times_X V) & \textit{if} \quad \mathrm{Im}(V) \subset U \\ 0 & \textit{else} \end{cases}$$

PROPOSITION 2.50. *We have*

a $i^* \dashv i_* \dashv i^!$, $j_! \dashv j^* \dashv j_*$

b *The functors $i^*, i_*, j^*, j_!$ are exact.*

c *The composites $i^*j_!, i^!j_!, i^!j_*, j^*i_*$ are all zero.*

d *The functors $i_*, j_*, j_!$ are fully faithful.*

e *The functors $j_*, j^*, i^!, i_*$ preserves injectives.*

REMARK 2.51. *If U is the empty scheme in the above setting, i.e. $i\colon Z \to X$ is a surjective closed immersion, which occurs when Z is the closed subscheme of X cut out by a nilpotent ideal, then i_* is an equivalence of categories with quasi-inverse i^*. In fact this follows from the fact that the functor $Y \mapsto Y \times_X Z$ is an equivalence of categories of etale schemes over X to etale schemes over Z. In particular this tells us that the reducedness property does not affect the category, hence the cohomology.*

More generally if $\pi\colon Y \to X$ is any universal homeomorphism then the same is true. It is known that π is a universal homeomorphism iff it is integral, surjective and radicial. Examples are $X \times_k k' \to$

X where k' is a purely inseparable extension of k, or a morphism $X' \to X$ where X is geometrically unibranch and X' is the normalization of X_{red}.

Remark 2.52. *Let $j\colon U \to X$ be in C/X for some site $(C/X)_E$. We shall show that j^* has a left adjoint $j_!$ with many of the properties as the special case of extension by zero functor. Let $p\colon C/U \to C/X$, $p(g\colon Y \to U) = (jg\colon Y \to X)$. The functor*

$$(f \longmapsto f \circ p)\colon Fun(C/X, Ab) \longrightarrow Fun(C/U, Ab)$$

is the just the functor $j^p\colon PSh(X) \to PSh(U)$. Then by Proposition 2.23 we get a left adjoint $j_!\colon PSh(U) \to PSh(X)$. Explicitly for $P \in PSh(U)$ and $V \in C/X$, $j_!P(V) = \operatorname{colim} P(V')$ where the colimit is taken over all commutative diagrams

$$\begin{array}{ccc} V' & \longleftarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

in C/X. The colimit breaks into

$$j_!P(V) = \bigoplus_{\phi} \operatorname{colim}_{S(\phi)} P(V')$$

where $\phi \in \operatorname{Hom}_X(V, U)$ and $S(\phi)$ is the set of squares with $(V \to V' \to U) = \phi$. Since $S(\phi)$ contains a final object $V = V'$ we see

$$j_!P(V) = \bigoplus_{\phi} P(V_\phi)$$

where V_ϕ is the object $V \xrightarrow{\phi} U$ of C/U. Thus $j_!$ is exact. Note that if j is an open immersion then $j_!P(V) = P(V)$ if $V \to X$ factors through U and is zero otherwise.

Finally we define $j_!$ on sheaves to be the composite

$$Sh(U) \longrightarrow PSh(U) \xrightarrow{j_!} PSh(X) \xrightarrow{\dagger} Sh(X)$$

and clearly it is exact and adjoint to j^*.

We also list here some standard constructions and properties for sheaves of modules.

Let A be a sheaf of commutative rings on $(C/X)_E$. As usual we can associate to any presheaf of sets the free sheaf of A-module generated by it, to two sheaves of A-modules the internal hom and tensor product with an adjunction relation. A sheaf F of A-modules on X_{et} is called pseudo-coherent at a geometric point $\overline{x}$ if there exists an etale neighborhood $U \to X$ of $\overline{x}$ and an exact sequence

$$(A|_U)^m \longrightarrow (A|_U)^n \longrightarrow F|_U \longrightarrow 0$$

of sheaves on U_{et}. Focus on stalks we have

Proposition 2.53. *Let $\overline{x}$ be a geometric point of X.*

- a *For any presheaf of sets, the stalk of the free sheaf generated at $\overline{x}$ is the free module generated by its stalk.*
- b *For any pseudo-coherent module, taking stalks commutes with internal hom.*
- c *Taking stalks commutes with tensor products.*

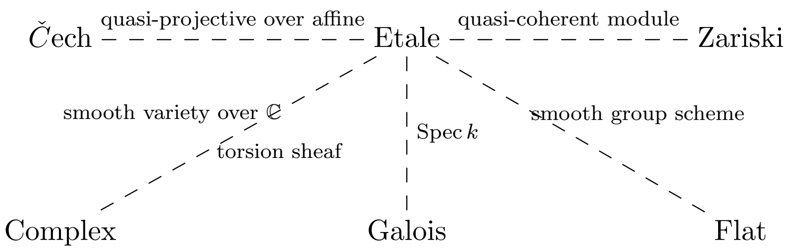
Proposition 2.54. *Let $\pi\colon X'_{E'} \to X_E$ be a morphism of sites. Then for any sheaves of abelian groups F on X and F' on X' we have*

$$\pi_* \mathcal{H}om(\pi^* F, F') = \mathcal{H}om(F, \pi_* F')$$

CHAPTER 3

Cohomology

Picture:



The general idea is to relate the etale cohomology with other cohomologies that we are more familiar with and that we know how to compute.

1. Cohomology

Most of the results in this section is to follow the general formalism of cohomology and derived functors. In particular we shall show that certain modules are acyclic for some functors hence we may apply the spectral sequence.

PROPOSITION 3.1. *The category* $\mathrm{Sh}(X_E)$ *has enough injectives.*

PROOF. This is true for any abelian category such that filtered colimits exist and are exact, arbitrary products exist and there is a family of generators. □

REMARK 3.2. *The category $Sh(X_E)$ rarely has enough projective objects.*

DEFINITION 3.3. *As usual we can define the derived functors $H^i(X_E, -)$, $H^i(U_E, -)$,* $\mathrm{Ext}^i(F, -)$*, $\mathcal{E}xt^i(F, -)$ and $R^i\pi_*$.*

Also the inclusion functor $i\colon Sh(X_E) \to PSh(X_E)$ is left exact and its right derived functors are written $\underline{H}^i(X_E, -)$.

REMARK 3.4.

a *As usual for any s.e.s. of sheaves there are two long exact sequences associated in* Ext *groups.*
b *$H^i(X_E, F)$ is a contravariant functor on X_E, i.e. if $\pi^*\colon Sh(X_E) \to Sh(X'_{E'})$ is exact then we get maps $H^i(X_E, F) \to H^i(X'_{E'}, \pi^* F)$ by the universal property of derived functors.*
c *These derived functors are related. For example, $H^i(X, -) = \mathrm{Ext}^i(\mathbb{Z}, -)$. Also $\underline{H}^i(X, F)$ is the presheaf $U \mapsto H^i(U, F)$.*

EXAMPLE 3.5. *Let $X = \mathrm{Spec}\, K$ and let $G = Gal(K_{sep}/K)$. If F is a sheaf on X_{et} corresponding to the discrete G-module M then $\Gamma(X, F) = M^G$ so $H^i(X, F) = H^i(G, M)$ is just the Galois cohomology.*

LEMMA 3.6. *Let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ be a left exact functor of abelian categories and assume $\boldsymbol{A}$ has enough injectives. Let T be a class of objects of $\boldsymbol{A}$ such that*

a *every object of $\boldsymbol{A}$ is a subobject of an object of T.*
b *if $A \oplus A' \in T$ so is A.*
c *if $0 \to A' \to A \to A'' \to 0$ is exact and A' and A are in T then so is A'' and apply f we still get an exact sequence.*

Then all injectives are in T and all elements of T are f-acyclic.

PROOF. Embed injectives as direct summand into T. □

EXAMPLE 3.7.

a *The class of all injectives in* $\boldsymbol{A}$ *above satisfies the conditions for* T.

b *A sheaf on a topological space* X *is called flabby if all the restriction maps are surjective. The class of flabby sheaves on* X *satisfies the conditions for* T *with* $f = \Gamma(X, -)$.

c *A sheaf* F *on a site* $(C/X)_E$ *is called flabby if* $H^i(U, F) = 0$ *for all* $U \in C/X$ *and* $i > 0$. *The class of flabby sheaves on* X_E *satisfies the conditions for* T *with* $f = \Gamma(X, -)$ *and* $f = \Gamma(U, -)$ *hence also* $F = \underline{H}^0(X, -)$. *We will see it also holds for* $f = \pi_*$ *so that we can use flabby sheaves to construct the Grothendieck spectral sequence.*

PROPOSITION 3.8. *For any sheaf* F *on* X_E *and any* $U \to X$ *in* C/X *the groups* $H^i(U, F)$ *and* $H^i(U, F|_U)$ *are canonically isomorphic.*

PROPOSITION 3.9. *Let* $\pi\colon X'_{E'} \to X_E$ *and let* $F \in Sh(X'_{E'})$. *Then* $R^i\pi_*F$ *is the sheafification of the presheaf* $U \mapsto H^i(U', F|_{U'})$ *where* $U' = U \times_X X'$.

COROLLARY 3.10. *If* F *is flabby then* $R^i\pi_*F = 0$ *for all* $i > 0$.

THEOREM 3.11. *Let* $\pi\colon Y \to X$ *be a quasi-compact morphism and* F *be a sheaf on* Y_{et}. *Let* $\overline{x}$ *be a geometric point of* X *such that* $\kappa(\overline{x})$ *is the separable closure of* $\kappa(x)$. *Let* $\widetilde{X} = \operatorname{Spec} \mathcal{O}_{X,\overline{x}}$ *and* $\widetilde{Y} = Y \times_X \widetilde{X}$ *and* $\widetilde{F}$ *the pullback of* F *to* $\widetilde{Y}$. *Then* $(R^i\pi_*F)_{\overline{x}} \cong H^i(\widetilde{Y}, \widetilde{F})$.

PROOF. It follows from definition and the next lemma. □

LEMMA 3.12. *Let* I *be a filtered category and* (X_i) *an inverse limit of schemes over* X *indexed by* I. *Assume all the schemes* X_i *are quasi-compact and all transition maps are affine. Then* $X_\infty = \lim X_i$ *exists and for any sheaf* F *on* X_{et} *we have*

$$\operatorname{colim} H^p((X_i)_{et}, F|_{X_i}) \xrightarrow{\sim} H^p((X_\infty)_{et}, F|_{X_\infty})$$

PROOF. See 03Q4. □

REMARK 3.13. *The above lemma holds for general sites as long as we have* $(X_\infty)_{E'} = \operatorname{colim}(X_i)_{E_i}$. *In particular the flat cohomology also commutes with colimits.*

REMARK 3.14. *If* π *is proper then for any torsion sheaf* F *on* Y_{et} *there is isomorphisms* $(R^p\pi_*F)_{\overline{x}} \cong H^p(Y_{\overline{x}}, F|_{Y_{\overline{x}}})$.

Recall the general result for Grothendieck spectral sequence.

PROPOSITION 3.15. *Let* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ *be abelian categories and let* $f\colon \boldsymbol{A} \to \boldsymbol{B}, g\colon \boldsymbol{B} \to \boldsymbol{C}$ *be left exact functors. If* $\boldsymbol{A}, \boldsymbol{B}$ *has enough injectives and* f *takes injectives to* g*-acyclics then there is a spectral sequence*

$$(R^pg)(R^qf) \Longrightarrow R^{p+q}(gf)$$

REMARK 3.16. *It might be beneficial to note that the* E_2 *page is* $(R^pg)(R^qf)$ *and the differential goes in the direction* $(2, -1)$. *In particular if* $R^n(gf) = 0$ *we should be carefully when concluding* $R^qf = 0$.

THEOREM 3.17.

a *(Leray spectral sequence) For any* $\pi\colon (C'/X')_{E'} \to (C/X)_E$ *there is a spectral sequence*

$$H^p(X_E, R^q\pi_*F) \Longrightarrow H^{p+q}(X'_{E'}, F)$$

b *For any* $X''_{E''} \xrightarrow{\pi'} X'_{E'} \xrightarrow{\pi} X_E$ *there is a spectral sequence*

$$(R^p\pi_*)(R^q\pi'_*)F \Longrightarrow R^{p+q}(\pi\pi')_*F$$

PROOF. Follows from the next lemma. □

We will prove this lemma in next section.

LEMMA 3.18. *Pushforward preserves flabby sheaves.*

REMARK 3.19. *If π^* is exact then π_* preserves injectives hence we get the spectral sequences for free.*

REMARK 3.20. *Recall if G is a finite/profinite group, an induced module is of the form $M_G(N) = \{f\colon G \to N \text{continuous}\}$ where N is an abelian group and G acts by $(\sigma f)\tau = f(\tau\sigma)$. In case $G = \mathrm{Gal}(k_s/k)$ for some field k, the induced G-modules correspond exactly to those sheaves on $X = \operatorname{Spec} k$ of the form u_*F where $u\colon \operatorname{Spec} k_s \to X$. Clearly induced modules are flabby. For any sheaf F on X_{et} the map $F \to u_*u^*F$ is injective. Hence we may use induced resolutions to compute derived functors.*

Now in general let X be a scheme and a sheaf F on X_{et} is called induced if it is of the form $\coprod_{x\in X}(u_x)_*(F_x)$ where F_x is a sheaf on $\overline{x}$. Again induced modules are flabby and we can use induced resolutions. An important fact is that every sheaf F on X_{et} has a canonical resolution by induced sheaves, called its Godement resolution, which is constructed as follows:

i $C^0(F) = u_*u^*F$ *where* $u\colon \coprod_{x\in X}\overline{x} \to X$. *There is a canonical map* $\epsilon\colon F \to C^0(F)$;

ii $C^1(F) = C^0(\operatorname{coker}\epsilon)$. *There is a canonical map* $d^0\colon C^0(F) \to C^1(F)$;

iii (*inductively*) $C^i(F) = C^0(\operatorname{coker} d^{i-2})$. *There is a canonical map* $d^{i-1}\colon C^{i-1}(F) \to C^i(F)$.

Then $F \to C^\bullet(F)$ is a flabby resolution of F, functorial in F and each functor $F \mapsto C^n(F)$ is exact. If X is Jacobson, only closed points are needed.

THEOREM 3.21 (Local-global spectral sequence for Exts). *There is a spectral sequence*

$$H^p(X_E, \mathcal{E}xt^q(F_1,F_2)) \Longrightarrow \mathrm{Ext}^{p+q}(F_1,F_2)$$

The proof relies on the next lemma, which will be proved in next section.

LEMMA 3.22. *$\mathcal{H}om(F_1,-)$ sends injectives to flabby sheaves.*

REMARK 3.23. *The sheaf $\mathcal{E}xt^p(F_1,F_2)$ is in fact the sheaf associated to the presheaf $U \mapsto \mathrm{Ext}^p(F_1|_U, F_2|_U)$.*

Now consider the situation $Z \xrightarrow{i} X \xleftarrow{j} U$ where j is open immersion and i is closed immersion such that $X = U \coprod Z$ as sets. We shall consider two related cohomology.

For any sheaf F on X_{et}, $i_*i^!F$ is the largest subsheaf of F that is zero outside Z. The group

$$\Gamma(X, i_*i^!F) = \Gamma(Z, i^!F) = \ker(F(X) \to F(U))$$

is called the group of sections of F with support in Z. The functor $F \mapsto \Gamma(Z, i^!F)$ is left exact and denote its right derived functors by $H^p_Z(X,F)$ called the cohomology groups of F with support in Z.

PROPOSITION 3.24. *For any sheaf F on X_{et} there is a long exact sequence*

$$0 \to (i^!F)(Z) \to F(X) \to F(U) \to \dots$$
$$\to H^p(X,F) \to H^p(U,F) \to H^{p+1}_Z(X,F) \to \dots$$

PROOF. For any sheaf F on X_{et} there is an exact sequence

$$0 \to j_!j^*F \to F \to i_*i^*F \to 0$$

now take $F = \mathbb{Z}$ and we get a long exact sequence

$$\dots \to \mathrm{Ext}^p(\mathbb{Z},F) \to \mathrm{Ext}^p(j_!\mathbb{Z},F) \to \mathrm{Ext}^{p+1}(i_*\mathbb{Z},F) \to \dots$$

now $\mathrm{Ext}^p(\mathbb{Z},F) = H^p(X,F)$. Since $\mathrm{Hom}_X(j_!\mathbb{Z},F) = \mathrm{Hom}_U(\mathbb{Z}, j^*F)$ and j^* preserves injectives, we see $\mathrm{Ext}^p(j_!\mathbb{Z},F) = H^p(U, F|_U)$. Since $\mathrm{Hom}_X(i_*\mathbb{Z},F) = \mathrm{Hom}_Z(\mathbb{Z}, i^!F) = H^0_Z(X,F)$ we get $\mathrm{Ext}^p(i_*\mathbb{Z},F) = H^p_Z(X,F)$. □

REMARK 3.25. *With a slight refinement we can show that for any triple $V \subset U \subset X$ where U, V are open subschemes and any sheaf F on X_{et} there is a long exact sequence*

$$\cdots \to H^p_{X-U}(X, F) \to H^p_{X-V}(X, F) \to H^p_{U-V}(U, F|_U) \to H^{p+1}_{X-U}(X, F) \to \dots$$

and it is functorial in the triples.

PROPOSITION 3.26 (Excision). *Let $Z \subset X$ and $Z' \subset X'$ be closed subschemes and let $\pi \colon X' \to X$ be etale such that $\pi|_{Z'}$ induces an isomorphism $Z' \cong Z$ and $\pi(X' - Z') \subset X - Z$. Then the maps $H^p_Z(X, F) \to H^p_{Z'}(X', \pi^* F)$ are all isomorphisms.*

PROOF. Since π is etale, by Remark 2.52 π^* is exact and preserves injectives. Hence it is enough to prove for $p = 0$, which is a diagram chasing. □

COROLLARY 3.27. *Let z be a closed point of X. Then $H^p_z(X, F) \xrightarrow{\sim} H^p_z(\operatorname{Spec} \mathcal{O}^h_{X,z}, F)$.*

Next we define the cohomology groups with compact support $H^p_c(X, F)$. Assume the scheme X is separated and finite type over a field k. The group of sections of F with compact support is defined to be

$$\Gamma_c(X, F) = \bigcup \ker(\Gamma(X, F) \to \Gamma(X - Z, F))$$

where Z runs through all closed subschemes of X which is proper over k. It is easy to see the union of the underlying closed set of two closed subschemes proper over k is still a closed subscheme proper over k given the reduced structure. Thus $\Gamma_c(X, F)$ is indeed an abelian subgroup of $\Gamma(X, F)$. The functor $\Gamma_c(X, -)$ is also left exact but its right derived functors might not carry enough information. For example if X is affine then $\Gamma_c(X, F) = \bigoplus_{x \in X \text{ closed}} H^0_x(X, F)$ and so $R^q\Gamma_c(F) = \bigoplus_x H^p_x(X, F)$. Instead we assume that X can be embedded $j \colon X \to \overline{X}$ as an open subscheme of a proper scheme $\overline{X}$ over k and define $H^p_c(X, F) = H^p(\overline{X}, j_! F)$. For the time being the definition depends on the embedding j.

