REALIZING $\widehat{\mathcal{O}}_{G_m^\sharp}$-COMODULES EXPLICITLY

We show the following theorem [BL22a, Theorem 3.5.8] by mimic the argument of [BL22b, Proposition 2.4.4].

Theorem 0.1. *There is an equivalence of categories*

$$\mathcal{D}(\mathrm{WCart}^{\mathrm{HT}}) \simeq \widehat{\mathcal{D}}_{(\Theta^p-\Theta)-\mathrm{nil}}(\mathbb{Z}_p[\Theta])$$

In particular, the Cartier dual of $\mathbb{G}_m^\sharp$ is the formal completion of $\mathrm{Spec}(\mathbb{Z}[\Theta])$ *along* $V(p, \Theta^p - \Theta)$.

Proof. Let $\eta : \mathrm{Spf}(\mathbb{Z}_p) \to \mathrm{WCart}^{\mathrm{HT}}$ be the faithfully flat cover. Identifying $\widehat{\mathcal{O}}_{G_m^\sharp}$ with $\mathbb{Z}[t^{-1}, (t-1)^n/n!]^\wedge$ and a simple computation we have a short exact sequence

$$0 \to \mathbb{Z}_p \to \widehat{\mathcal{O}}_{G_m^\sharp} \xrightarrow{\frac{d}{d\log(t)}} \widehat{\mathcal{O}}_{G_m^\sharp} \to 0.$$

By faithfully flatness of η, this gives a fiber sequence

$$\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}} \to \eta_*\mathcal{O}_{\mathrm{Spf}(\mathbb{Z}_p)} \xrightarrow{\Theta_{\mathbb{Z}_p}} \eta_*\mathcal{O}_{\mathrm{Spf}(\mathbb{Z}_p)}$$

where $\Theta_{\mathbb{Z}_p}$ denote the Sen operator for $\eta_*\mathcal{O}_{\mathrm{Spf}(\mathbb{Z}_p)}$. Since η_* and η^* both commutes with filtered colimits, it follows from the projection formula and the above fibre sequence that $R\Gamma(\mathrm{WCart}^{\mathrm{HT}}, -)$ also commutes with filtered colimits. Hence for any $\mathcal{E} \in \mathcal{D}(\mathrm{WCart}^{\mathrm{HT}})$, we have $\mathcal{E}_\eta \simeq R\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \eta_*\mathcal{E}_\eta) \simeq R\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \eta_*\mathcal{O}_{\mathrm{Spf}(\mathbb{Z}_p)} \otimes \mathcal{E})$ carrying an endomorphism $\Theta_{\mathcal{E}}$ (given by $\Theta_{\mathbb{Z}_p} \otimes \mathrm{Id}_{\mathcal{E}}$). Since $\Theta_{\mathbb{Z}_p}^p - \Theta_{\mathbb{Z}_p}$ is locally nilpotent $\mod p$, so is $\Theta_{\mathcal{E}}$ by using $R\Gamma(\mathrm{WCart}^{\mathrm{HT}}, -)$ commuting with filtered colimits. In conclusion, we get a functor

$$\mathcal{D}(\mathrm{WCart}^{\mathrm{HT}}) \to \widehat{\mathcal{D}}_{(\Theta^p-\Theta)-\mathrm{nil}}(\mathbb{Z}_p[\Theta])$$

by $\mathcal{E} \mapsto (\mathcal{E}_\eta, \Theta_{\mathcal{E}})$. And the above fiber sequence shows that this functor is fully faithful.

As for essential surjectivity, suppose $M \in \widehat{\mathcal{D}}_{(\Theta^p-\Theta)-\mathrm{nil}}(\mathbb{Z}_p[\Theta])$, i.e. p-adically complete $\mathbb{Z}_p$-module M with a Θ-action which $\Theta^p - \Theta$ is locally nilpotent (mod p). We endow M a $\widehat{\mathcal{O}}_{G_m^\sharp}$ comodule structure by the following function $t^\Theta : M \to \widehat{\mathcal{O}}_{G_m^\sharp} \otimes M$:

$$t^\Theta(m) = \sum_{n=0}^{\infty} \frac{(t-1)^n}{n!} \otimes (\Theta(\Theta-1)\cdots(\Theta-n+1)(m)).$$

It is well defined by the natural of $\Theta^p - \Theta \equiv \prod_{0 \le n < p}(\Theta - n) \mod p$ being locally nilpotent. It is then not hard to verify that this realizes M as a $\widehat{\mathcal{O}}_{G_m^\sharp}$-comodule and it is an inverse. ∎

Remark 0.2. Heuristically, one can think

$$\begin{aligned} t^{\Theta}(m) &= \exp(\log(t)\Theta)(m) \\ &:= \sum_{n=0}^{\infty} \frac{(\log(t))^n}{n!} \otimes \Theta^n(m) \end{aligned}$$

which agrees with the formula in [BL22a, Proposition 3.7.1]. However, it is not very clear that the infinite sum converges here.

In particular, one obtains the following result in [BL22a, Proposition 3.5.11].

Proposition 0.3. *Let $\eta : \mathrm{Spf}(\mathbb{Z}_p) \to \mathrm{WCart}^{\mathrm{HT}}$ be the faithfully flat cover. Then for any $\mathcal{E} \in \mathcal{D}(\mathrm{WCart}^{\mathrm{HT}})$, we have a canonical fibre sequence:*

$$R\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \mathcal{E}) \to \mathcal{E}_\eta \xrightarrow{\Theta_{\mathcal{E}}} \mathcal{E}_\eta$$

where $\Theta_{\mathcal{E}}$ is the Sen operator.

Proof. By the Theorem above we have

$$R\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \mathcal{E}) \to \mathcal{E}_\eta \xrightarrow{\Theta_{\mathbb{Z}_p} \otimes \mathrm{Id}_{\mathcal{E}_\eta}} \mathcal{E}_\eta$$

It suffices to show that $\Theta_{\mathbb{Z}_p} \otimes \mathrm{Id}_{\mathcal{E}}$ is isomorphic to $\Theta_{\mathcal{E}}$. Since $\mathcal{E}_\eta = R\Gamma(\eta_* \mathcal{E}_\eta)$ is the fibre of

$$\widehat{\mathcal{O}}_{G_m^\sharp} \otimes \mathcal{E}_\eta \xrightarrow{\frac{d}{d\log(t)} \otimes \mathrm{Id}_{\mathcal{E}_\eta}} \widehat{\mathcal{O}}_{G_m^\sharp} \otimes \mathcal{E}_\eta$$

it suffices to show we have the following commutative diagram with vertical arrows isomorphic

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{G_m^\sharp} \otimes M & \xrightarrow{\frac{d}{d\log(t)} \otimes \mathrm{Id}} & \widehat{\mathcal{O}}_{G_m^\sharp} \otimes M \\ \simeq \uparrow & & \uparrow \simeq \\ \widehat{\mathcal{O}}_{G_m^\sharp} \otimes M & \xrightarrow{\frac{d}{d\log(t)} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Theta_M} & \widehat{\mathcal{O}}_{G_m^\sharp} \otimes M \end{array}$$

We then win by the usual trick of trivializing Hopf algebra's comodules and the above Theorem. ☐

References

[BL22a] Bhargav Bhatt and Jacob Lurie, *Absolute prismatic cohomology*, arXiv preprint arXiv:2201.06120 (2022).

[BL22b] ———, *Prismatic f-gauges*, Lecture notes available at https://www. math. ias. edu/~ bhatt/teaching/mat549f22/lectures. pdf (2022).