PROPOSITION 3.28. *Let $j \colon X \to \overline{X}$ be as above.*

a $H^0_c(X, F) = \Gamma_c(X, F)$.
b *The functors $H^p_c(X, -)$ form a δ-functor.*
c *For any proper closed subscheme Z of X there is a canonical morphism of δ-functors $H^p_Z(X, -) \to H^p_c(X, -)$.*

PROOF. Let $j_0 F$ be the extension by 0 presheaf of F, then it is separated. Hence for any $U \to \overline{X}$ etale we have

$$(j_! F)(U) = \operatorname{colim}_{\mathcal{U}/U} \check{H}^0(\mathcal{U}/U, j_0 F)$$

Note that for any etale covering $\mathcal{U} \to \overline{X}$ we can refine it to $\{U, V\}$ where $\operatorname{Im}(U) \subset X$ and $\overline{X} - X \subset \operatorname{Im}(V)$. Thus every element in the global section $H^0_c(X, F) = (j_! F)(\overline{X})$ is represented by $s \in F(U)$ such that $s|_{U \times_{\overline{X}} V} = 0$. Use $\{U, X \times_{\overline{X}} V\}$ to give X a covering we see s could be glued to $a \in F(X)$ with $a|_{X \times_{\overline{X}} V} = 0$. Thus we conclude

$$H^0_c(X, F) = \bigcup \ker(\Gamma(X, F) \to \Gamma(V \times_{\overline{X}} X))$$

where $V \to \overline{X}$ is etale and contains $\overline{X} - X$ in its image.

Suppose $s \in \Gamma_c(X, F)$ so that there is some proper closed subscheme Z such that $s|_{X-Z} = 0$. Then Z is closed in $\overline{X}$ and so $s|_{V \cap X} = 0$ if $V = \overline{X} - Z$. Thus $s \in H^0_c(X, F)$. Conversely suppose $s \in H^0_c(X, F)$ so that $s|_{V \times X} = 0$ for some V. Let V' be the image of V, it is open in $\overline{X}$ and contains $\overline{X} - X$. Thus $Z = \overline{X} - V'$ is a proper closed subscheme in X. Since $V \times X \to V' \cap X$ is an etale covering, $s|_{V \times X} = 0$ implies $s|_{X-Z} = 0$ and $s \in \Gamma_c(X, F)$.

The long exact sequence formulation comes from $j_!$ being exact.

There is a canonical map $H^0_Z(X, F) \to H^0_c(X, F)$ and it induces maps on cohomology since derived functors are universal. □

REMARK 3.29. *Let Z be a closed subscheme of X. For any sheaf on X there is an exact sequence*

$$0 \longrightarrow j_! j^* F \longrightarrow F \longrightarrow i_* i^* F \longrightarrow 0$$

where $j\colon X - Z \to X$ is open immersion. Thus there is a long exact sequence

$$\cdots \to H^p_c(X - Z, F) \to H^p_c(X, F) \to H^p_c(Z, F) \to \dots$$

More generally if $X = X_0 \supset X_1 \supset \cdots \supset X_r \neq \emptyset$ is a sequence of closed subschemes of X then there is a spectral sequence

$$E_1^{q,p} = H_c^{p+q}(X_p - X_{p+1}, F) \Longrightarrow H_c^{p+q}(X, F)$$

PROPOSITION 3.30. *Let A be a Noetherian ring.*

a *If F is a sheaf of injective A-module on X_{et} then $F_{\overline{x}}$ is an injective A-module.*

b *If F is pseudo-coherent at $\overline{x}$ then $\mathcal{E}xt^p_A(F, G)_{\overline{x}} = \mathrm{Ext}^p_A(F_{\overline{x}}, G_{\overline{x}})$.*

c *$\mathcal{E}xt^p_A(F, G) = 0$ for $p > 0$ if F is locally free of finite rank, or if F is pseudo-coherent and G is an injective A-module.*

2. Čech Cohomology

As usual for a covering of X in the E-topology $\mathcal{U} = (U_i \xrightarrow{\phi_i} X)$ we can define the Čech complex $C^\bullet(\mathcal{U}/X, P)$ for any presheaf P, and denote its cohomology groups by $\check{H}^p(\mathcal{U}/X, P)$. There is a canonical map $P(X) \to \check{H}^0(\mathcal{U}/X, P)$ which is an isomorphism whenever P is a sheaf.

We have a natural analog of refinement of coverings and any refinement induces maps on cohomology groups (which only depends on the two coverings). Define the Čech cohomology groups of P over X to be the filtered colimit over a suitable set of coverings. Similarly we have $\check{H}^p(U, P)$ for any $U \in C/X$ and let $\underline{\check{H}}^p(X_E, P)$ denote the presheaf $U \mapsto \check{H}^p(U, P)$.

PROPOSITION 3.31. *The functors $\check{H}^p(\mathcal{U}/U, -)$ are the right derived functors of $\check{H}^0(\mathcal{U}/U, -)\colon PSh(X_E) \to Ab$ for any $U \in C/X$.*

LEMMA 3.32. *$\check{H}^p(\mathcal{U}/U, P) = 0$ for $p > 0$ if P is injective.*

COROLLARY 3.33. *The Čech cohomology groups $\check{H}^p(X, -)$ compute the derived functor cohomology $H^p(X, -)$ on sheaves iff for every s.e.s. of sheaves there is a functorially associated long exact sequence of Čech cohomology groups. This will be true for example if for every surjection $F \to F''$ of sheaves the map*

$$\mathrm{colim}(\prod F(U_{i_0 \dots i_p}) \to \prod F''(U_{i_0 \dots i_p}))$$

is surjective where the limit is taken over all suitable coverings of X.

EXAMPLE 3.34. *Let $Y \to X$ be a finite Galois covering with Galois group G. Then there is an isomorphism of the complex of inhomogeneous cochains of G with values in $P(Y)$ and the Čech complex $C^\bullet(Y/X, P)$ of the covering $Y \to X$. Thus $\check{H}^p(Y/X, P)$ is canonically isomorphic to $H^p(G, P(Y))$.*

PROPOSITION 3.35. *Let $U \to X$ be in C/X and $\mathcal{U}$ be a covering of U. There are spectral sequences*

$$\check{H}^p(\mathcal{U}/U, \underline{H}^q(F)) \Longrightarrow H^{p+q}(U, F)$$
$$\check{H}^p(U, \underline{H}^q(F)) \Longrightarrow H^{p+q}(U, F)$$
$$\underline{H}^p(X, \underline{H}^q(F)) \Longrightarrow \underline{H}^{p+q}(X, F)$$

PROPOSITION 3.36. *For all $U \to X$ in C/X we have $\check{H}^0(U, \underline{H}^q(F)) = 0$ for $q > 0$.*

Intuitively this says for any $s \in H^q(U, F)$, $q > 0$ there is a covering $U' \to U$ such that $s|_{U'} = 0$.

COROLLARY 3.37. *For any sheaf F and any $U \to X$ in C/X there are isomorphisms*

$$\check{H}^0(U, F) \cong H^0(U, F)$$
$$\check{H}^1(U, F) \cong H^1(U, F)$$

and an exact sequence

$$0 \to \check{H}^2(U,F) \to H^2(U,F) \to \check{H}^1(U,\underline{H}^1(F)) \to \check{H}^3(U,F) \to H^3(U,F)$$

PROPOSITION 3.38. *Let F be a sheaf on X_E. TFAE.*

a *F is flabby.*
b *$\check{H}^q(\mathcal{U}/U,F)=0$ for all $q>0$ and $U \to X$ in C/X and any cofinal systems of coverings.*
c *$\check{H}^q(U,F)=0$ for all $q>0$.*

COROLLARY 3.39.

a *If F is flabby then $F|_U$ is flabby for any $U \to X$ in C/X.*
b *If $\pi\colon X'_{E'} \to X_E$ and F is flabby then so is π_*F.*
c *If F is injective then $\mathcal{H}om(G,F)$ is flabby for any G.*

Next we consider about the relation between Čech cohomology and derived cohomology.

PROPOSITION 3.40. *Let F be a quasi-coherent $\mathcal{O}_X$-modules on X_{Zar} and assume X is separated. Then there are canonical isomorphisms $\check{H}^p(X_{Zar},F) \xrightarrow{\sim} H^p(X_{Zar},F)$ for all p.*

THEOREM 3.41. *Let X be a quasi-compact scheme such that every finite subset of X is contained in an affine open (for example X is quasi-projective over an affine scheme) and let F be a sheaf on X_{et}. Then there are canonical isomorphisms $\check{H}^p(X_{et},F) \cong H^p(X_{et},F)$.*

THEOREM 3.42 (Hochschild-Serre spectral sequence). *Let $\pi\colon X' \to X$ be a finite Galois covering with Galois group G and let F be a sheaf on X_{et}. There is a spectral sequence*

$$H^p(G,H^q(X'_{et},F)) \Longrightarrow H^{p+q}(X_{et},F)$$

PROOF. Note $\Gamma(X',-)^G=\Gamma(X,-)$ and $H^p(G,I(X'))=\check{H}^p(X'/X,I)$. □

REMARK 3.43.

a *The same argument shows that if F is a sheaf for the flat site then there is also a spectral sequence.*
b *If $X' \to X$ is an infinite Galois covering with Galois group G, by considering finite quotients and taking inverse limit we still could get a spectral sequence.*

EXAMPLE 3.44. *Let X be a regular integral quasi-compact scheme. We shall compute some of the cohomology groups of $\mathbb{G}_m$ on the etale site. Recall that there is an exact sequence*

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow g_*\mathbb{G}_{m,\eta} \longrightarrow D_X \longrightarrow 0$$

so we are left to compute $H^r(X_{et},D_X)$ and $H^r(X_{et},g_*\mathbb{G}_{m,\eta})$.

The Leray spectral sequence for $i_v\colon v \to X$ is

$$H^p(X,R^q(i_v)_*\mathbb{Z}) \Longrightarrow H^{p+q}(v,\mathbb{Z})$$

Note that $H^0(v,\mathbb{Z})=\mathbb{Z}, H^1(v,\mathbb{Z})=0, H^2(v,\mathbb{Z})=\mathrm{Hom}(G_v,\mathbb{Q}/\mathbb{Z})$. Also $R^1(i_v)_*\mathbb{Z}=0$. Thus

$$H^0(X,(i_v)_*\mathbb{Z})=\mathbb{Z}, H^1(X,(i_v)_*\mathbb{Z})=0, H^2(X,(i_v)_*\mathbb{Z}) \hookrightarrow \mathrm{Hom}(G_v,\mathbb{Q}/\mathbb{Z})$$

We have similar results for D_X since the cohomology commutes with direst sums in this case.

Similarly for $g\colon \eta \to X$ the Leray spectral sequence is

$$H^p(X,R^qg_*\mathbb{G}_{m,\eta}) \Longrightarrow H^{p+q}(\eta,\mathbb{G}_m)$$

Note the stalk of $R^qg_*\mathbb{G}_{m,\eta}$ at $\overline{x}$ is $H^q(K_{\overline{x}},\mathbb{G}_m)$ where $K_{\overline{x}}$ is the field of fractions of the strictly Henselization of X at $\overline{x}$. Thus $R^1g_*\mathbb{G}_{m,\eta}=0$ by Hilbert's 90. Write $K=K(X)$, it follows that

$$H^0(g_*\mathbb{G}_{m,\eta})=K^*, H^1(g_*\mathbb{G}_{m,\eta})=H^1(K,\mathbb{G}_m)=0, H^2(g_*\mathbb{G}_{m,\eta}) \hookrightarrow H^2(K,\mathbb{G}_m)$$

Put the results together we get exact sequence

$$0 \to \Gamma(X,\mathcal{O}_X)^* \to K^* \to \bigoplus_v \mathbb{Z} \to H^1(X,\mathbb{G}_m) \to 0$$

and in particular $H^1(X, \mathbb{G}_m) = \mathrm{Cl}(X) = \mathrm{Pic}(X) = H^1(X_{zar}, \mathbb{G}_m)$.

We shall mainly focus on two special cases.

Case (a): X *is a smooth curve over an algebraic closed field* k. *Then* $H^r(X_{et}, D_X) = 0$ *for* $r > 0$. *As we would see in next chapter that in this case both* K *and* $K_{\overline{v}}$ *are all* C_1 *fields so* $H^r(K, \mathbb{G}_m) = H^r(K_{\overline{v}}, \mathbb{G}_m) = 0$ *for all* $r > 0$. *Then it follows easily that* $H^r(X, \mathbb{G}_m) = 0$ *for* $r > 1$.

Case (b): X *is a proper smooth curve over a finite field* k. *Then* $H^r(K, \mathbb{G}_m) = 0$ *for* $r > 2$. *The class field theory for function fields implies*

$$H^2(X, \mathbb{G}_m) = 0, H^3(X, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}, H^r(X, \mathbb{G}_m) = 0 \text{ for } r > 3$$

REMARK 3.45. *When computing the stalk, the following fact might be useful: Let* $Y \to X, Z \to X$ *be morphisms of schemes. Then points* s *of* $Y \times_X Z$ *are in one to one correspondence with* $(x, y, z, \mathfrak{p})$ *where* $x \in X, y \in Y, z \in Z$ *such that* y *and* z *map to* x *and* $\mathfrak{p}$ *is a prime ideal of* $\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z}$ *and in addition the stalk of* s *is just* $(\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z})_{\mathfrak{p}}$.

PROPOSITION 3.46 (Mayer-Vietoris). *Let* U, V *be opens in* X *and let* F *be a sheaf on some site* $(C/X)_E$. *Then there is a long exact sequence*

$$\cdots \to H^{n-1}(U, F) \oplus H^{n-1}(V, F) \xrightarrow{\phi} H^{n-1}(U \cap V, F) \longrightarrow H^n(U \cup V, F) \to H^n(U, F) \oplus H^n(V, F) \to \ldots$$

where $\phi(s, t) = s - t$.

PROPOSITION 3.47. *Let* A *be a ring. Then the diagram commutes*

$$\begin{array}{ccc} Sh_A(X_{et}) & \xrightarrow{\textit{forget}} & Sh(X_{et}) \\ \downarrow R\Gamma & & \downarrow R\Gamma \\ A - Mods & \xrightarrow{\textit{forget}} & Ab \end{array}$$

i.e. as abelian groups the cohomology in A-Mods is the same as in Ab.

3. Comparison of Topologies

Changing C/X

PROPOSITION 3.48. *Let* C/X *be a subcategory of* C'/X *and let* $f\colon (C'/X)_E \to (C/X)_E$ *be the morphism induced by identity. Then*

- a *The functor* f_* *is exact and* $F \to f_* f^* F$ *is an isomorphism for any sheaf* F *on* $(C/X)_E$.
- b *The functor* f^* *is fully faithful.*
- c *The canonical maps*

$$H^i(X, f_* F') \to H^i(X, F') \text{ and } H^i(X, F) \to H^i(X, f^* F)$$

are isomorphisms.

REMARK 3.49. *The proposition implies in particular that the small E-site gives the same cohomology groups as the big E-site.*

Changing E

PROPOSITION 3.50. *Let* $E_2 \subset E_1$ *be two classes of morphisms satisfying the conditions at the beginning of Chapter* 2. *Let* C_2/X *be a subcategory of* C_1/X *and let* $f\colon (C_1/X)_{E_1} \to (C_2/X)_{E_2}$ *be the morphism induced by identity. Assume that for each* U *in* C_2/X *and every covering of* U *in* E_1*-topology there is a covering of* U *in* E_2*-topology that refines it. Then* f_* *is exact hence* $H^i(X_{E_2}, f_* F) \xrightarrow{\sim} H^i(X_{E_1}, F)$ *for any* F *on* X_{E_1}.

EXAMPLE 3.51. *The above proposition applies in the following cases of $E_2 \subset E_1$:*

$$\text{affine etale} \subset \text{separated etale} \subset \text{qc etale} \subset \text{etale}$$

and qc etale $\subset$ smooth by Proposition 1.70, and finite type flat $\subset$ flat LFT and quasi-finite flat $\subset$ flat LFT by Proposition 1.38.

Noetherian Sites

DEFINITION 3.52. *A site is called Noetherian if every covering $(U_i \to U)$ has a finite subcovering, for example if X is qc and C/X is the category of schemes of finite type over X then any class E whose morphisms are open would give a Noetherian site.*

PROPOSITION 3.53. *Let $(C/X)_E$ be a Noetherian site. Then the category of sheaves in the usual sense is equivalent to the category of sheaves with only finite coverings taken into consideration.*

REMARK 3.54.

a *Combine all the results together, to compute the cohomology of some sheaves on the big etale site, it suffices to consider the small etale site with finite coverings and the covering map being affine etale.*

b *On a Noetherian site the presheaf colimit of a (pseudo)filtered system of sheaves is a sheaf by Remark 2.35. Cohomology commutes with (pseudo)filtered colimits.*

Quasi-coherent Modules

PROPOSITION 3.55. *Let $f\colon X_{fl} \to X_{Zar}$ be the induced map from identity. Let F be a quasi-coherent $\mathcal{O}_X$-module and $W(F)$ be the corresponding sheaf on X_{fl}. Then there are canonical isomorphisms*

$$H^i(X_{Zar}, F) \xrightarrow{\sim} H^i(X_{fl}, W(F))$$

REMARK 3.56. *The proof essentially comes from the f.f. descent. Similarly we also have $H^i(X_{Zar}, F) \xrightarrow{\sim} H^i(X_{et}, W(F))$.*

Flat and Etale

THEOREM 3.57. *Let G be a smooth quasi-projective commutative group scheme over X, then the canonical maps*

$$H^i(X_{et}, G) \longrightarrow H^i(X_{fl}, G)$$

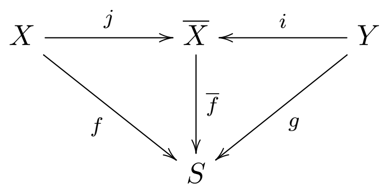
are isomorphisms.

REMARK 3.58. *The theorem holds if G is represented by a smooth commutative algebraic space. This implies for any sheaf F on X_{et}, the canonical maps $F \to f_*f^*F$ and $H^i(X_{et}, F) \to H^i(X_{fl}, f^*F)$ are isomorphisms.*

Etale and Complex

We first introduce an important lemma which is useful when inducting on the dimension.

DEFINITION 3.59. *A morphism $f\colon X \to S$ is called an elementary fibration if there is a commutative diagram*



such that

a *j is an open immersion dense in each fibre, and $Y = \overline{X} - X$ with i closed immersion;*

b *$\overline{f}$ is smooth and projective with geometrically irreducible fibres of dimension 1;*

c *g is finite etale and each fibre of g is nonempty.*

This would imply, for example, X is dense in $\overline{X}$ and $\overline{f}$ is surjective. One should think of $\overline{X}$ as a family of curves of same genus over S and X the same family with each curve removing the same number of points.

An Artin neighborhood relative to S is an S-scheme X with a sequence

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow X_0 = S$$

with each f_i an elementary fibration.

LEMMA 3.60. *Let X be a smooth scheme over an algebraically closed field k. Then any closed point x of X has an open neighborhood U that is an Artin neighborhood relative to k.*

We also recall the Riemann Existence Theorem.

LEMMA 3.61. *Let X be a scheme LFT over $\mathbb{C}$ and let X^{an} be the associated complex analytic space. The functor $Y \mapsto Y^{an}$ gives an equivalence between the category of finite etale coverings Y/X and the category of finite topological locally analytic isomorphism coverings of X^{an}.*

It is quite reasonable to expect the etale cohomology and classical complex cohomology have some relations over $\mathbb{C}$ since etale cohomology should be an analogue of the complex cohomology in algebraic geometry. Just as for the fundamental groups, for nontorsion coefficient groups their behavior might be different since $\pi_1(X_{et})$ is profinite. For example if X is smooth proper curve of genus g over $\mathbb{C}$ then $H^1(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^{2g}$ but

$$H^1(X_{et}, \mathbb{Z}) = \mathrm{Hom}_{\text{cont.}}(\pi_1(X_{et}, \mathbb{Z})) = 0$$

However as we shall see

$$H^1(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^{2g} = H^1(X_{et}, \mathbb{Z}/n\mathbb{Z})$$

THEOREM 3.62. *Let X be a smooth scheme over $\mathbb{C}$. For any finite abelian group M, $H^i(X(\mathbb{C}), M) = H^i(X_{et}, M)$.*

Sketch of proof: Induct on i.

For $i = 0$ we need to show $X(\mathbb{C})$ and X have the same number of connected components. This could be shown by applying the elementary fibration and inducting on the dimension.

For $i = 1$ the theorem says that there is a one to one correspondence between the Galois coverings of $X(\mathbb{C})$ with automorphism group M and the similar coverings of X. This is a direct corollary of the Riemann Existence Theorem.

For $i > 1$ let X^{an}_{et} be the small site associated to the class of morphisms of complex analytic spaces that are locally analytic isomorphisms. Then we have a natural map of sites $X^{an}_{et} \to X(\mathbb{C})$ which induces isomorphisms on cohomology $H^i(X^{an}_{et}, M) = H^i(X(\mathbb{C}), M)$. Also there is a map $an \colon X^{an}_{et} \to X_{et}$ under which the inverse image of an etale covering U over X is U^{an}. This makes sense since the implicit function theorem implies that $U^{an} \to X^{an}$ is a local isomorphism. Thus there is a Leray spectral sequence

$$H^i(X_{et}, R^j(an)_* M) \Longrightarrow H^{i+j}(X^{an}_{et}, M)$$

and it remains to show $R^j(an)_* M = 0$ for $j > 0$, which follows from the next lemma. □

LEMMA 3.63. *Let $\gamma \in H^i(X^{an}_{et}, F)$ for some $i > 0$ and F locally constant sheaf with finite fibres. Then for any $x \in X(\mathbb{C})$ there exists an etale morphism $U \to X$ whose image contains x and such that $\gamma|_{U^{an}_{et}} = 0$.*

PROOF. We shall induct on $n = \dim(X)$. We may assume F is constant and that $f \colon X \to S$ is an elementary fibration. All computations will be relative to the topology of local analytic isomorphisms.

There are Leray spectral sequences

$$H^i(S_{et}^{an}, R^j f_* F) \Longrightarrow H^{i+j}(X_{et}^{an}, F) \text{ and } (R^p \overline{f}_*)(R^q j_*)F \Longrightarrow R^{p+q} f_* F$$

then j_*F is the same constant sheaf on $\overline{X}$, $R^1 j_* F$ is the pushforward of a constant sheaf on Y and $R^q j_* F = 0$ for $q > 1$ by similar arguments as purity which would be discussed later. Also as $\overline{f}$ is proper and smooth, $R^i \overline{f}_*$ preserves locally constant sheaves by similar arguments as smooth proper base change which would be discussed later. It turns out that f_*F is again constant, $R^1 f_* F$ is locally constant with finite fibres and $R^i f_* F = 0$ for $i > 1$. Therefore we get a long exact sequence

$$\cdots \to H^i(S_{et}^{an}, f_*F) \to H^i(X_{et}^{an}, F) \to H^{i-1}(S_{et}^{an}, R^1 f_* F) \to \dots$$

By induction there is an etale map $U' \to S$ whose image contains $f(x)$ and such that $\gamma|_{U'^{an}_{et}} = \gamma'|_{U'^{an}_{et}} = 0$ for any given $\gamma \in H^{i-1}(S_{et}^{an}, R^1 f_* F)$ and $\gamma' \in H^i(S_{et}^{an}, f_* F)$ with $i > 1$. Thus we may take $U = U' \times_S X$ for some suitable U'. □

REMARK 3.64. *The theorem might be generalized to varieties over $\mathbb{C}$ (not necessarily smooth) and torsion sheaves on X_{et}.*

4. Principal Homogeneous Spaces

All group schemes will be flat and locally of finite type (hence f.f.).

DEFINITION 3.65. *Let G be a group scheme over X, then an action of G on an X-scheme S is a morphism $S \times_X G \to S$ inducing group action of $G(T)$ on $S(T)$ for every T over X. For example the multiplication $G \times_X G \to G$ defines an action of G on itself.*

PROPOSITION 3.66. *Let G act on S. TFAE.*

a *S is f.f. and LFT over X and the morphism*

$$S \times_X G \longrightarrow S \times_X S \ , \ (s, g) \longmapsto (s, sg)$$

is an isomorphism.

b *there is a covering $(U_i \to X)$ for the flat site such that $S_{(U_i)}$ is isomorphic to $G_{(U_i)}$ with its action.*

DEFINITION 3.67. *A scheme S with a G-action satisfying the equivalent conditions above is called a principal homogeneous space or torsor for G over X. A torsor isomorphic to G itself is called a trivial torsor.*

Note that S is trivial iff $S(X)$ is nonempty. Denote $PHS(G/X)$ the set of all isomorphism classes of torsors for G over X and the trivial torsors give a distinguished element. The proposition above says every torsor is locally trivial for the flat site.

PROPOSITION 3.68. *If G is smooth/etale/proper and so on over X then so is any G-torsor.*

COROLLARY 3.69. *If G is smooth then every G-torsor is locally trivial for the etale site.*

EXAMPLE 3.70. *If G is a commutative finite group then a G-torsor is a Galois covering of X with Galois group G thus $PHS(G/X) = \mathrm{Hom}_{cont.}(\pi_1(X), G)$.*

In general to compute $PHS(G/X)$ we need three steps. Firstly we define the concept of a sheaf being a G-torsor and investigate which sheaf torsors are representable by schemes. Secondly we show that the sheaf torsors are classified by certain cohomology group. Finally we compute the cohomology group.

DEFINITION 3.71. *Let G be a group scheme over X and let S be a sheaf of sets on $(LFT/X)_{fl}$ with a G-action. Then S is called a principal homogeneous space or torsor for G if it is locally trivial for the flat site. Clearly a G-torsor scheme S defines a G-torsor sheaf, and two schemes are isomorphic as G-torsors iff they are isomorphic as sheaf torsors.*

THEOREM 3.72. *A G-torsor sheaf S on X_{fl} is representable in the following cases:*

a G *is affine over* X.
b G *is smooth and separated over* X *and* X *has dimension* ≤ 1.
c G *is smooth and proper over* X *with geometrically connected fibers and is regular.*
d G *is quasi-projective over* X *and* S *becomes trivial on some* X' *finite f.f. over* X.
e G *is an abelian scheme projective over* X *and* S *defines a torsion element of* $\check{H}^1(X_{fl}, G)$.

Let G be a sheaf of groups on X_E and let $\mathcal{U} = (U_i \to X)$ be a covering. A 1-cocycle for $\mathcal{U}$ with values in G is a family $(g_{ij} \in G(U_{ij}))$ satisfying

$$(g_{ij}|_{U_{ijk}})(g_{jk|_{U_{ijk}}}) = (g_{ik}|_{U_{ijk}})$$

Two cocycles g and g' are cohomologous if there is a family $(h_i \in G(U_i))$ such that

$$g'_{ij} = (h_i|_{U_{ij}})g_{ij}(h_j|_{U_{ij}})^{-1}$$

this is an equivalence relation and the set of cohomology classes is written $\check{H}^1(\mathcal{U}/X, G)$. It is a set with a distinguished element $(g_{ij} = 1)$. The set $\check{H}^1(X, G)$ is the direct limit over all coverings. If G is abelian, the definition is the same as the usual Čech cohomology since filtered colimits of sets and modules agree.

A sequence $1 \to G' \to G \to G'' \to 1$ of sheaves of groups is exact if for every U in C/X, $G'(U)$ is the kernel of $G(U) \to G''(U)$ and every $s \in G''(U)$ can be locally lift to a section of G.

PROPOSITION 3.73. *To any exact sequence of sheaves of groups as above there is associated exact sequence of pointed sets*

$$1 \to G'(X) \to G(X) \to G''(X) \xrightarrow{d} \check{H}^1(X, G') \to \check{H}^1(X, G) \to \check{H}^1(X, G'')$$

Let S be a sheaf torsor for G and $(U_i \to X)$ be a flat covering that trivializes S. In particular $S(U_i)$ is nonempty. Choose $s_i \in S(U_i)$ then there is a unique $g_{ij} \in G(U_{ij})$ such that $(s_i|_{U_{ij}})g_{ij} = (s_j|_{U_{ij}})$. Then (g_{ij}) defines a 1-cocycle and the corresponding cohomology class is independent of the choice of s_i or the isomorphic torsor. Thus S defines an element $c(S) \in \check{H}^1(X, G)$.

PROPOSITION 3.74. *The map $S \to c(S)$ is bijective between isomorphism classes of sheaf torsors for G and $\check{H}^1(X, G)$ under which the trivial class corresponds to the distinguished element.*

COROLLARY 3.75. *There is a canonical injection $PHS(G/X) \to \check{H}^1(X_{fl}, G)$. If G is commutative, it is also an injection $PHS(G/X) \to H^1(X_{fl}, G)$. If G/X satisfies $(a), (b), (c)$ of Theorem 3.72 then the map $PHS(G/X) \to \check{H}^1(X, G)$ is an isomorphism of pointed sets.*

REMARK 3.76. *When G is commutative, it is possible to give a direct description of the composition law of sheaf torsors induced by the map $S \mapsto c(S)$ and the addition on $\check{H}^1(X_{fl}, G)$. If S is a sheaf of sets on which G acts then S/G is defined to be the sheaf associated to the presheaf $U \mapsto S(U)/G(U)$. If S_1, S_2 are torsors then let $S_1 \vee S_2 = S_1 \times S_2/G$ where G acts on the product by $(s_1, s_2)g = (s_1 g^{-1}, s_2 g)$. It is again a torsor and $c(S_1 \vee S_2) = c(S_1) + c(S_2)$.*

PROPOSITION 3.77 (Hilbert's 90). *The canonical maps*

$$H^1(X_{Zar}, \mathbb{G}_m) \longrightarrow H^1(X_{et}, \mathbb{G}_m) \longrightarrow H^1(X_{fl}, \mathbb{G}_m)$$

are isomorphisms. In particular $H^1(X_{et}, \mathbb{G}_m) = \operatorname{Pic}(X)$.

PROOF. Consider flat site for example. Let $f\colon X_{fl} \to X_{Zar}$ be the map induced by identity. It suffices to show $R^1 f_* \mathbb{G}_m = 0$. Taking stalk we get $H^1((\mathcal{O}_{X,x})_{fl}, \mathbb{G}_m)$. So it follows from the next lemma. □

LEMMA 3.78. *Let A be a local ring and $U = \operatorname{Spec} A$. Then*

$$H^1(U_{fl}, GL_n) = 0$$

PROOF. Here GL_n is the sheaf $U \mapsto GL_n(\Gamma(U, \mathcal{O}_U))$. Let $\alpha \in \check{H}^1(U_{fl}, GL_n)$. Then there is some affine V faithfully flat of finite type over U such that $\alpha \in \check{H}^1(V/U, GL_n)$. Let $B = \Gamma(V, \mathcal{O}_V)$ and let $\beta \in GL_n(V \times_U V) = GL_n(B \otimes_A B)$ be a cocycle lift of α. Then β may be regarded as an isomorphism $(B \otimes_A B)^n \to (B \otimes_A B)^n$ or $B^n \otimes_A B \to B \otimes_A B^n$. The fact that β is a cocycle implies that (B^n, β) is a descent data. Thus there is an A-module M with $M \otimes_A B = B^n$. Since B^n is flat and finite over B by descent so is A. Hence M is finite free and β is a coboundary. $\square$

As a corollary if we take $X = \operatorname{Spec} k$ then $H^1(k, \mathbb{G}_m) = \operatorname{Pic}(k) = 0$.

With everything interpreted into torsors, this lemma states that vector bundles on the flat or etale site are the same as on the Zariski site. To see this recall there is a general principle:

Let T be a sheaf of sets on X_{et} and $G = \mathcal{A}ut(T)$. Then there is a natural bijection

$$\{\text{isomorphism class of } G\text{-torsors}\} \overset{1:1}{\longleftrightarrow} \{\text{isomorphism class of forms of } T\}$$
$$P \longmapsto (P \times T)/G$$
$$\mathcal{I}som(F, T) \longleftarrow F$$

where a sheaf S is called a form of T if it is locally isomorphic to T.

Clearly if we take $T = \mathcal{O}_X^n$ then the forms of T are just vector bundles while the left side is GL_n-torsors.

Kummer Theory

For any scheme X and n, the Kummer sequence gives rise to

$$0 \to \mu_n(X) \to \Gamma(X, \mathcal{O}_X)^* \xrightarrow{n} \Gamma(X, \mathcal{O}_X)^* \to H^1(X, \mu_n) \to \operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X)$$

Thus $H^1(X, \mu_n)$ is the set (modulo isomorphism) of pairs (L, ϕ) where L is an invertible sheaf on X and $\phi\colon \mathcal{O}_X \xrightarrow{\sim} L^{\otimes n}$. The image of (L, ϕ) in $\operatorname{Pic}(X)[n]$ is $[L]$. If L is trivial by $\psi\colon \mathcal{O}_X \to L$, then (L, ϕ) is the image of some $a \in \Gamma(X, \mathcal{O}_X)^*$ where

$$\mathcal{O}_X \xrightarrow{\phi} L^{\otimes n} \xrightarrow{(\psi^{-1})^{\otimes n}} \mathcal{O}_X^{\otimes n} \xrightarrow{\text{can}} \mathcal{O}_X$$

is multiplication by a^{-1}. The μ_n-torsor corresponding to (L, ϕ) is $S = \mathcal{S}pec B$ where B is the coherent $\mathcal{O}_X$-algebra $\bigoplus_{i=0}^{n-1} L^{\otimes i}$ with multiplication $B \otimes B \to B$ given by $\phi^{-1} \circ \text{can}$ or can depends on whether $i + j \geq n$. In particular if $X = \operatorname{Spec} A$ is affine and there is an isomorphism $\psi\colon A \to L$ such that $(\psi(1))^{\otimes n} = a\phi(1)$ for $a \in A^*$ then $B = A[T]/(T^n - a)$.

If n is invertible in $\Gamma(X, \mathcal{O}_X)$, for example X is over a field k and n is prime to $\operatorname{char} k$, then noncanonically there is an identification $\mu_n \to \mathbb{Z}/n\mathbb{Z}$.

Now assume X is a proper smooth curve of genus g over an algebraically closed field k whose char is prime to n. Recall $\operatorname{Pic}(X) = \operatorname{Pic}^0(X) \oplus \mathbb{Z}$ and $\operatorname{Pic}^0(X) = J(k)$ where J is the Jacobian of X. As J is an abelian variety of dimension g, $J(k)[n] = (\mathbb{Z}/n\mathbb{Z})^{2g}$ and $J(k)$ is divisible. Thus we get

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^{2g}$$

which agrees with the computation over $\mathbb{C}$. If in addition we could show $H^2(X, \mathbb{G}_m) = 0$ then $H^2(X, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

For the Artin-Schreier sequence and infinitesimal sequence we could have similar descriptions. Refer to Milne's book for more details.

Weak Mordell-Weil Theorem

The above theory has many arithmetic applications.

THEOREM 3.79. *Let A be an abelian variety over a number field K. For any integer n, $A(K)/nA(K)$ is finite.*

Proof. There is an open subset U of the spectrum of the ring of integers in K such that A has good reduction at every point of U, that is, A is the generic fibre of an abelian scheme A' over U. We may assume n is invertible on U. Then there is an exact sequence of sheaves on U_{et}

$$0 \longrightarrow N \longrightarrow A' \xrightarrow{n} A' \longrightarrow 0$$

since multiplication by n is etale on A' by looking at each fibre. The cohomology group of the sequence is

$$\cdots \to A'(U) \xrightarrow{n} A'(U) \to H^1(U_{et}, N) \to H^1(U, A') \to \dots$$

Since $A = A' \times_U \operatorname{Spec} K$ we know $A'(K) = A(K)$ and by valuative criterion for properness $A'(K) = A'(U)$. Thus it suffices to show $H^1(U, N)$ is finite. We may replace U by an open and again by a Galois covering. Thus we are in the situation where N is constant sheaf killed by n and $\Gamma(U, \mathcal{O}_U)$ contains the n-th roots of unity. Since cohomology commutes with direst sum in this case, we may just assume $N = \mathbb{Z}/n\mathbb{Z}$. Now consider the s.e.s.

$$0 \to \Gamma(U, \mathcal{O}_U)^*/\Gamma(U, \mathcal{O}_U)^{*n} \to H^1(U_{et}, \mathbb{Z}/n\mathbb{Z}) \to \operatorname{Pic}(U)_n \to 0$$

Fundamental theorems in algebraic number theory tell us that $\Gamma(U, \mathcal{O}_U)$ is finitely generated and has finite ideal class group. Hence the middle term is also finite. □

CHAPTER 4

The Brauer Group

Our main goal in this chapter is to prove the next theorem.

THEOREM. *Let X be a smooth curve over an algebraically closed field and let $K(X)$ be its function field. For any $x \in X$ closed let $K_{\overline{x}}$ be the fraction field of $\mathcal{O}_{X,\overline{x}}$. Then we have*

$$H^i(L, \mathbb{G}_m) = 0 \text{ for } i > 0$$

if $L = K(X)$ or $L = K_{\overline{x}}$.

For $i = 1$ it follows from Hilbert's 90. For $i = 2$ the group $H^2(L, \mathbb{G}_m)$ is usually called the Brauer group of L, classifying similarity classes of central simple algebras over L.

We will firstly generalize these notions to schemes and then prove this theorem.

1. Azumaya Algebras

In schemes what plays the role of central simple algebras over fields is called Azumaya algebras.

DEFINITION 4.1. *Let R be a local ring and A an R-algebra which is not necessarily commutative. We assume A has an identity element and the map $R \to A$ sending r to $r \cdot 1$ identifies R with a subring of the center of A. Let A° denote the opposite algebra to A, i.e. the algebra with the multiplication reversed. Then A is called an Azumaya algebra over R if it is of finite rank as an R-module and the map $A \otimes_R A^\circ \to \mathrm{End}_{R-mod}(A)$ sending $a \otimes a'$ to $(x \mapsto axa')$ is an isomorphism.*

PROPOSITION 4.2.

- a *If A is an Azumaya algebra over R and R' is a local R-algebra then $A \otimes_R R'$ is an Azumaya algebra over R'.*
- b *If A is free of finite rank as an R-module and $\overline{A} = A \otimes_R (R/\mathfrak{m})$ is an Azumaya algebra over $R/\mathfrak{m}$ then A is an Azumaya algebra over R.*
- c *If A and A' are Azumaya algebras over R then so is $A \otimes_R A'$.*
- d *The matrix ring $M_n(R)$ is an Azumaya algebra over R.*

PROPOSITION 4.3 (Skolem-Noether). *Let A be an Azumaya algebra over R. Every automorphism of A as an R-algebra in of the form $a \mapsto uau^{-1}$ with $u \in A^*$.*

COROLLARY 4.4. *The automorphism group of $M_n(R)$ as an R-algebra is $PGL_n(R) = GL_n(R)/R^*$.*

Now let X be a scheme.

DEFINITION 4.5. *An $\mathcal{O}_X$-algebra A is called an Azumaya algebra over X if it is coherent as an $\mathcal{O}_X$-module and for all points $x \in X$, A_x is an Azumaya algebra over $\mathcal{O}_x$.*

PROPOSITION 4.6. *Let A be a coherent $\mathcal{O}_X$-module. TFAE.*

- a *A is an Azumaya algebra over X.*
- b *A is locally free as an $\mathcal{O}_X$-module and $A(x) = A_x \otimes \kappa(x)$ is a central simple algebra over $\kappa(x)$ for all $x \in X$.*
- c *A is locally free as an $\mathcal{O}_X$-module and the canonical morphism $A \otimes_{\mathcal{O}_X} A' \to \mathcal{E}nd_{\mathcal{O}_X-mod}(A)$ is an isomorphism.*

d *There is a covering* $(U_i \to X)$ *for the etale/flat topology on* X *such that for each* i, $A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} = M_{r_i}(\mathcal{O}_{U_i})$ *for some* r_i.

PROPOSITION 4.7 (Skolem-Noether). *Let* A *be an Azumaya algebra on* X. *Any automorphism* ϕ *of* A *is locally for the Zariski topology on* X, *an inner automorphism, i.e. there is a covering* (U_i) *of opens in* X *such that* $\phi|_{U_i}$ *is of the form* $a \mapsto uau^{-1}$ *for some* $u \in \Gamma(U_i, A)^*$.

Let GL_n be the presheaf $S \mapsto GL_n(\Gamma(S, \mathcal{O}_S)) = M_n(\Gamma(S, \mathcal{O}_S))^*$. Then it is representable by an affine scheme, hence is a sheaf for the flat site or coarser topology.

Let PGL_n be the presheaf $S \mapsto \mathrm{Aut}(M_n(\mathcal{O}_S))$. Then it is also representable hence is a sheaf for the flat site or coarser topology. To see this note that any automorphism of $M_n(\mathcal{O}_S)$ as an $\mathcal{O}_S$-algebra may be regarded as an endomorphism of $M_n(\mathcal{O}_S)$ as an $\mathcal{O}_S$-module. Thus PGL_n is a subpresheaf of M_{n^2}. The condition that an endomorphism be an automorphism of algebras is described by polynomials, hence PGL_n is represented by a closed subscheme of M_{n^2}.

As a corollary of Skolem-Noether we have

LEMMA 4.8. *There is a s.e.s. of sheaves of groups on* X_{Zar}, X_{et}, X_{fl}

$$1 \longrightarrow \mathbb{G}_m \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1$$

We shall denote the map $H^1(X_{et}, PGL_n) \to H^2(X_{et}, \mathbb{G}_m)$ by δ_n.

2. Brauer Groups

DEFINITION 4.9. *Let* X *be a scheme. The cohomological Brauer group* $Br^{Coh}(X)$ *is defined to be* $H^2(X, \mathbb{G}_m)_{tor}$.

We shall try to understand $Br^{Coh}(X)$ geometrically by relating it with Azumaya algebras.

DEFINITION 4.10. *Let* $Br(X)$ *be*

$$\bigcup_n (\mathrm{Im}\, \delta_n \colon H^1(X_{et}, PGL_n) \to H^2(X, \mathbb{G}_m))$$

This should be the collection of all Azumaya algebras. However in priori we do not even know this is a subgroup.

Note that $H^1(X_{et}, PGL_n)$ is also the isomorphism classes of PGL_n-torsors. Using Čech cohomology we could make δ_n explicit. Let $[T] \in H^1(X_{et}, PGL_n)$ be a PGL_n-torsor split by some $U \to X$. On $U \times_X U$ the descent data is given by some element in $\Gamma(U \times_X U, PGL_n)$ satisfying the cocycle condition. After refining U, we can lift the descent data to $s \in \Gamma(U \times_X U, GL_n)$ and the cocycle condition implies that

$$\pi_{23}^* s \circ \pi_{12}^* s \circ (\pi_{13}^* s)^{-1} \in \Gamma(U \times_X U \times_X U, \mathbb{G}_m)$$

and this element is a 2-cocycle representing $\delta_n([T]) \in \check{H}^2(X_{et}, \mathbb{G}_m) \hookrightarrow H^2(X_{et}, \mathbb{G}_m)$. It is the obstruction to lifting T to a GL_n-torsor.

Now to relate Azumaya algebras with PGL_n-torsors recall the general principle that forms of a sheaf T corresponds to $\mathcal{A}ut(T)$-torsors. By Skolem-Noether we know the sheaf $M_n(\mathcal{O}_X)$ has automorphism groups PGL_n hence PGL_n-torsors are the same as forms of $M_n(\mathcal{O}_X)$ which are just Azumaya algebras by Proposition 4.6.

REMARK 4.11. *Actually we also have* $\mathcal{A}ut_X(\mathbf{P}_X^{n-1}) = PGL_n$ *hence* PGL_n*-torsors and Azumaya algebras also correspond to forms of* $\mathbf{P}_X^{n-1}$ *which are called Severi-Brauer schemes. By considering the moduli of certain ideals in Azumaya algebra one can get Severi-Brauer scheme.*

The description of δ_n in terms of Čech cohomology inspires the following definition.

DEFINITION 4.12. *Let $U \to X$ be an etale covering and $\alpha \in \Gamma(U \times_X U \times_X U, \mathbb{G}_m)$ representing $[\alpha] \in H^2(X_{et}, \mathbb{G}_m)$. An α-twisted sheaf is a quasi-coherent sheaf F on U_{et} with an isomorphism $\phi\colon \pi_1^* F \to \pi_2^* F$ such that*

$$\pi_{23}^*\phi \circ \pi_{12}^*\phi = \alpha \cdot \pi_{13}^*\phi$$

Let $QCoh(U/X, \alpha)$ be the category whose objects are α-twisted sheaves and morphisms are morphisms of sheaves commuting with ϕ.

It is easy to see if $[\alpha] = [\beta]$ then there is an equivalence $QCoh(U/X, \alpha) = QCoh(X, \beta)$. In particular if $[\alpha]$ is trivial then $QCoh(U/X, \alpha) = QCoh(X)$ by descent.

PROPOSITION 4.13. *Let α, β be 2-cocycles for some U/X and $\mathbb{G}_m$ as above.*

a $[\alpha] \in Br(X) \iff \exists\alpha$*-twisted vector bundle.*

b $QCoh(X, \alpha)$ *is an abelian category.*

c *there are functors*

$$\otimes\colon QCoh(X,\alpha) \times QCoh(X,\beta) \to QCoh(X, \alpha+\beta)$$
$$\mathrm{Hom}\colon QCoh(X,\alpha) \times QCoh(X,\beta) \to QCoh(X, \beta-\alpha)$$

d *there are functors*

$$\mathrm{Sym}^n, \wedge^n\colon QCoh(X,\alpha) \to QCoh(X, n\alpha)$$

COROLLARY 4.14. *$Br(X)$ is a group.*

PROOF. Addition corresponds to tensor product of vector bundles and inverse corresponds to dual of vector bundles. □

PROPOSITION 4.15. *Let α be a 2-cocycle for $\mathbb{G}_m$. Then*

$$[\alpha] \text{ is trivial } \iff \exists\alpha\text{-twisted line bundle}$$

PROOF. In this case ϕ is given by some $a \in \Gamma(U \times_X U, \mathbb{G}_m)$ and $\delta(a) = \alpha$. □

COROLLARY 4.16. *Suppose E is an α-twisted vector bundle of rank n. Then $[\alpha]$ is n-torsion.*

PROOF. Consider $\wedge^n E$. □

COROLLARY 4.17. *We have $Br(X) \subset Br^{Coh}(X)$. In particular if $X = \operatorname{Spec} k$ then they agree.*

Now it is easy to see given α a 2-cocycle representing $[\alpha] \in Br(X)$ and an α-twisted vector bundle E, $\mathcal{E}nd(E)$ is just the corresponding Azumaya algebra and $\mathbf{P}(E)$ the Severi-Brauer scheme.

DEFINITION 4.18 (Reduced norm). *Let E be an α-twisted vector bundle of rank n and $\mathcal{E}nd(E)$ be the corresponding Azumaya algebra. Define*

$$Nm\colon \mathcal{E}nd(E) \to \mathcal{E}nd(\wedge^n E) = \mathcal{O}_X$$

given by functorial of $\wedge$.

PROPOSITION 4.19. *Given $f \in \mathcal{E}nd(E)(X)$. Then f is invertible iff $Nm(f)$ is a unit.*

3. Proof of Theorem

DEFINITION 4.20. *A field k is called C_1 field (quasi-algebraically closed) if any homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ of degree $d < n$ has a nontrivial zero.*

THEOREM 4.21 (Tsen). *Let k be a C_1 field. Then $Br(k) = 0$.*

PROOF. Given $[\alpha] \in Br(k)$ we want to find an α-twisted line bundle. Let E be an α-twisted vector bundle of rank $n > 1$. Consider the $Nm\colon \mathcal{E}nd(E)(X) \to k$, this gives a homogeneous polynomial in

n^2-variables with degree n. Thus it admits a nontrivial solution f. Regard $f \in \mathcal{E}nd(E)$ and let $E' = \ker f$. Then E' is an α-twisted vector bundle of rank $< n$. Keep going until we will get α-twisted line bundle. □

Thus for any C_1 field k we have $H^2(k, \mathbb{G}_m) = 0$. For higher degrees we need to work a little bit more.

LEMMA 4.22. *Any algebraic extension of C_1 field is again C_1.*

PROOF. We can assume k'/k is finite of degree m with k C_1. Let $f \in k'[x_1, \ldots, x_n]$ homogeneous of degree $d < n$. Then $Nm_{k'/k}f(x)$ is a homogeneous polynomial with coefficients in k, in nm variables and of degree dm. Then a nontrivial zero of $Nm_{k'/k}f(x)$ will give rise to a nontrivial solution of f. □

PROPOSITION 4.23. *Let K be a C_1 field. Then $H^i(K, \mathbb{G}_m) = 0$ for $i > 0$.*

PROOF. This is the standard process as in class field theory. We may reduce to the the case of finite field extension by inflation-restriction. Then it suffices to prove for $H^i(L/K, \mathbb{G}_m)$ for all L/K finite separable Galois.

As K is C_1, $H^i(L/K, \mathbb{G}_m) = 0$ for $i = 1, 2$ by Hilbert's 90 and the inflation-restriction sequence.

If L/K is cyclic then the cohomology groups are 2-periodic by Tate's theorem so we are done.

If $\mathrm{Gal}(L/K)$ is solvable then we can take a normal subgroup H with G/H cyclic and induct on the size of the Galois group using inflation-restriction sequence.

In general for $\mathrm{Gal}(L/K)$ finite consider its p-Sylow subgroup G_p, which is solvable. Then the restriction map $H^i(G, M) \to \bigoplus_p H^i(G_p, M)$ is injective. □

Now let K either be the function field $K(X)$ of a smooth curve X over an algebraically closed field k, or the fraction field $K_{\overline{x}}$ of $\mathcal{O}_{X,\overline{x}}$. We need to show K is C_1.

Assume $K = K(X)$. Let $f \in K[x_1, \ldots, x_n]$ be homogeneous of degree $d < n$. We want to find a nontrivial zero of f in K^n. Choose an ample divisor D on X and let

$$Y = \Gamma(X, \mathcal{O}(mD))^n \to Z = \Gamma(X, \mathcal{O}(dmD + D'))$$

where the map is given by f and D' is the poles of coefficients of f. By Riemann-Roch, as $d < n$ and D ample, for m large enough

$$\dim(Y) \sim mn > md \sim \dim(Z)$$

Since f is a polynomial this is a map of affine spaces over K, hence the dimension of any nonempty fibre is bigger than 0. Obviously $0 \in f^{-1}(0)$ hence $f^{-1}(0)$ is nonempty hence we can find another nonzero element.

For the other case where K is the fraction field of a strictly Henselian DVR, refer to Serge Lang, *On Quasi Algebraic Closure.*

Here we conclude our computation results for a smooth curve X over algebraically closed field.

THEOREM 4.24. *Assume n is prime to $\mathrm{char}(k)$. If X is proper then*

$$H^i(X_{et}, \mu_n) = \begin{cases} \mu_n & \text{if } i = 0 \\ \mathrm{Pic}(X)[n] & \text{if } i = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i > 2 \end{cases}$$

CHAPTER 5

The Cohomology of Curves and Surfaces

Slogan: The smallest useful class of sheaves containing the finite constant sheaves and preserved under direct images of proper morphisms is the class of constructible sheaves. The category of constructible sheaves is abelian and they are precisely those that can be represented by etale algebraic spaces of finite type.

Poincare duality for curves.

Lefschetz pencil to compute the cohomology of surfaces.

1. Constructible Sheaves : Pairings

DEFINITION 5.1. *A sheaf F on X_{et} is called finite if $F(U)$ is finite for all quasi-compact U. It has finite stalks if $F_{\overline{x}}$ is finite for all geometric points. These two assumptions do not imply each other. A sheaf F is locally constant if there is a covering $(U_i \to X)$ such that $F|_{U_i}$ is constant for all i.*

PROPOSITION 5.2. *Let F be a locally constant sheaf on X_{et}. If F has finite stalks, then it is finite and is represented by a finite etale group scheme $\widetilde{F}$ over X.*

PROOF. Let (U_i) be a covering of X such that $F|_{U_i}$ is constant. Then the abelian group corresponding to it must be finite. For any qc U the covering $(U_i \times_X U)$ contains a finite subcovering. Therefore $F(U)$ is a subgroup of a finite product $\prod F(U_i \times_X U)$ where each one is a finite group after possibly refining the covering.

Let $X' = \coprod U_i$ then $F|_{X'}$ is represented by a finite etale group scheme $\widetilde{F}'/X'$. The canonical isomorphism $p_1^*(F|_{X'}) \to p_2^*(F|_{X'})$ where p_i are projections $X' \times_X X' \to X'$ defines a descent datum on $\widetilde{F}'$. By fpqc descent we see $\widetilde{F}'$ with its descent datum arises from a group scheme $\widetilde{F}$ finite etale over X. □

For convenience we shall write lcc for locally constant sheaves with finite stalks.

REMARK 5.3. *The proposition above has an obvious converse, the sheaf defined by a finite etale group scheme is lcc.*

Let X be connected. Recall that there is a category equivalence between finite $\pi_1(X, \overline{x})$-sets and finite etale X-schemes. By the proposition above this induces a category equivalence between finite $\pi_1(X, \overline{x})$-modules and lcc sheaves under which a sheaf F sends to its stalk $F_{\overline{x}}$ and a module M sends to a sheaf F such that $F(U) = \mathrm{Hom}_{\pi_1}(\mathrm{Hom}_X(\overline{x}, U), M)$ for any U finite etale over X. For such F there is a finite etale $X' \to X$ such that $\widetilde{F} \times_X X'$ is a disjoint union of X' and $F|_{X'}$ is constant.

Actually every sheaf F on X_{et} can be represented by an algebraic space.

DEFINITION 5.4. *Let F be a sheaf on X_{et}. It is called a constructible sheaf if for every irreducible closed subscheme Z of X there is a nonempty open U in Z such that $F|_U$ is lcc.*

As mentioned before, the cohomology of nontorsion sheaf disagrees with what we expected in complex case. Thus we shall firstly consider the category $Sh_{\mathbb{Z}/n\mathbb{Z}}(X_{et})$, the sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules. For a lcc sheaf F let $\check{F} = \mathcal{H}om(F, \mathbb{Z}/n\mathbb{Z})$ called its dual sheaf. The functor $F \mapsto \check{F}$ is exact and preserves locally free sheaves. For any scheme X whose residue field characteristics are prime to n,

μ_n is a locally free $\mathbb{Z}/n\mathbb{Z}$-module of rank 1. Define

$$(\mathbb{Z}/n\mathbb{Z})(r) = \begin{cases} \mu_n^{\otimes r} & \text{if } r > 0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } r = 0 \\ (\mathbb{Z}/n\mathbb{Z})(-r)^\vee & \text{if } r < 0 \end{cases}$$

they are called r-th Tate twists. For any $\mathbb{Z}/n\mathbb{Z}$-module F let $F(r) = F \otimes (\mathbb{Z}/n\mathbb{Z})(r)$. If $\Gamma(X, \mathcal{O}_X)$ admits all n-th roots of unity then F is noncanonically isomorphic to $F(r)$ and

$$H^s(X, F) \otimes (\mathbb{Z}/n\mathbb{Z})(r) = H^s(X, F(r))$$

The reason that twists come into play could be explained in three aspects, namely computation, orientation and weights. First of all essentially almost all computable results come from our previous results for smooth curves over algebraically closed fields, where we identify $\mathbb{Z}/n\mathbb{Z}$ with μ_n and use the Kummer sequence. Secondly, when showing similar results in topology over $\mathbb{C}$ like Poincare duality we always need to fix an orientation, which is just an identification of μ_2 with $\mathbb{Z}/2\mathbb{Z}$. Now in the etale case over algebraically closed field, identifying μ_n with $\mathbb{Z}/n\mathbb{Z}$ is just like choosing an orientation among all n choices of them, and if we want to move more intrinsically we need to keep track of the choices. See Tony Feng, *Poincare duality for etale cohomology* for a detailed discussion. Lastly, as we shall see later, the twists remember different actions of $\mathbb{Z}/n\mathbb{Z}$ hence will give rise to different weights. This is important in particular when proving the *Riemann Hypothesis* for Weil Conjecture.

Now we shall glue cohomology of torsion sheaves and obtain some results of $\mathbb{Q}_\ell$-sheaves.

DEFINITION 5.5. *Let ℓ be a prime. A sheaf of $\mathbb{Z}_\ell$-modules on X_{et} is a projective system $F = (F_n)$ of sheaves such that F_n is a sheaf of $\mathbb{Z}/\ell^n$-sheaf and the transition map $F_{n+1} \to F_n$ induces isomorphism $F_{n+1}/\ell^n F_{n+1} \xrightarrow{\sim} F_n$. Bu induction $F_m/\ell^n F_m = F_n$ for any $m > n$. The (compact supported) cohomology of F is defined to be*

$$H^r_{(c)} = \lim_n H^r_{(c)}(X, F_n)$$

The ring $\mathbb{Z}_\ell$ acts on $H^r(X, F)$ and on

$$\operatorname{Hom}(F, F') = \lim_m \operatorname{colim}_n \operatorname{Hom}(F_n, F'_m) = \lim_m (F_m, F'_m)$$

The sheaf F is called constructible or lcc if each F_n is.

If X is connected then $\pi_1(X, \overline{x})$ has a continuous action on $F_{\overline{x}} = \lim(F_n)_{\overline{x}}$ when $F_{\overline{x}}$ is given the ℓ-adic topology, and $F \mapsto F_{\overline{x}}$ induces an equivalence between the category of lcc $\mathbb{Z}_\ell$-sheaves and the category of $\pi_1(X, \overline{x})$-modules that are finitely generated over $\mathbb{Z}_\ell$.

Similarly we can define twists $F(r)$ by twisting each F_n. Again if X admits all ℓ-power roots of unity then there are noncanonically isomorphism and the twisting commutes with cohomology.

DEFINITION 5.6. *The category of $\mathbb{Q}_\ell$-sheaves is defined to be the localization of the category of $\mathbb{Z}_\ell$-sheaves at morphisms whose kernel and cokernel are killed by some power of ℓ. For a $\mathbb{Q}_\ell$-sheaf F its cohomology is defined to be the tensor product of its cohomology as $\mathbb{Z}_\ell$-sheaves with $\mathbb{Q}_\ell$.*

If X is connected then $F \mapsto F_{\overline{x}} \otimes \mathbb{Q}_\ell$ induces an equivalence between the category of lcc $\mathbb{Q}_\ell$-sheaves and the category of finite dimensional $\mathbb{Q}_\ell$-vector spaces on which $\pi_1(X, \overline{x})$ acts continuously. Hence we get a continuous representation

$$\rho \colon \pi_1(X, \overline{x}) \longrightarrow GL_n(\mathbb{Q}_\ell)$$

As π_1 is compact, there is always a stable lattice hence ρ is conjugate to a representation into $GL_n(\mathbb{Z}_\ell)$.

The whole notions above can be generalized to a finite field extension Ω of $\mathbb{Q}_\ell$ and A the integral closure of $\mathbb{Z}_\ell$ in Ω.

From now on we shall mainly focus on the sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules and leave it to the reader to check the results remain true for $\mathbb{Z}_\ell$-sheaves or $\mathbb{Q}_\ell$-sheaves. For this the following lemma is useful.

Definition 5.7. *A $\mathbb{Z}_\ell$-sheaf F is called lisse if each F_n is flat and lcc.*

Lemma 5.8. *Let F be a lisse sheaf such that $H^r(X, F_n)$ is finite for all r and n. Then $H^r(X, F)$ is finite $\mathbb{Z}_\ell$-modules and there are long exact sequences*

$$\cdots \to H^r(X, F) \xrightarrow{\ell^s} H^r(X, F) \to H^r(X, F_s) \to H^{r+1}(X, F) \to \dots$$

Proof. Note that the category of profinite abelian groups is dual to the category of discrete torsion abelian groups by trivial Pontryagin duality, so inverse limits of exact sequences of finite abelian groups are exact. Since each F_n is flat, tensoring the s.e.s.

$$0 \longrightarrow \mathbb{Z}/\ell^n \xrightarrow{\ell^s} \mathbb{Z}/\ell^{n+s} \longrightarrow \mathbb{Z}/\ell^s \longrightarrow 0$$

with F_{n+s} we get s.e.s.

$$0 \longrightarrow F_n \longrightarrow F_{n+s} \longrightarrow F_s \longrightarrow 0$$

Fix s and vary n we get compatible short exact sequences. Taking cohomology for each n and passing to inverse limit over all n we get exact sequence as desired.

As $H^r(X, F)$ is an inverse limit of ℓ-power-torsion finite groups, no nonzero element of it is divisible by all powers of ℓ. Thus

$$\lim H^{r+1}(X, F)[\ell^n] = 0 \text{ and } \lim H^r(X, F)/\ell^n H^r(X, F) = H^r(X, F)$$

This means $H^r(X, F)$ is complete $\mathbb{Z}_\ell$-module and by a Hensel's lemma argument we see that $H^r(X, F)$ is generated over $\mathbb{Z}_\ell$ by any subset that generates it modulo ℓ. □

To prepare for the Poincare duality, let us discuss some pairings here. Recall that in an abelian category $\mathcal{A}$ with enough injectives, $\mathrm{Ext}^r_{\mathcal{A}}(B, C)$ can be interpreted as homotopy classes of maps $B \to I^\bullet[r]$ where $I^\bullet$ is an injective resolution of C. If $B \to B^\bullet$ is any resolution then it is also the homotopy classes of maps $B^\bullet \to I^\bullet[r]$. Thus we can define a canonical pairing

$$\mathrm{Ext}^r(A, B) \times \mathrm{Ext}^s(B, C) \longrightarrow \mathrm{Ext}^{r+s}(A, C)$$

by taking injective resolutions and compose the corresponding maps.

In particular if $\mathcal{A}$ is $Sh(X_{et})$ or $Sh(X_{et}, \mathbb{Z}/n\mathbb{Z})$ and A is $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ then the pairings are

$$H^r(X, F) \times \mathrm{Ext}^s(F, G) \longrightarrow H^{r+s}(X, G)$$

Also if $j\colon X \hookrightarrow \overline{X}$ is an open immersion then for sheaves F and G on X we have

$$\mathrm{Ext}^s(j_! F, j_! G) = \mathrm{Ext}^s(F, G)$$

hence we get a canonical pairing

$$H^r(\overline{X}, j_! F) \times \mathrm{Ext}^s(F, G) \to H^{r+s}(\overline{X}, j_! G)$$

The Ext pairing behaves well with respect to finite etale coverings.

Lemma 5.9. *Let $\pi\colon X' \to X$ be a finite etale map of constant degree d and X is connected. For any sheaf F on X there is a trace map $\mathrm{tr}\colon \pi_*\pi^* F \to F$ functorial in F such that for any F' on X' the map*

$$\mathrm{Hom}_{X'}(F', \pi^* F) \to \mathrm{Hom}_X(\pi_* F', F)\ ,\ \phi \mapsto \mathrm{tr} \circ \pi_*(\phi)$$

is an isomorphism. Thus π_* is a left adjoint to π^*, and is just $\pi_!$, and tr *is the adjunction map. The composites*

$$F \to \pi_*\pi^* F \xrightarrow{\mathrm{tr}} F \text{ and } H^r(X, F) \to H^r(X', F|_{X'}) \xrightarrow{\mathrm{tr}} H^r(X, F)$$

are just multiplication by d.

Proof. We may assume X' connected by considering its connected components. Then we may find $X'' \to X$ be a Galois morphism with Galois group G which factors through X'. Then $X'' \to X'$

is also Galois with Galois group $H \subset G$. For any U etale over X we have

$$\Gamma(U, F) \subset \Gamma(U', F) \subset \Gamma(U'', F)$$

and $\Gamma(U, F) = \Gamma(U'', F)^G$ where U' and U'' are base changes of U to X' and X''. For any $s \in \Gamma(U, \pi_*\pi^*F) = \Gamma(U', F)$ we define

$$\mathrm{tr}(s) = \sum_{\sigma \in G/H} \sigma(s|_{U''})$$

then this is fixed by G hence it lies in $\Gamma(U, F)$. Clearly tr defines a map $\pi_*\pi^*F \to F$ whose composite with $F \to \pi_*\pi^*F$ is multiplication by d.

If X' is disjoint union of d copies of X then it is obvious that $\mathrm{Hom}_{X'}(F', \pi^*F) = \mathrm{Hom}_X(\pi_*F', F)$ by tr. Now since $\mathcal{H}om$ is a sheaf we may pass to a finite etale covering of X, i.e. $X'' \to X$. To descent the result on X'' one may need proper base change to be discussed in next chapter. To show $X'' \times_X X'$ is also copies of X'' one may apply Lemma 1.72.

In

$$H^r(X, F) \xrightarrow{res} H^r(X', \pi^*F) \xrightarrow{\sim} H^r(X, \pi_*\pi^*F) \xrightarrow{tr} H^r(X, F)$$

the composite of the first two is induced by $F \to \pi_*\pi^*F$ and the composite of all three is induced by $(F \to \pi_*\pi^*F \xrightarrow{tr} F) = [d]$. □

It turns out that the condition of being finite etale of constant degree is nothing more. Actually if $f\colon Y \to X$ is finite etale with X connected then $\sum_{y \in f^{-1}x} d(y)$ is constant where $d(y)$ is the extension degree of $\kappa(y)/\kappa(x)$. To see this note that f is finite flat LFT map of locally Noetherian schemes hence it is finite locally free of constant rank as X connected. Then consider $f_*\mathcal{O}_Y$ as finite free $\mathcal{O}_X$-module. Taking stalks we get $\prod_{y \in f^{-1}x} \mathcal{O}_{Y,\overline{y}}^{d(y)}$ but $\mathcal{O}_{Y,\overline{y}} = \mathcal{O}_{X,\overline{x}}$ hence f has constant degree.

Proposition 5.10. *Let $\pi\colon X' \to X$ be a finite etale map of constant degree d and X connected. Suppose there is a pullback diagram*

$$\begin{array}{ccc} X' & \xhookrightarrow{j'} & \overline{X}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\ X & \xhookrightarrow{j} & \overline{X} \end{array}$$

with $\overline{\pi}$ finite and j', j open immersions. For any sheaf F on X we have

$$(F \to \pi_*\pi^*F \xrightarrow{tr} F) = [d]$$

and look at stalks we have $\overline{\pi}_* j'_! = j_!\pi_*$ hence

$$(j_!F \to \overline{\pi}_* j'_! \pi^*F \xrightarrow{tr} j_!F) = [d]$$

If in addition $\overline{X}$ is proper then

$$H^r_c(X, F) \xrightarrow{res} H^r_c(X', F|_{X'}) \xrightarrow{tr} H^r_c(X, F) = [d]$$

Moreover we have

$$\begin{array}{ccccc} H^r_c(X', F') & \times & \mathrm{Ext}^s_{X'}(F', \pi^*F) & \longrightarrow & H^{r+s}_c(X', \pi^*F) \\ \uparrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle tr} \\ H^r_c(X, \pi_*F') & \times & \mathrm{Ext}^s_X(\pi_*F', F) & \longrightarrow & H^{r+s}_c(X, F) \end{array}$$

commutes.

Proof. We only need to prove the diagram is commutative. Take injective resolutions of F and $j'_!F'$ we only need to consider the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\overline{X}'}(\mathbb{Z}, j'_!F') \times \mathrm{Hom}_{X'}(F', \pi^*F) & \longrightarrow & \mathrm{Hom}_{\overline{X}'}(\mathbb{Z}, j'_!\pi^*F) \\
\sim\uparrow \qquad\qquad \downarrow tr & & \downarrow tr \\
\mathrm{Hom}_{\overline{X}}(\mathbb{Z}, j_!\pi_*F') \times \mathrm{Hom}_X(\pi_*F', F) & \longrightarrow & \mathrm{Hom}_{\overline{X}}(\mathbb{Z}, j_!F)
\end{array}$$

which is commutative by functoriality of tr. □

Ext pairings are also related to cup products.

Proposition 5.11. *Let $\mathcal{A}$ be an abelian category with tensor products and let f_1, f_2, f_3 be left exact functors $Sh(X_{et}) \to \mathcal{A}$ such that any induced sheaf is f_i-acyclic. Suppose that there is a morphism of bi-functors $f_1(F_1) \otimes f_2(F_2) \to f_3(F_1 \otimes F_2)$, then there is a unique family of morphisms of bi-functors*

$$(R^r f_1 F_1) \otimes (R^s f_2 F_2) \longrightarrow R^{r+s} f_3(F_1 \otimes F_2)$$

written as $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cup \gamma_2$ satisfying

a *for $r = s = 0$ it is the given morphism.*

b *if $0 \to F'_1 \to F_1 \to F''_1 \to 0$ is split-exact on stalks then for $\gamma_1 \in R^r f_1 F''_1$ and $\gamma_2 \in R^s f_2 F_2$ we have $(d\gamma_1) \cup \gamma_2 = d(\gamma_1 \cup \gamma_2)$.*

c *if $0 \to F'_2 \to F_2 \to F''_2 \to 0$ is split-exact on stalks then for $\gamma_1 \in R^r f_1 F_1$ and $\gamma_2 \in R^s f_2 F''_2$ we have $\gamma_1 \cup (d\gamma_2) = (-1)^r d(\gamma_1 \cup \gamma_2)$.*

Example 5.12. *The natural pairing*

$$\Gamma(X, F_1) \times \Gamma(X, F_2) \to \Gamma(X, F_1 \otimes F_2)$$

induces a unique family of pairing

$$H^r(X, F_1) \times H^r(X, F_2) \to H^{r+s}(X, F_1 \otimes F_2)$$

Similarly we have pairings

$$R^r \pi_* F_1 \times R^s \pi_* F_2 \to R^{r+s} \pi_*(F_1 \otimes F_2)$$

and

$$H^r_{Z_1}(X, F_1) \times H^s_{Z_2}(X, F_2) \to H^{r+s}_{Z_1 \cap Z_2}(X, F_1 \otimes F_2)$$

All of them will be referred as cup products.

Remark 5.13. *For any sheaves F and G on X_{et} and any covering $U \to X$ there is a pairing of Čech complexes*

$$C^\bullet(U, F) \times C^\bullet(U, G) \to C^\bullet(U, F \otimes G)$$

that sends $f = (f_{i_0 \cdots i_r}) \in C^r(U, F)$ and $g = (g_{i_0 \cdots i_s}) \in C^s(U, G)$ to

$$(f \cup g)_{i_0 \cdots i_r + s} = f_{i_0 \cdots i_r} \otimes g_{i_r \cdots i_{r+s}}$$

After passing to the colimit of all coverings and taking cohomology we get a pairing

$$\check{H}^r(X, F) \times \check{H}^s(X, G) \to \check{H}^{r+s}(X, F \otimes G)$$

When Čech cohomology agrees with derived functor cohomology it is easy to check that these pairings are just the cup products.

Now for any sheaves F and G on X there are canonical maps

$$G \to \mathcal{H}om(F, F \otimes G) \text{ and } H^s(X, \mathcal{H}om(F, F \otimes G)) \to \mathrm{Ext}^s(F, F \otimes G)$$

where the second map comes from the edge morphism in the Local-Global spectral sequence. On composing we get a canonical map $H^s(X, G) \to \mathrm{Ext}^s(F, F \otimes G)$.

Proposition 5.14. *We have a commutative diagram*

$$\begin{array}{ccccc} H^r(X,F) & \times & H^s(X,G) & \longrightarrow & H^{r+s}(X,F\otimes G) \\ \| & & \downarrow & & \| \\ H^r(X,F) & \times & \operatorname{Ext}^s(F,F\otimes G) & \longrightarrow & H^{r+s}(X,F\otimes G) \end{array}$$

where the first line pairing is cup product and second line is Ext pairing. In the case X is an open subscheme of a proper scheme we have similar results for compactly supported cohomology.

2. The Cohomology of Curves

Throughout this section unless otherwise specified X will be a smooth projective/proper curve over an algebraically closed field k, whose char is prime to n. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$. We shall only consider sheaves of Λ-modules and write Ext^r_U for $\operatorname{Ext}^r_{Sh_\Lambda(U_{et})}$.

Theorem 5.15 (Poincare Duality). *For any nonempty open U of X there is a canonical isomorphism $\eta_U\colon H^2_c(U,\mu_n)\to\Lambda$.*

For any constructible sheaf F on U the groups $H^r_c(U,F)$ and $\operatorname{Ext}^r_U(F,\mu_n)$ are finite for all r and zero for $r>2$. The canonical pairings

$$H^r_c(U,F)\times \operatorname{Ext}^{2-r}(F,\mu_n)\to H^2_c(U,\mu_n)\xrightarrow{\eta_U}\Lambda$$

are perfect for all r.

Proof. **Step 5**. As F is lcc, F is pseudo-coherent. Also μ_n is an injective Λ-module. By Proposition 3.30 we are done. □

Proposition 5.16. *If in addition F is lcc then $\operatorname{Ext}^s(F,\mu_n)=H^s(U,\check{F}(1))$. If F is lcc $\mathbb{Q}_\ell$-sheaf for $\ell\neq\operatorname{char}(k)$ then the pairings*

$$H^r_c(U,F)\times H^{2-r}(U,\check{F}(1))\to H^2_c(U,\mathbb{Q}_\ell(1))=\mathbb{Q}_\ell$$

are perfect pairings of finite dimensional $\mathbb{Q}_\ell$-vector spaces.

Proof. From the Local-Global spectral sequence

$$H^r(U,\mathcal{E}xt^s(F,\mu_n))\Longrightarrow \operatorname{Ext}^{r+s}(F,\mu_n)$$

and as in the proof of the theorem $\mathcal{E}xt^s(F,\mu_n)=0$ for $s>0$.

For $\mathbb{Q}_\ell$-sheaves, pass to an inverse limit and tensor with $\mathbb{Q}_\ell$. □

Corollary 5.17. *Let k be a finite field and let n be prime to $\operatorname{char}(k)$. Let X be a smooth projective curve over k and let U be a nonempty open. Then there is a canonical isomorphism $\eta_U\colon H^3_c(U,\mu_n)\to\Lambda$ and for any lcc sheaf F the canonical pairings*

$$H^r_c(U,F)\times H^{3-r}(U,\check{F}(1))\to H^3_c(U,\mu_n)\xrightarrow{\eta_U}\Lambda$$

are perfect pairings of finite groups.

Proof. Let $\Gamma=\operatorname{Gal}(k_s/k)$ and let σ be the canonical topological generator of Γ. For any finite n-torsion Γ-module M, $H^0(\Gamma,M)$ and $H^1(\Gamma,M)$ are respectively the kernel M^Γ and cokernel M_Γ of $\sigma-1$ and $H^r(\Gamma,M)=0$ for $r\geq 2$. Moreover if $\check{M}=\operatorname{Hom}(M,\Lambda)$ then the canonical pairings

$$H^r(\Gamma,M)\times H^{1-r}(\Gamma,\check{M})\to H^1(\Gamma,\Lambda)=\Lambda$$

are perfect.

Let $\overline{X}$ and $\overline{U}$ be the base change to k_s. Apply the Hochschild-Serre spectral sequence to $\overline{X}/X$ and $\overline{U}/U$ to get s.e.s.

$$0\longrightarrow H^{r-1}_{(c)}(\overline{U},F)_\Gamma\longrightarrow H^r_{(c)}(U,F)\longrightarrow H^r_{(c)}(\overline{U},F)^\Gamma\longrightarrow 0$$

In particular $H^3_c(U,\mu_n) = H^2_c(\overline{U},\mu_n)_\Gamma \xrightarrow{\eta_{\overline{U}}} \Lambda$ and this is η_U. Then check the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{r-1}_c(\overline{U},F)_\Gamma & \longrightarrow & H^r_c(U,F) & \longrightarrow & H^r_c(\overline{U},F)^\Gamma & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \\ 0 & \to & (H^{3-r}(\overline{U},\check{F}(1))^\Gamma)^\vee & \to & H^{3-r}(U,\check{F}(1))^\vee & \to & (H^{2-r}(\overline{U},\check{F}(1))_\Gamma)^\vee & \to & 0 \end{array}$$

commutes and apply the five lemma. □

REMARK 5.18. *Let U,k,F,n be as in the setting of the theorem and assume U affine. For some $j\colon V \hookrightarrow U$, $F|_V$ is locally constant and a modification of Proposition 5.16 shows that $H^2(U,j_*(F|_V))$ is dual to $H^0_c(U,j_*(\check{F}|_V)(1))$ which is zero since no section of $j_*(\check{F}|_V)$ has support on a proper closed subset of U. Since the kernel and cokernel of $F \to j_*j^*F$ have support in dimension zero, it follows that $H^2(U,F) = 0$. By a normalization argument we could show that for any U affine curve over k not necessarily smooth and F torsion sheaf on U whose torsion is prime to* char(k)*, we would have $H^r(U,F) = 0$ for $r \geq 2$.*

REMARK 5.19. *Recall in Remark 2.51 if U is over some field k and k' is a purely inseparable extension of k then the functor $F \mapsto F|_{U\times_k k'}$ is an equivalence of category of sheaves. Thus all the results above hold if the ground field k is only separably closed.*

REMARK 5.20. *Using class field theory, Artin and Verdier refined the proof of the theorem and showed that if X is the spectrum of the ring of integers in a number field then there is a canonical isomorphism $\eta\colon H^3(X,\mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$ modulo 2-torsion and for any constructible sheaf F the pairings*

$$H^r(X,F) \times \mathrm{Ext}^{3-r}_X(F,\mathbb{G}_m) \to H^3(X,\mathbb{G}_m) \xrightarrow{\eta} \mathbb{Q}/\mathbb{Z}$$

are perfect pairings of finite groups modulo 2-torsion.

CHAPTER 6

Fundamental Theorems

Slogan: A smooth proper variety of dimension d over a separably closed field behaves cohomologically like a smooth manifold of complex dimension d.

In particular we shall check the etale cohomology satisfies the requirements made in the definition of a Weil cohomology theory, i.e. finiteness, vanishing in degree$> 2d$, Poincare duality, Kunneth formula and Lefschetz fixed point formula.

1. Cohomological Dimension

DEFINITION 6.1. *Let ℓ be a prime. A sheaf F is (ℓ-)torsion if for all qc U, $F(U)$ is (ℓ-)torsion. The ℓ-cohomological dimension $cd_\ell(C/X)_E$ of a site $(C/X)_E$ is the smallest integer n or ∞ such that $H^i(X_E, F) = 0$ for all $i > n$ and ℓ-torsion sheaves.*

THEOREM 6.2 (Tate). *If K/k is of transcendence degree n then $cd_\ell(K) \leq cd_\ell(k) + n$.*

THEOREM 6.3. *If X is a scheme of finite type over a separably closed field k then $cd_\ell(X_{et}) \leq 2\dim(X)$.*

PROOF. Induct on $n = \dim(X)$. □

COROLLARY 6.4. *Let X be a scheme of finite type over a field k. For all $\ell \neq \mathrm{char}(k)$, $cd_\ell(X_{et}) \leq cd_\ell(k) + 2\dim(X)$.*

PROOF. Use Hochschild-Serre spectral sequence. □

2. Proper Base Change

THEOREM 6.5. *Let $\pi\colon X \to S$ be a proper morphism and $F \in Sh(X_{et})$ be constructible. Then $R^i\pi_*F$ are constructible for $i \geq 0$ and $(R^i\pi_*F)_{\overline{s}} = H^i(X_{\overline{s}}, F|_{X_{\overline{s}}})$ for all geometric points $\overline{s}$.*

COROLLARY 6.6. *For $\pi\colon X \to S$ proper, the formation of R^if_*F for F constructible (torsion) commutes with base change.*

PROOF. Construct a morphism using adjunction and check isomorphism on stalks. □

Key ideas for the Proper Base Change Theorem.

- Step 1 Reduce to the case where π is a relative curve.
- Step 2 Devissage to the case where $F = \mu_n$.
- Step 3 Consider the exact sequence

$$0 \to \pi_*\mu_n \to \pi_*\mathbb{G}_m \to \pi_*\mathbb{G}_m \to R^1\pi_*\mu_n \to R^1\pi_*\mathbb{G}_m \to R^1\pi_*\mathbb{G}_m \to R^2\pi_*\mu_n \to 0$$

  where the vanishing of higher direct image for $\mathbb{G}_m$ is the relative analogue of the result for curves over algebraic closed field.
- Goal Show $R^i\pi_*\mu_n$ is representable by quasi-finite S-schemes. Via Grothendieck, the geometric input is that $R^1\pi_*\mathbb{G}_m = \mathrm{Pic}(X/S)$ is representable by LFT S-scheme. The sheaf $\mathrm{Pic}(X/S)$ is the sheafification of the presheaf

$$T \longmapsto \{\text{line bundles on } X_T\}/\{\pi_T^*\{\text{line bundles on } T\}\}$$

Then we are left to study

$$\ker(\mathrm{Pic}(X/S) \xrightarrow{[n]} \mathrm{Pic}(X/S)) \text{ and } \mathrm{coker}(\mathrm{Pic}(X/S) \xrightarrow{[n]} \mathrm{Pic}(X/S))$$

and these two are both representable by scheme-theoretic kernel/cokernel which are quasi-finite S-schemes. This follows from the structure theory that the identity component is divisible.

3. Finiteness

4. Smooth Base Change

For any scheme X let $\mathrm{char}(X)$ denote the set of primes occurring as $\mathrm{char}\,\kappa(x)$ for some $x \in X$. The torsion of a sheaf F is prime to $\mathrm{char}(X)$ if $p\colon F \to F$ is injective for all $p \in \mathrm{char}(X)$ nonzero.

THEOREM 6.7 (Smooth Base Change Theorem). *Consider a fibre product*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array}$$

with g smooth and π qc. Then for any Torsion sheaf F on Y_{et} whose torsion is prime to $\mathrm{char}(X)$ *the natural base change morphism*

$$g^*(R^i\pi_*F) \longrightarrow R^i\pi'_*(g'^*F)$$

is an isomorphism.

COROLLARY 6.8. *Let $\pi\colon Y \to X$ be proper and smooth and F lcc sheaf on Y_{et} whose torsion is prime to* $\mathrm{char}(X)$. *Then $R^i\pi_*F$ is lcc. In particular if X is connected then the groups $H^i(Y_{\overline{x}}, F|_{Y_{\overline{x}}})$ are all isomorphic.*

PROOF. We first give a criterion for constructible sheaves to be lcc.

Let $x_1 \in X$ and x_0 specialize x_1, i.e. $x_0 \in \overline{\{x_1\}}$. Choose geometric points $\overline{x}_1$ and $\overline{x}_0$. Then for any sheaf G on X_{et} we can have a cospecialization map $G_{\overline{x}_0} \to G_{\overline{x}_1}$ if we can treat etale neighborhoods of $\overline{x}_0$ as etale neighborhoods of $\overline{x}_1$ in a compatible way. Claim if G is constructible, then it is lcc iff all cospecialization maps are isomorphisms.

Now by proper base change we know $R^i\pi_*F$ is already constructible. It suffices to show all cospecialization maps are isomorphisms. Fix a pair x_0 and x_1 then we can find a strictly Henselian normal ring A with separably closed field of fractions and a morphism $\mathrm{Spec}\,A \to X$ sending the generic point to x_1 and closed point to x_0 such that the cospecialization map on $\mathrm{Spec}\,A$ induced by the inclusion of A into its field of fractions comes from the given cospecialization map. Since π is proper $R^i\pi_*F$ commutes with base change hence we may assume $X = \mathrm{Spec}\,A$. Denote Y_1 the generic fibre of $Y \to X$ then $(R^i\pi_*F)_{\overline{x}_1} = H^i(Y_1, F)$ and $(R^i\pi_*F)_{\overline{x}_0} = H^i(Y, F)$ hence we only need to show that the restriction $H^i(Y,F) \to H^i(Y_1, F)$ is an isomorphism.

Consider the commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ x_1 & \xrightarrow{g} & X \end{array}$$

By Leray spectral sequence for g' it suffices to show

$$g'_*g'^*F = F \ , \ R^ig'_*(g'^*F) = 0 \ i > 0$$

These conditions are local on Y_{et} hence after taking an etale covering we may assume F is constant, $F = M_Y$ for some finite abelian group M. Apply the smooth base change to this diagram we get

$$\pi^*(R^i g_* M_{x_1}) = R^i g'_* M_{Y_1}$$

Since X is normal, $g_* M_{x_1} = M_X$. Since x_1 is separably closed, $R^i g_* = 0$ for $i > 0$. This finishes the proof. □

Corollary 6.9. *Let $k \subset K$ be two separably closed fields and X a scheme over k. Then for any torsion sheaf F whose torsion is prime to* $\operatorname{char} k$ *we have $H^i(X, F) = H^i(X_K, F|_{X_K})$.*

Proof. We may replace these fields by their algebraic closures. Then $K = \operatorname{colim} A_i$ where A_i is smooth k-algebras. □

With these two corollaries we can somehow treat the ℓ-etale cohomology of varieties in characteristic p where $\ell \neq p$. Now given any smooth proper variety X over k of char p, the general idea is:

Step 1: Try to lift it to char 0, i.e. find some DVR R with $R/\mathfrak{m} = k$ and smooth proper R-scheme $\mathcal{X}$ with $\mathcal{X}_k = X$. The obstruction for formal lifting lives in $H^2(X, T_X)$ and the obstruction for lifting formally ample line bundles lives in $H^2(X, \mathcal{O}_X)$. It is known that the lift exists for curves, abelian varieties, $K3$ surfaces(Deligne) and hypersurfaces and complete intersections in $\mathbf{P}^n$.

Step 2: Compute the cohomology of $\mathcal{X}_K$ where $K = \operatorname{Frac} R$. By smooth proper base change we know these cohomologies are related to $\mathcal{X}_k = X$. Then reduce to $\mathcal{X}_{\overline{K}}$. Let L be the algebraic closure of $\mathbb{Q}$ adding all the coefficients defining $\mathcal{X}$ then we have inclusions $L \subset \overline{K}$ and $L \subset \mathbb{C}$. Then reduce to X_L and $X_{\mathbb{C}}$ and use Artin comparisons.

Before diving into the technical proof we first think about what would happen topologically. By the structure theorem of smooth morphisms we know locally the smooth morphism is of the form $X \times (\text{open ball}) \to X$ hence by homotopy equivalence the cohomology does not change. This is the idea for the acyclic morphism defined below.

Definition 6.10. *A morphism $g \colon Y \to X$ is called n-acyclic for some $n \geq -1$ if for all X' etale of finite type over X and torsion sheaf F on X' whose torsion is prime to* $\operatorname{char}(X)$ *the map $H^i(X', F) \to H^i(Y', F|_{Y'})$ is an isomorphism for $0 \leq i \leq n$ and injective for $i = n+1$ where $Y' = X' \times_X Y$.*

The morphism g is called acyclic if it is n-acyclic for all n and is called universally (n-)acyclic if $g_{X'}$ is (n-)acyclic for all X-schemes X'.

The morphism g is called (universally) locally (n-)acyclic if for every geometric point $\overline{y}$ of Y the map $\widetilde{g} \colon \widetilde{Y} = \operatorname{Spec} \mathcal{O}_{Y,\overline{y}} \to \widetilde{X} = \operatorname{Spec} \mathcal{O}_{X,\overline{y}}$ is (universally) (n-)acyclic.

From the Leray spectral sequence for $g' \colon Y' \to X'$ we see g being n-acyclic is equivalent to $F \xrightarrow{\sim} g'_* g'^* F$ and $R^i g'_*(g'^* F) = 0$ for $1 \leq i \leq n$ if $n \geq 0$, or $F \to g'_* g'^* F$ is injective if $n = -1$. Also the case $n = -1$ is needed for induction argument and it is actually a surjectivity condition since any surjective map is (-1)-acyclic and any qc (-1)-acyclic map is surjective.

Lemma 6.11. *A morphism $g \colon Y \to X$ is n-acyclic iff the definition condition holds for any X' quasi-finite over X.*

Proof. One direction is clear. For the converse it suffices to show $F \cong g'_* g'^* F$ and $(R^i g'_*) g'^* F = 0$ for $i \leq n$ assuming $n \geq 0$.

Firstly we shall need a sublemma.

Sublemma: If $f\colon X' \to X$ is quasi-finite then there is a family of commutative diagrams

$$\begin{array}{ccc} X'_i & \xrightarrow{f_i} & X_i \\ \downarrow{\scriptstyle h'_i} & & \downarrow{\scriptstyle h_i} \\ X' & \xrightarrow{f} & X \end{array}$$

in which f_i is finite and h_i is qc etale and $\{h'_i\}$ is a covering of X'_{et}.

Proof. Fix any $x' \in X'$ and let $x = f(x')$. Let $\widetilde{X} = \operatorname{Spec} \mathcal{O}^h_{X,x}$ and $\widetilde{X}' = \widetilde{X} \times_X X'$. By equivalent criteria for Henselian rings we can find an open subscheme $\widetilde{U}$ of $\widetilde{X}'$ whose image in X' contains x' and $\widetilde{U}$ is finite over $\widetilde{X}$. As $\widetilde{X} = \lim U_v$ with U_v etale over X, one can see that for some v there is an open subscheme $U_{x'}$ of $U_v \times_X X'$ finite over U_v and $\widetilde{U} = \widetilde{X} \times_{U_v} U_{x'}$. Then just take all such U_v and $U_{x'}$. □

By the sublemma we can assume $f\colon X' \to X$ is finite. According to proper base change theorem $g^* f_* F = f'_* g'^* F$ and so

$$H^0(X', F) = H^0(X, f_*F) \to H^0(Y, g^*f_*F) = H^0(Y, f'_*g'^*F) = H^0(Y', g'^*F)$$

is isomorphism if g is 0-acyclic. Similarly we have $H^0(U, F|_U) = H^0(U', (g'^*F)|_{U'})$ for any U qc etale over X'. Hence $F = g'_*g'^*F$. This argument also applies if g is only (-1)-acyclic.

If $n \geq 1$ then since f_* and f'_* are exact we have

$$f_*(R^i g'_*)g'^*F = R^i(fg')_*g'^*F = R^i(gf')_*g'^*F = (R^ig_*)f'_*g'^*F = (R^ig_*)g^*f_*F$$

which is zero for $i \leq n$ by considering f_*F on X_{et}. It follows that $(R^ig'_*)g'^*F = 0$ for $i \leq n$. □

Lemma 6.12. *A morphism $g\colon Y \to X$ is locally n-acyclic iff for any diagram of pullbacks*

$$\begin{array}{ccccc} Y'' & \xrightarrow{f'} & Y' & \xrightarrow{j'} & Y \\ \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X'' & \xrightarrow{f} & X' & \xrightarrow{j} & X \end{array}$$

with j etale of finite type and f quasi-finite and any torsion sheaf F on X''_{et} whose torsion is prime to $\operatorname{char}(X)$ *the base change morphism*

$$g'^*R^if_*F \longrightarrow R^if'_*(g''^*F)$$

is an isomorphism for $i \leq n$ and injective for $i = n+1$.

Proof. Suppose g is locally n-acyclic. For any geometric point $\overline{y}'$ of Y' let $x' = g'(y')$ and $\overline{x}' = \overline{y}'$ and consider the diagram

$$\begin{array}{ccc} \widetilde{Y}'' & \xrightarrow{\widetilde{f}'} & \widetilde{Y}' \\ \downarrow{\scriptstyle \widetilde{g}''} & & \downarrow{\scriptstyle \widetilde{g}'} \\ \widetilde{X}'' & \xrightarrow{\widetilde{f}} & \widetilde{X}' \end{array}$$

where $\widetilde{X}' = \operatorname{Spec} \mathcal{O}_{X',\overline{x}'}, \widetilde{Y}' = \operatorname{Spec} \mathcal{O}_{Y',\overline{y}'}, \widetilde{X}'' = X'' \times_{X'} \widetilde{X}'$ and $\widetilde{Y}'' = Y'' \times_{Y'} \widetilde{Y}' = \widetilde{Y}' \times_{\widetilde{X}'} \widetilde{X}''$. Then

$$(g'^*(R^if_*F))_{\overline{y}'} = (R^if_*F)_{\overline{x}'} = H^i(\widetilde{X}'', F|_{\widetilde{X}''}) \text{ and } (R^if'_*(g''^*F))_{\overline{y}'} = H^i(\widetilde{Y}'', F|_{\widetilde{Y}''})$$

Since $\widetilde{g}'$ is n-acyclic and $\widetilde{f}$ is quasi-finite the map

$$H^i(\widetilde{X}'', F|_{\widetilde{X}''}) \longrightarrow H^i(\widetilde{Y}'', F|_{\widetilde{Y}''})$$

is bijective for $i \leq n$ and injective for $i = n+1$.

Conversely we may assume X affine. Let $\widetilde{g} \colon \widetilde{Y} \to \widetilde{X}$ correspond to a geometric point $\overline{y}$ of Y. We must show that for any $\widetilde{X}'$ etale of finite type over $\widetilde{X}$ and F some torsion sheaf on $\widetilde{X}'_{et}$ the map $H^i(\widetilde{X}', F) \to H^i(\widetilde{Y}', F|_{\widetilde{Y}'})$ satisfies the properties. Write $\widetilde{X} = \lim X_v$ with X_v affine and etale over X. Then for some v_0 we can find some $X'_0 \to X_0$ etale of finite type such that $\widetilde{X}' = X'_0 \times_{X_0} \widetilde{X}$. Then $\widetilde{X}' = \lim X'_v$ and denote $h_v \colon \widetilde{X}' \to X'_v$. Now the sheaf F on $\widetilde{X}'$ is a direct limit of constructible sheaves and any constructible sheaf on $\widetilde{X}'$ comes from the pullback of a sheaf on X'_v for some v. Hence we may assume F is of the form $h_0^* F_0$ for some F_0 on X'_0. Now apply the hypothesis to the diagram

$$\begin{array}{ccccc} Y'_0 & \xrightarrow{f'} & Y_0 & \longrightarrow & Y \\ \big\downarrow{\scriptstyle g'_0} & & \big\downarrow{\scriptstyle g_0} & & \big\downarrow{\scriptstyle g} \\ X'_0 & \xrightarrow{f} & X_0 & \longrightarrow & X \end{array}$$

we find that $g_0^*(R^i f_* F_0) \to R^i f'_*(g_0'^* F_0)$ is bijective for $i \leq n$ and injective for $i = n+1$. But $\overline{y}$ can be regarded canonically as a geometric point of Y_0 so taking stalks at $\overline{y}$ gives relations for $H^i(\widetilde{X}', F) \to H^i(\widetilde{Y}', F|_{\widetilde{Y}'})$. □

Corollary 6.13. *A morphism $g \colon Y \to X$ that is LFT is locally n-acyclic if $\widetilde{g}$ is n-acyclic for all $\overline{y}$ such that y is closed in its fibre.*

Corollary 6.14. *If g is universally locally acyclic then we have the base change theorem.*

Proof. Firstly assume π is compactifiable, for example affine of finite type. Then the diagram becomes

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle j} \\ \overline{Y}' & \xrightarrow{\overline{g}} & \overline{Y} \\ \big\downarrow{\scriptstyle \overline{\pi}'} & & \big\downarrow{\scriptstyle \overline{\pi}} \\ X' & \xrightarrow{g} & X \end{array}$$

where j is open immersion and $\overline{\pi}$ is proper. Then

$$g^* R^i \overline{\pi}_*(R^j j_* F) = (R^i \overline{\pi}'_*) \overline{g}^*(R^j j_* F) = (R^i \overline{\pi}'_*)(R^j j'_*)(g'^* F)$$

by proper base change and the local acyclicity for $\overline{g}$. This proves the lemma because we have commutative diagram of spectral sequences

$$\begin{array}{ccc} g^* R^i \overline{\pi}_*(R^j j_* F) & \xrightarrow{\sim} & (R^i \overline{\pi}'_*)(R^j j'_*)(g'^* F) \\ \Big\Downarrow & & \Big\Downarrow \\ g^* R^{i+j} \pi_* F & \longrightarrow & R^{i+j} \pi'_*(g'^* F) \end{array}$$

Since the proposition is local on X we may just assume X is affine. For any V open affine of Y or an open of an open affine we can write $V = \lim V_i$ where each V_i is affine and of finite type over X or an open of such a scheme. Since each V_i is compactifiable the result holds for V_i and since the higher direct images commute with certain inverse limits we know the result holds for V as well. Then apply the Mayer-Vietoris sequence to conclude. □

We are now left to show smooth maps are universally local acyclic. First we need a criterion for acyclicity.

Proposition 6.15. *If $g\colon Y \to X$ is qc and locally $(n-1)$-acyclic such that all geometric fibres $Y_{\overline{x}} \to \overline{x}$ with $\kappa(\overline{x})$ algebraic over $\kappa(x)$ are n-acyclic, then g is n-acyclic.*

We only sketch the proof. Consider sheaves on X' being the pushforward from a geometric point. Then reduce to the geometric point to show the results for such sheaves. In general any constructible sheaf on X' can be embedded into a finite direct sum of sheaves pushing forward from geometric points and use the previous argument.

Corollary 6.16. *A morphism $g\colon Y \to X$ is locally acyclic if for any geometric point $\overline{y}$ of Y and $\overline{x}$ of $\widetilde{X}$ such that $\kappa(\overline{x})$ algebraic over $\kappa(x)$ the geometric fibre $\widetilde{Y}_{\overline{x}} \to \overline{x}$ is acyclic.*

Theorem 6.17. *Any smooth morphism is locally acyclic and hence universally locally acyclic.*

Proof. Locally any smooth morphism factors into

$$Y \xrightarrow{g_0} \mathbb{A}^n_X \xrightarrow{g_1} \mathbb{A}^{n-1}_X \to \cdots \to \mathbb{A}^1_X \xrightarrow{g_n} X$$

where g_0 is etale. Clearly composition of locally acyclic morphisms is locally acyclic. Thus we only need to show each g_i is locally acyclic. For g_0 this is trivial. Hence we may assume Y is the affine line over X.

Let y be a point of Y closed in its fibre and $x = g(y)$. Assume $\kappa(\overline{y}) = \kappa(y)_{\mathrm{sep}}$ and $\kappa(\overline{x})$ is the separable closure of $\kappa(x)$ in $\kappa(\overline{y})$. We must show that for any geometric point $\overline{z}$ of $\widetilde{X} = \operatorname{Spec} \mathcal{O}_{X,\overline{x}}$ with $\kappa(\overline{z})$ algebraic over $\kappa(z)$ the geometric fibre $\widetilde{Y}_{\overline{z}} \to \overline{z}$ of $\widetilde{g}\colon \widetilde{Y} = \operatorname{Spec} \mathcal{O}_{Y,\overline{y}} \to \widetilde{X}$ is acyclic. We may replace X by $\widetilde{X} = \operatorname{Spec} A$ with $A = \mathcal{O}_{X,\overline{x}}$ strictly Henselian and then $Y = \operatorname{Spec} A[T]$ and y is a closed point of the closed fibre $\mathbb{A}^1_{\kappa(x)}$ of $Y \to X$. Note $\kappa(y)$ could either be the same as $\kappa(x)$ or be a finite purely inseparable extension. After a suitable base change we may kill the purely inseparable extension since it does not affect the cohomology. Hence we may assume $\overline{y}$ corresponds to a rational point of the closed fibre and after a translation we may assume $\widetilde{Y} = A\{T\}$ where $A\{T\}$ is the Henselization of $A[T]$ at the ideal generated by maximal ideal of A and T. Thus it suffices to show if A is Henselian then any geometric fibre of $\operatorname{Spec} A\{T\}$ over A is acyclic.

Finally to finish the proof we apply the computation results for curves over separably closed fields plus some extra argument to reduce to the case of A being excellent. Details omitted. □

5. Purity

6. The Weak Lefschetz Theorem

7. Kunneth Formula

We want to build up the Kunneth Formula for $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ sheaves. As usual we look into $\mathbb{Z}/\ell^n\mathbb{Z}$ sheaves first. Before stating the main theorem let us recall some background facts from derived categories.

Let Λ be a finite/torsion ring and consider the ringed site (X_{et}, Λ) and the category of sheaves of Λ-modules on this site, denoted by $Sh^\Lambda(X_{et})$. Denote $D(X)$ for the derived category of $Sh^\Lambda(X_{et})$. Then we have

Facts:

1 $Sh^\Lambda(X_{et})$ is a Grothendieck abelian category. In particular there exists K-flat resolutions (06YL):

For any complex $F^\bullet$ there exists a K-flat complex $P^\bullet$ whose terms are flat modules and a morphism $P^\bullet \to F^\bullet$ which is termwise surjective and quasi-isomorphism;

and K-injective resolutions (079P):

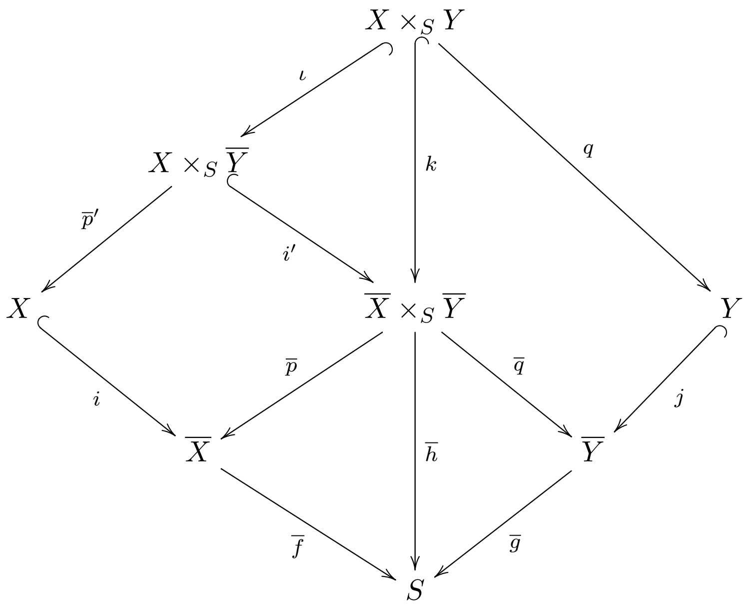
For any complex $F^\bullet$ there exists a K-injective complex $I^\bullet$ whose terms are injective modules and a morphism $F^\bullet \to I^\bullet$ which is termwise injective and quasi-isomorphism.

2 We have the derived tensor product $- \otimes^{\mathrm{L}} -$ in $D(X)$.

3 For any morphism $f\colon (X_{et}, \Lambda) \to (Y_{et}, \Lambda)$ we have the derived pushforward Rf_* and derived pullback Lf^* between $D(X)$ and $D(Y)$ and Lf^* is left adjoint to Rf_* (09T5) and Lf^* preserves K-flat complexes. Actually since the structure sheaf of rings is always Λ, pullback is exact so Lf^* is just f^*.

Let S be a qc scheme and $f\colon X \to S$ be a compactifiable morphism (separated of finite type, 0F41) with a compactification $X \xhookrightarrow{j} \overline{X} \xrightarrow{\overline{f}} S$. For any complex of sheaves $F^\bullet$ on X write $R_c f_* F^\bullet$ for $R\overline{f}_*(j_! F^\bullet)$. By definition this is $\overline{f}_* I(j_! F^\bullet)$ where $I(G^\bullet)$ is any K-injective resolution of $G^\bullet$. This is well-defined up to a quasi-isomorphism.

Suppose S qc and $f\colon X \to S$ and $g\colon Y \to S$ are compactifiable. Then we have a commutative diagram



Given any complex of sheaves F on X and G on Y we want to construct a canonical map in derived categories

$$R_c f_* F \otimes^{\mathrm{L}} R_c g_* G \longrightarrow R_c h_*(F \boxtimes^{\mathrm{L}} G)$$

where $F \boxtimes^{\mathrm{L}} G = p^* F \otimes^{\mathrm{L}} q^* G$. By definition this is a map

$$R\overline{f}_*(i_! F) \otimes^{\mathrm{L}} R\overline{g}_*(j_! G) \longrightarrow (R\overline{h}_*) k_!(p^* F \otimes^{\mathrm{L}} q^* G)$$

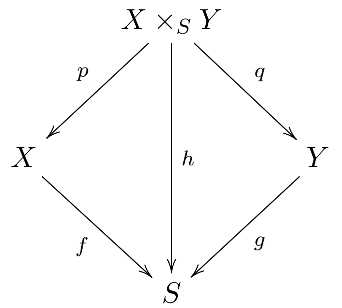
Use the adjunction $\overline{h}^* \dashv R\overline{h}_*$ and the fact that (derived) pullback commutes with derived tensor product it suffices to show

$$k_!(p^* F \otimes^{\mathrm{L}} q^* G) \xrightarrow{\sim} \overline{p}^* i_! F \otimes^{\mathrm{L}} \overline{q}^* j_! G$$

where this map comes from adjunction $(-)_! \dashv (-)^*$. Note that taking stalks is actually a pullback. Hence we may check on stalks to show it is an isomorphism. More precisely, take a K-flat resolution T of F. Then $p^* T$ is again a K-flat resolution of $p^* F$. Moreover $i_! T$ is also K-flat resolution of $i_! F$ since $i_!$ is exact and we can check by definition of acyclicity on stalks.

The main theorem is as follows.

Theorem 6.18 (Kunneth Formula). *Consider the diagram*



where S is qc and f, g compactifiable. Let F and G be complex of sheaves on X and Y. Then the natural map in derived categories

$$R_c f_* F \otimes^{\mathrm{L}} R_c g_* G \longrightarrow R_c h_* (F \boxtimes^{\mathrm{L}} G)$$

is a quasi-isomorphism.

Proof.

$$\begin{aligned} R_c f_* F \otimes^{\mathrm{L}} R_c g_* G &\xrightarrow{[1]} R_c f_* (F \otimes^{\mathrm{L}} f^* R_c g_* G) \\ &\xrightarrow{[2]} R_c f_* (F \otimes^{\mathrm{L}} R_c p_* (q^* G)) \\ &\xrightarrow{[3]} R_c f_* (R_c p_* (p^* F \otimes^{\mathrm{L}} q^* G)) \\ &\xrightarrow{[4]} R_c h_* (p^* F \otimes^{\mathrm{L}} q^* G) \end{aligned}$$

□

Next we will explain what are these quasi-isomorphisms [1], [2], [3], [4].

Lemma 6.19 (0F0G). *Let $f\colon X \to Y$ be a proper morphism. Then for any $E \in D(X_{et}, \mathbb{Z}/n\mathbb{Z}), K \in D(Y_{et}, \mathbb{Z}/n\mathbb{Z})$ the natural map*

$$R f_* E \otimes^{\mathrm{L}} K \longrightarrow R f_* (E \otimes^{\mathrm{L}} f^* K)$$

is a quasi-isomorphism.

Sketch of proof: Take stalks and use proper base change (0F0C) to reduce to the case where Y is separable closed field and X has finite cohomological dimension (095U). Then apply principal 09PB and reduce to show derived pushforward commutes with direct sums. This follows from 09Z1 and 07K7. Check these tags and make sure you believe in what is happening here! □

Corollary 6.20 (Projection Formula). *Let $f\colon X \to S$ be compactifiable with S qc. For any $F \in D(X_{et}, \mathbb{Z}/n\mathbb{Z})$ and $G \in D(S_{et}, \mathbb{Z}/n\mathbb{Z})$ the natural map in derived categories*

$$R_c f_* F \otimes^{\mathrm{L}} G \longrightarrow R_c f_* (F \otimes^{\mathrm{L}} f^* G)$$

is a quasi-isomorphism.

Proof. Let $X \overset{i}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} S$ be a compactification. Then by the lemma above

$$R_c f_* F \otimes^{\mathrm{L}} G = R\overline{f}_* (i_! F) \otimes^{\mathrm{L}} G \xrightarrow{\text{qis}} R\overline{f}_* (i_! F \otimes^{\mathrm{L}} \overline{f}^* G)$$

so it suffices to show the natural map

$$i_! (F \otimes^{\mathrm{L}} i^* H) \longrightarrow i_! F \otimes^{\mathrm{L}} H$$

is a quasi-isomorphism for any $H \in D(\overline{X}_{et}, \mathbb{Z}/n\mathbb{Z})$. This can be checked on stalks. □

Proof of [1], [3]: Apply this corollary with $\mathbb{Z}/n\mathbb{Z}$ replaced by Λ.

□

Lemma 6.21 (0F0C). *Consider a Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where f is proper. For any $E \in D(X_{et}, \mathbb{Z}/n\mathbb{Z})$ the natural morphism

$$g^* Rf_* E \longrightarrow (Rf'_*) g'^* E$$

is a quasi-isomorphism.

Sketch of proof: Use 0A3V to show fibres have bounded dimension. Apply 095U to show the pushforwards have finite cohomological dimension. Take a K-injective resolution and use the usual proper base change theorem to show we can compute both sides using this resolution (07K7). Apply again the usual proper base change theorem. □

Proof of [2]: (See the commutative diagram before the theorem).

$$f^* R_c g_* G = f^* R\overline{g}_* j_! G \xrightarrow{\sim} R\overline{p}'_* (\overline{q} i')^* j_! G = R\overline{p}'_* \iota_! q^* G = R_c p_* q^* G$$

where the third equality comes from the fact that base change commutes with extension by zero (check on stalks). □

Lemma 6.22. *Consider a Cartesian diagram*

$$\begin{array}{ccc} U & \overset{i'}{\hookrightarrow} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \overset{i}{\hookrightarrow} & Y \end{array}$$

where f is proper. Then for any complex of sheaves G on U the natural map

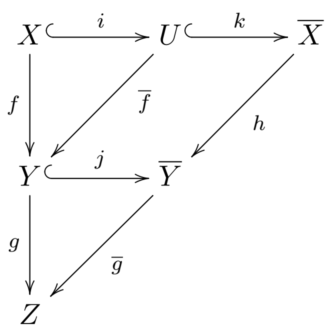
$$i_! Rf'_* G \longrightarrow (Rf_*) i'_! G$$

is a quasi-isomorphism.

Sketch of proof: Take a K-injective resolution I of G and check that if I is an injective object then $R^n f_* i'_! I = 0$ for $n > 0$. This can be shown by taking stalks and applying the usual proper base change theorem. Then by 095U and 07K7 we can compute both sides using I. Again the usual proper base change theorem shows that $i_! f'_* I \to f_* i'_! I$ is termwise isomorphism. □

Proof of [4]: It is enough to show if $f\colon X \to Y$ and $g\colon Y \to Z$ are compactifiable then $R_c g_* R_c f_* \xrightarrow{\sim} R_c(gf)_*$.

Consider the commutative diagram



where $\overline{Y}$ is a compactification of Y over Z and $\overline{X}$ is a compactification of X over $\overline{Y}$ and U is the fibre product $Y \times_{\overline{Y}} \overline{X}$.

Then $R_c g_* R_c f_* = R\overline{g}_* j_! R\overline{f}_* i_! \xrightarrow{\sim} R\overline{g}_* Rh_* k_! i_! = R(\overline{g}h)_*(ki)_! = R_c(gf)_*$ where the second equality comes from the lemma above. □

Remark 6.23. *All the results/lemmas above involving a tag can be essentially reduced to these two facts*:

a *(0A3V,095U) If f is proper then f_* has finite cohomological dimension on torsion abelian sheaves.*

b *(07K7) If F left exact and $R^nF = 0$ for some n then every complex of right F-acyclic objects computes RF.*

Next we will introduce some explicit corollaries and a few variations of the Kunneth Formula.

Corollary 6.24. *If F and G are acyclic above and in addition $R_c^r f_* F$ is flat for all r then there are canonical isomorphisms*

$$\bigoplus_{r+s=m} R_c^r f_* F \otimes R_c^s g_* G \xrightarrow{\sim} R_c^m h_*(F \boxtimes^{\mathrm{L}} G)$$

Proof. In general if $F^\bullet$ and $G^\bullet$ are acyclic above complexes then there is a spectral sequence

$$E_2^{r,s} = \bigoplus_{i+j=s} \mathrm{Tor}_{-r}^{\Lambda}(H^i(F^\bullet), H^j(G^\bullet)) \Longrightarrow H^{r+s}(F^\bullet \otimes^{\mathrm{L}} G^\bullet)$$

We may reduce to the case $F^\bullet$ and $G^\bullet$ are bounded above and take a bounded above K-flat resolution $P^\bullet$ of $F^\bullet$ and look at the double complex $P^\bullet \otimes G^\bullet$. Then the corollary follows from the fact (07K7) that if F is acyclic above complex of sheaves then $R_c f_* F$ is acyclic above. □

Corollary 6.25. *Let X and Y be proper over a separably closed field. Let F and G be sheaves on X and Y. Assume F and $H^r(X, F)$ are all flat, then we have isomorphisms*

$$\bigoplus_{r+s=m} H^r(X, F) \otimes H^s(Y, G) \longrightarrow H^m(X \times Y, F \boxtimes G)$$

Remark 6.26. *In the case above the natural isomorphism could be obtained using a cup product. To make this explicit one can use Čech cohomology.*

Remark 6.27. *The steps* [1] *to* [4] *are formal. If for some other morphisms and sheaves we have the corresponding projection formula and base change theorem then we would get the Kunneth formula for free. For example, if f is smooth and g qc, Λ is prime to* $\mathrm{char}(S)$*, F is a complex of sheaves on X quasi-isomorphic to some perfect complex T, G is an acyclic below complex of sheaves on Y such that Rg_*G quasi-isomorphic to some perfect complex P, then the natural map*

$$Rf_*F \otimes^{\mathrm{L}} Rg_*G \longrightarrow Rh_*(p^*F \otimes^{\mathrm{L}} q^*G)$$

is a quasi-isomorphism (0944).

We can also require f to be proper and impose some conditions on the sheaves.

Also see 0F1P: let X and Y be schemes of finite type over a separably closed field k and let F and G be Λ-modules with Λ torsion and prime to $\operatorname{char} k$ *then*

$$R\Gamma(X,F)\otimes^{\mathrm{L}} R\Gamma(Y,G)\to R\Gamma(X\times Y, F\boxtimes^{\mathrm{L}} G)$$

is a quasi-isomorphism.

In particular if k is a separably closed field, $X=\mathbb{A}^n_k$ and $Y=\mathbb{A}^m_k$, $F=G=\Lambda=\mathbb{Z}/\ell\mathbb{Z}$, ℓ invertible in k, then by induction we would have $H^i(\mathbb{A}^n_k,\mathbb{Z}/\ell\mathbb{Z})=0$ if $i>0$ and $H^0(\mathbb{A}^n_k,\mathbb{Z}/\ell\mathbb{Z})=\mathbb{Z}/\ell\mathbb{Z}$. This could also be done by showing $\mathbb{A}^n_X\to X$ is acyclic.

Lemma 6.28. *Let X be a variety of dimension m over a separably closed field k. For any flat constructible sheaf F on X, $R_c\Gamma(X,F)$ is quasi-isomorphic to a complex of finitely generated projective modules bounded in* $[0,2m]$.

Proof. Since F is constructible, by finiteness theorem we see $H^*_c(X,F)$ are finite, in particular finitely generated. Since X has dimension m, $H^i_c(X,F)=0$ if $i>2m$. Thus we see the complex $R_c\Gamma(X,F)$ is acyclic above $2m$, bounded below 0 with finitely generated cohomology groups. Then we can find a bounded above complex P with each P^i finitely generated flat and P quasi-isomorphic to $R_c\Gamma(X,F)$.

Now for any sheaf G on the base, by the projection formula we have

$$P\otimes G\xrightarrow{\sim} R_c\Gamma(X,F)\otimes^{\mathrm{L}} G\xrightarrow{\sim} R_c\Gamma(F\otimes G|_X)$$

thus

$$H^i(P\otimes G)=H^i_c(F\otimes G|_X)=0$$

if $i<0$. Let B be the image of $P^{-1}\to P^0$. Then after tensoring G, $B\otimes G$ is still the cokernel of the map $P^{-2}\otimes G\to P^{-1}\otimes G$ thus injects into $P^0\otimes G$. Since P^0 is flat, this would imply that B is also flat. Replace P^0 by P^0/B and we are done with the wanted complex. □

Definition 6.29. *A complex of modules over a ring A is called perfect if it is bounded of finitely generated projective modules.*

We shall write $\mathbf{H}_c(X,F)$ for the perfect complex quasi-isomorphic to $R_c\Gamma(X,F)$ as in Lemma 6.28.

For a general setup, let Ω be a finite field extension of $\mathbb{Q}_\ell$ and A be the integral closure of $\mathbb{Z}_\ell$ in Ω. Then A has a unique nonzero prime ideal $\mathfrak{m}$ which gives rise to the unique extension of valuation on Ω. Under this valuation Ω is complete. In particular if we set $\Lambda_n=A/\mathfrak{m}^n$, then $A=\lim\Lambda_n$.

Proposition 6.30. *Let $f\colon X\to S$ be a variety of dimension m over a separably closed field. Let $F=(F_n)$ be a constructible sheaf of A-modules with each F_n flat Λ_n-module. Then it is possible to choose the complexes $\mathbf{H}_c(X,F_n)$ so that for all n there are morphisms of complexes $\mathbf{H}_c(X,F_{n+1})\to\mathbf{H}_c(X,F_n)$ inducing isomorphisms*

$$\mathbf{H}_c(X,F_{n+1})\otimes\Lambda_n\xrightarrow{\sim}\mathbf{H}_c(X,F_n)$$

Sketch of proof: By definition we have $F_{n+1}\otimes_{\Lambda_{n+1}}\Lambda_n\xrightarrow{\sim}F_n$ so by projection formula there are quasi-isomorphisms

$$\mathbf{H}_c(X,F_{n+1})\otimes_{\Lambda_{n+1}}\Lambda_n\xrightarrow{\sim}R_c\Gamma(X,F_n)\xleftarrow{\sim}\mathbf{H}_c(X,F_n)$$

Then we can apply the following lemma to get an actual quasi-isomorphism morphism from left to right.

Lemma 6.31. *Let A be any Noetherian ring. Let $M\xrightarrow{\phi}L\xleftarrow{\pi}N$ be morphisms of complexes with ϕ and π quasi-isomorphisms. If M is perfect then there exists a quasi-isomorphism $\psi\colon M\to N$. Moreover if π is surjective then $\pi\psi=\phi$.*

Now we want this quasi-isomorphism to become an isomorphism. This could be done using the following lemma.

Lemma 6.32. *Let A be a local Artin ring and A_0 some quotient ring of A. Let M be perfect A-complex and N be perfect A_0 complex such that there is a quasi-isomorphism $\psi \colon M_0 \to N$. Then there exists a perfect complex L of A-modules and a quasi-isomorphism $\phi \colon M \to L$ and an isomorphism $L_0 \cong N$ whose composition with ϕ_0 is ψ.*

□

Let $\mathbf{H}_c(X, F) = \lim \mathbf{H}_c(X, F_n)$. Similar to LEMMA it is a perfect complex of A-modules and $\mathbf{H}_c(X, F) \otimes \Lambda_n \xrightarrow{\sim} \mathbf{H}_c(X, F_n)$. Moreover

$$H^r(\mathbf{H}_c(X, F)) = H^r(\lim \mathbf{H}_c(X, F_n)) = \lim H^r(\mathbf{H}_c(X, F_n))$$
$$= \lim H_c^r(X, F_n) = H_c^r(X, F)$$

Corollary 6.33. *Let S be a separably closed field. Let $f \colon X \to S, g \colon Y \to S$ be compactifiable. For any flat constructible sheaves F and G of A-modules on X and Y there is a natural quasi-isomorphism*

$$\mathbf{H}_c(X, F) \otimes \mathbf{H}_c(Y, G) \xrightarrow{\sim} \mathbf{H}_c(X \times_S Y, F \boxtimes G)$$

Thus there are exact sequences

$$0 \to \bigoplus_{r+s=m} H_c^r(F) \otimes H_c^s(G) \to H_c^m(F \boxtimes G) \to \bigoplus_{i+j=m+1} \mathrm{Tor}_1^A(H_c^i(F), H_c^j(G)) \to 0$$

Sketch of proof: As A is PID, higher Tor groups vanish. Hence the exact sequences come from the spectral sequences in Corollary 6.24.

We have quasi-isomorphisms

$$\mathbf{H}_c(F_n) \otimes \mathbf{H}_c(G_n) \xrightarrow{\sim} R_c h_*(F_n \boxtimes G_n) \xleftarrow{\sim} \mathbf{H}_c(F_n \boxtimes G_n)$$

By Lemma 6.31 we get quasi-isomorphisms

$$\psi_n \colon \mathbf{H}_c(X, F_n) \otimes \mathbf{H}_c(Y, G_n) \xrightarrow{\sim} \mathbf{H}_c(X \times_S Y, F_n \boxtimes G_n)$$

One checks that all constructions are functorial hence $\psi_{n+1} \otimes \Lambda_n$ is homotopic to ψ_n. After some modification we can get $\psi_{n+1} \otimes \Lambda_n = \psi_n$.

□

Corollary 6.34. *There are natural isomorphisms*

$$H_c^*(X, F \otimes \Omega) \otimes H_c^*(Y, G \otimes \Omega) \xrightarrow{\sim} H_c^*(X \times_S Y, (F \boxtimes G) \otimes \Omega)$$

where Ω is the fraction field of A.

8. Cycle Class Map and Chern Classes

Let k be an algebraically closed field. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ be the constant sheaf with n prime to $\mathrm{char}(k)$. Let X be a smooth variety over k. Recall that

1 A prime r-cycle on X is a closed integral subscheme of codimension r. An algebraic r-cycle is an element of the free abelian group $C^r(X)$ generated by prime r-cycles. An algebraic cycle is an element of the graded group $C^*(X)$.

2 A prime r-cycle W and a prime s-cycle Z intersect properly if each irreducible component of $W \cap Z$ has codimension $r + s$. In this case $W \cdot Z$ is defined and belongs to $C^{r+s}(X)$ (0AZL). Two algebraic cycles intersect properly if every pair of prime cycles occurring in them intersects properly.

3 For any flat map $\pi\colon Y \to X$ of constant relative dimension, the pullback $\pi^*\colon C^*(X) \to C^*(Y)$ is defined and it is just the fiber product (02RA). For any proper map $\pi\colon Y \to X$, the pushforward $\pi_*\colon C^*(Y) \to C^*(X)$ is defined. For Z a prime cycle on Y, $\pi_* Z = 0$ if $\dim(\pi(Z)) < \dim(Z)$ and $\pi_* Z = d \cdot \pi(Z)$ where d is the degree of $\pi|_Z$ otherwise (02R3). If π is proper flat of constant relative dimension then we have the projection formula (0B0C)

$$\pi_*(\pi^* W \cdot Z) = W \cdot \pi_* Z$$

Denote $H^*(X,\Lambda)$ to be the graded abelian group $\bigoplus_r H^{2r}(X,\Lambda(r))$. The cup product makes $H^*(X)$ into a (anti)commutative graded ring, as can be seen explicitly using Čech cohomology. We shall define a homomorphism of graded groups $\mathrm{cl}_X\colon C^r(X) \to H^{2r}(X,\Lambda(r))$. If Z is a smooth prime r-cycle then (Z,X) is a smooth pair of codimension r and we have the Gysin sequence

$$H^0(Z,\Lambda) \longrightarrow H^{2r}(X,\Lambda(r))$$

then $\mathrm{cl}_X(Z)$ is defined to be the image of $1 \in H^0(Z,\Lambda)$ under this map.

Lemma 6.35. *For any reduced closed subscheme Z of pure codimension r in X, $H^s_Z(X,\Lambda) = 0$ for $s < 2r$.*

Proof. If Z is smooth, this follows from the Gysin sequence. We shall induct on r. If $r = \dim(X)$ then Z is finitely many points hence smooth. In general choose U open in X such that $U \cap Z$ is smooth and dense in each irreducible component of Z and $X - Z \subset U$. Using the exact sequence for triple LEMMA we get

$$\cdots \to H^s_{X-U}(X,\Lambda) \to H^s_Z(X,\Lambda) \to H^s_{U\cap Z}(U,\Lambda) \to H^{s+1}_{X-U}(X,\Lambda) \to \dots$$

Now $X - U$ has codimension at least $r+1$ in X and by induction hypothesis $H^s_{X-U}(X,\Lambda) = 0$ for $s < 2(r+1)$. Also $H^s_{U\cap Z}(U,\Lambda) = 0$ for $s < 2r$ as $U \cap Z$ is smooth. □

Now let Z be any prime r-cycle. Choose an open subset U in X as in the proof of the lemma. Then by Gysin sequence we have

$$H^0(U\cap Z,\Lambda) \xrightarrow{\sim} H^{2r}_{U\cap Z}(U,\Lambda(r)) \xleftarrow{\sim} H^{2r}_Z(X,\Lambda(r)) \longrightarrow H^{2r}(X,\Lambda(r))$$

and $\mathrm{cl}_X(Z)$ is defined to be the image of $1 \in H^0(U \cap Z,\Lambda)$ under these maps. It is independent of the choice of U. Extend linearly we get the map cl_X.

Lemma 6.36. *Let $\pi\colon Y \to X$ be a map of smooth varieties over k and Z be an algebraic cycle on X. If for every prime cycle Z' occurring in Z, $Y \times_X Z'$ is integral of the same codimension, then $\pi^* Z$ is defined and $\mathrm{cl}_Y(\pi^* Z) = \pi^* \mathrm{cl}_X(Z)$.*

Proof. We may assume Z is prime and choose open subvarieties on X and Y on which Z and $Z \times_X Y$ are smooth. Then use the functoriality for long exact sequences for triples LEMMA. □

Lemma 6.37. *Let $i\colon Z \to X$ be a closed immersion of smooth varieties of codimension c. For any $W \in C^r(Z)$ we have $i_*(\mathrm{cl}_Z(W)) = \mathrm{cl}_X(W)$ where i_* is the Gysin map $H^{2r}(Z,\Lambda(r)) \to H^{2(r+c)}(X,\Lambda(r+c))$.*

Proof. Varieties are catenary so codimension is additive. Check by functoriality that if $i_1\colon W \to Z$ then $i_* i_{1*} = (i \circ i_1)_*$ as Gysin maps. □

Lemma 6.38. *Let X and Y be smooth varieties. For any $W \in C^*(X)$ and $Z \in C^*(Y)$ we have*

$$\mathrm{cl}_{X\times Y}(W \times Z) = p^* \mathrm{cl}_X(W) \cup q^* \mathrm{cl}_Y(Z)$$

Proof. Note that over algebraically closed fields, products of varieties are again varieties. We may assume W and Z are prime cycles with Z smooth since we can choose dense open and by purity lower cohomology groups stay stable.

Let $i\colon X\times Z\to X\times Y$. Then

$$\begin{aligned} i_*i^*\operatorname{cl}_{X\times Y}(W\times Y) &= i_*\operatorname{cl}_{X\times Z}(i^*(W\times Y))\\ &= i_*\operatorname{cl}_{X\times Z}(W\times Z)\\ &= \operatorname{cl}_{X\times Y}(W\times Z)\end{aligned}$$

where we have used the lemmas above. The assumptions for i^* are satisfied since Z and Y are smooth, hence the projections $X\times Z\to X$ and $X\times Y\to X$ are flat of constant relative dimension.

On the other hand

$$\begin{aligned} i_*i^*\operatorname{cl}_{X\times Y}(W\times Y) &= \operatorname{cl}_{X\times Y}(W\times Y)\cup\operatorname{cl}_{X\times Y}(X\times Z)\\ &= \operatorname{cl}_{X\times Y}(p^*W)\cup\operatorname{cl}_{X\times Y}(q^*Z)\\ &= p^*\operatorname{cl}_X(W)\cup q^*\operatorname{cl}_Y(Z)\end{aligned}$$

where the first equality follows essentially from the local purity $R^{2c}i^!F=i^*F(-c)$ and $R^ji^!F=0$ for $j\neq 2c$. □

LEMMA 6.39. *Let W and Z be algebraic cycles on smooth variety X such that W intersects with Z properly. Then*

$$\operatorname{cl}_X(W\cdot Z)=\operatorname{cl}_X(W)\cup\operatorname{cl}_X(Z)$$

PROOF. We may assume W and Z are prime. Then $W\times Z$ intersects with Δ_X properly in $X\times X$. Thus

$$\begin{aligned} \operatorname{cl}_X(W\cdot Z) &= \operatorname{cl}_X(\Delta^*(W\times Z))\\ &= \Delta^*\operatorname{cl}_{X\times X}(W\times Z)\\ &= \Delta^*(p^*\operatorname{cl}_X(W)\cup q^*\operatorname{cl}_X(Z))\\ &= \operatorname{cl}_X(W)\cup\operatorname{cl}_X(Z)\end{aligned}$$

□

REMARK 6.40. *The map $\operatorname{cl}_X\colon C^1(X)\to H^2(X,\Lambda(1))$ is the composite of the canonical maps*

$$C^1(X)\to\operatorname{Pic}(X)\ \text{and}\ \operatorname{Pic}(X)\to H^2(X,\Lambda(1))$$

where the first map comes from the fact that smooth varieties are regular hence all local rings are UFDs (0BE9).

To see this note for any smooth prime cycle Z of codimension 1 we have two exact sequences coming from the Kummer sequence

$$\begin{array}{ccccc} H^1_Z(X,\mathbb{G}_m) & \longrightarrow & H^1_Z(X,\mathbb{G}_m) & \xrightarrow{\alpha} & H^2_Z(X,\Lambda(1))\\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle i_*}\\ H^1(X,\mathbb{G}_m) & \longrightarrow & H^1(X,\mathbb{G}_m) & \xrightarrow{\beta} & H^2(X,\Lambda(1))\end{array}$$

where the vertical maps are induced by $H^0_Z(X,-)\to H^0(X,-)$ hence just being the Gysin maps. Now since Z is prime of codimension 1 $H^1_Z(X,\mathbb{G}_m)$ is just $\mathbb{Z}$ and the map δ sends 1 to the line bundle associated to Z in $H^1(X,\mathbb{G}_m)=\operatorname{Pic}(X)$. Since Z is smooth by purity we have $H^2_Z(X,\Lambda(1))=H^0(Z,\Lambda)=\Lambda=\mathbb{Z}/n\mathbb{Z}$. Hence $\operatorname{cl}_X(Z)=i_*(1)=i_*\alpha(1)=\beta\delta(1)=\beta(Z)$.

EXAMPLE 6.41. *Consider $\mathbf{P}^m_k$. Note that for any linear subspace L^r of $\mathbf{P}^m_k$ of codimension $r>0$, the Gysin map $\Lambda=H^0(L^r,\Lambda)\xrightarrow{i_*}H^{2r}(\mathbf{P}^m_k,\Lambda(r))$ is an isomorphism. Thus $H^{2r}(\mathbf{P}^m_k,\Lambda(r))$ is generated by $\operatorname{cl}(L^r)$. Since $\operatorname{Pic}(\mathbf{P}^m_k)=\mathbb{Z}$ is generated by the class of any hyperplane, $\operatorname{cl}(L^1)$ is independent of L^1. Also L^r is the transverse intersection of r hyperplanes, Lemma 6.39 tells us that*

$$\operatorname{cl}(L^r)=\operatorname{cl}(L^1)\cup\cdots\cup\operatorname{cl}(L^1)$$

is also independent of L^r. Thus the map

$$\Lambda[T]/T^{m+1} \longrightarrow H^*(\mathbf{P}_k^m)$$

that sends T^r to $\mathrm{cl}(L^r)$ is an isomorphism of graded rings.

Next we discuss the Chern classes. We shall only consider quasi-projective smooth varieties.

Let X be a quasi-projective smooth variety. Let E be a vector bundle on X. Associated to E the projective bundle $\mathbf{P}(E)$. It is a variety with a morphism $\mathbf{P}(E) \to X$ such that each fiber $\mathbf{P}(E)_x$ is canonically isomorphic to $\mathbf{P}(E_x)$.

Proposition 6.42. *Let E be a vector bundle of rank m over X and let $P = \mathbf{P}(E) \xrightarrow{p} X$ be the associated projective bundle. Let $\xi \in H^2(P, \Lambda(1))$ be the image of the canonical line bundle $\mathcal{O}_P(1)$ on P under the map $\mathrm{Pic}(P) \to H^2(P, \Lambda(1))$. Then the map*

$$H^*(X)[T]/T^m \longrightarrow H^*(P)$$

which is p^* on $H^*(X)$ and sends T to ξ is an isomorphism of graded $H^*(X)$-modules.

Proof. Suppose firstly that E is a trivial bundle. Then $P = \mathbf{Proj}(\mathcal{O}_X[x_1, \ldots, x_m]) = X \times \mathbf{P}^{m-1}$. Since X is smooth, we can do the Kunneth formula for Λ on X and $\mathbf{P}^{m-1}$ by Remark 6.27. Thus we get isomorphisms

$$\bigoplus_{r+s=t} H^r(X, \Lambda) \otimes H^s(\mathbf{P}^{m-1}, \Lambda) \overset{p^* \cup q^*}{\longrightarrow} H^t(X \times \mathbf{P}^{m-1}, \Lambda)$$

hence we get isomorphisms of graded rings after tensoring the twists

$$H^*(X)[T]/T^m \xrightarrow{\sim} H^*(X) \otimes H^*(\mathbf{P}^{m-1}) \xrightarrow{\sim} H^*(X \times \mathbf{P}^{m-1})$$

This shows the result in the trivial bundle case. Since locally any vector bundle is trivial, we then use Mayer-Vietoris sequences to conclude. □

Now there are unique elements $c_r(E) \in H^{2r}(X)$ such that

$$c_0(E) = 1 \ , \ c_r(E) = 0 \text{ for } r > m$$

$$\sum_{r=0}^{m} (-1)^r c_r(E) \xi^{m-r} = 0$$

Definition 6.43. *The $c_r(E)$ is called the r-th Chern class of E, $c(E) = \sum c_r(E)$ is called the total Chern class of E and $c(E)(t) = \sum c_r(E) t^r$ is called the Chern polynomial of E.*

Theorem 6.44. *The Chern classes satisfy the following properties and are uniquely characterized by them.*

a *If $\pi \colon Y \to X$ is a morphism of quasi-projective smooth varieties and E is a vector bundle on X then $c_r(\pi^* E) = \pi^* c_r(E)$.*

b *If E is a line bundle on X then $c_1(E) = p_X(E)$ where $p_X(E)$ is the image of $E \in \mathrm{Pic}(X) \to H^2(X, \Lambda(1))$.*

c *If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles on X then*

$$c(E)(t) = c(E')(t) \cdot c(E'')(t)$$

Proof. Splitting principle. □

Remark 6.45. *Property (c) actually tells us that the Chern classes factor through the Grothendieck group $K(X)$ of vector bundles on X. Since X is smooth, $K(X)$ is the same as the Grothendieck group of coherent $\mathcal{O}_X$-modules. Thus a prime cycle Z on X defines an element $\gamma(Z) = [\mathcal{O}_Z] \in K(X)$ and $Z \mapsto \gamma(Z)$ extends linearly to $\gamma \colon C^*(X) \to K(X)$.*

Let $K^r(X)$ be the subgroup of $K(X)$ generated by all coherent sheaves supported in codimension $\geq r$. Then the groups $K^r(X)$ define a decreasing filtration of $K(X)$ and the associated graded group

$GK^*(X)$ becomes a ring under the product law defined by Tor:

$$[M][N] = \sum (-1)^i [\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(M,N)]$$

Then the graded homomorphism $\gamma' \colon C^*(X) \to GK^*(X)$ preserves products hence induce a surjective homomorphism of graded rings

$$\phi \colon CH^*(X) \to GK^*(X)$$

After suitable modifications of the Chern classes we can get a map of graded rings $\psi \colon GK^*(X) \to H^*(X)$. Define cl'_X to be the composite

$$C^*(X) \to CH^*(X) \xrightarrow{\phi} GK^*(X) \xrightarrow{\psi} H^*(X)$$

then actually this is just cl_X.

9. Poincare Duality

Let k be an algebraically closed field. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ where n is prime to $\mathrm{char}\, k$. We shall write $\mathrm{cl}_X(P)$ for the cycle class inside the compactly supported cohomology group.

Theorem 6.46 (Poincare Duality). *Let X be a smooth variety of dimension d over k.*

- a *There is a unique isomorphism $\eta \colon H_c^{2d}(X, \Lambda(d)) \to \Lambda$ such that $\mathrm{cl}_X(P) \mapsto 1$ for any closed point P of X.*
- b *For any constructible sheaf F of Λ-modules on X the canonical pairings*

$$H_c^r(X,F) \times \mathrm{Ext}_X^{2d-r}(F, \Lambda(d)) \longrightarrow H_c^{2d}(X, \Lambda(d)) \xrightarrow{\eta} \Lambda$$

are non-degenerate.

Corollary 6.47. *If F is lcc then the cup product pairings*

$$H_c^r(X,F) \times H^{2d-r}(X, \check{F}(d)) \longrightarrow H_c^{2d}(X, \Lambda(d)) \xrightarrow{\sim} \Lambda$$

are non-degenerate.

Lemma 6.48. *For any variety X of dimension d over k we have $H_c^{2d}(X, \Lambda(d)) = \Lambda$.*

Lemma 6.49. *Let $\pi \colon Y \to X$ be a separated qc etale morphism where X is a smooth variety of dimension d over k. Let P be a closed point of Y and let $Q = \pi(P)$. Then the map*

$$\pi_* \colon H_c^{2d}(Y, \Lambda(d)) \longrightarrow H_c^{2d}(X, \Lambda(d))$$

induced by $R_c^0 \pi_* \Lambda(d) = \pi_! \pi^* \Lambda(d) \longrightarrow \lambda(d)$ sends $\mathrm{cl}_Y(P)$ to $\mathrm{cl}_X(Q)$.

Proof of (b) by induction on dimension. The case $d = 1$.

Write $\phi^r(X,F)$ for the map $\mathrm{Ext}_X^{2d-r}(F, \Lambda(d)) \to H_c^r(X,F)^\vee$ induced by the pairing.

Step 1: Let $\pi \colon X' \to X$ be a finite etale map of smooth varieties over k. For any constructible sheaf F on X', $\phi^r(X',F)$ is an isomorphism if and only if $\phi^r(X, \pi_* F)$ is an isomorphism.

10. Rationalities